NONLINEAR THEORY OF SOUND INSTABILITY IN PIEZOELECTRICS

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A nonlinear theory is developed for the amplification of small-amplitude traveling sound waves in piezoelectric semiconductors in a constant electric field. Nonlinear effects are taken into account by amplitude iteration. It is shown that the method should be applied differently in the cases of strong and weak decay (or buildup) of the sound waves. The laws of propagation are found for waves whose amplification coefficient and phase velocity depend on the amplitude.

A linear theory of the amplification of traveling sound waves in piezoelectric semiconductors placed in a homogeneous electric field was constructed by White.[1] The purpose of the present work is to construct a nonlinear theory of sound amplification for this case.

The role of nonlinear effects in the propagation of traveling waves has been considered previously.[2,3] It was shown that nonlinear effects can lead to the existence of stationary waves that are propagated without being amplified or damped. In the present work (Sec. 2), a nonlinear theory is constructed for nonstationary waves of small amplitude. The problem is considered as to how the stationary waves are formed. A traveling sound wave, upon being amplified, goes over asymptotically into a stationary wave if the nonlinear correction to the damping coefficient of sound is greater than zero. A calculation of this correction is performed by the method of amplitude iteration, and the conditions are determined for which it is positive. The law governing the transition of the small amplitude traveling wave into a stationary wave is determined.

Throughout the work, the nonlinear effects of electronic origin are considered as the source of the nonlinearity. In most cases they are the most significant. The corresponding estimates are given in Sec. 3.

Section 1 is introductory and devoted to the transformation of the initial equations to the laboratory system of coordinates. This problem is most important to us since insufficiently clear understanding of it frequently leads to misunderstandings.

1. BASIC EQUATIONS OF THE PROBLEM

The set of equations that describe the sound propagation in piezoelectric semiconductors comprises the equations of elasticity theory and the equations of electrodynamics, and also the equations for the electric-induction vector and the current-density vector. Let us consider some point of a continuous medium. Before deformation, its coordinates were a_i ($i = 1, 2, 3$); after deformation, this point has taken the position $x_i = a_i + u_i(a_k, t)$, where u_i is the displacement vector. Thus, each point of the medium can be characterized either by the Lagrangian coordinates a_i, t, or by the Eulerian coordinates x_i, t. Initially, it is convenient to express certain of the equations mentioned above in the Lagrangian system of coordinates. Our goal is to rewrite all the equations in the Eulerian system, since it is the laboratory system.

The equations of elasticity theory in the Lagrangian system have the form ([4] p. 743):

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ik}}{\partial a_k}, \tag{1.1}$$

where ρ_0 is the density of the medium before deformation and σ_{ik} is the stress tensor. Here and below, summation is carried out over repeated subscripts.

The equation of continuity can be written in the form ([4], p. 18)

$$\rho\Delta = \rho_0, \tag{1.2}$$

where ρ is the density of the medium after deformation. Δ is the determinant of the matrix $\partial x_i/\partial a_k$

$= \delta_{ik} + \partial u_i/\partial a_k$. In the Eulerian system, this equation is described in the following fashion:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left[\rho\left(\frac{\partial u_i}{\partial t}\right)_a\right] = 0. \quad (1.3)$$

The transition from (1.2) to (1.3) can be easily made by using the Jacobian identity ([5], p. 246):

$$\frac{\partial \Delta}{\partial(\partial x_h/\partial a_m)} = \Delta \frac{\partial a_m}{\partial x_k}. \quad (1.4)$$

Poisson's equation, written in the Eulerian system, is

$$\frac{\partial \mathscr{D}_i}{\partial x_i} = 4\pi e n, \quad (1.5)$$

where $\mathscr{D}_i$ is the electric induction vector, e is the electron charge, and $n = n_e - n_i$ is the excess concentration of electrons n_e over the concentration of ions n_i.[1)]

The equation of continuity for electrons in both the Eulerian and the Lagrangian systems of coordinates has the same form

$$e\frac{\partial n_e}{\partial t} + \frac{\partial j_{ei}}{\partial x_i} = 0 \quad \text{(Eulerian system)} \quad (1.6)$$

$$e\frac{\partial n_e^{(a)}}{\partial t} + \frac{\partial j_{ei}^{(a)}}{\partial a_i} = 0 \quad \text{(Lagrangian system)}. \quad (1.7)$$

Here j_{ei} is the electron current density, the index a denotes that the corresponding quantity applies to the Lagrangian system, $n_e^{(a)}$ is the number of electrons enclosed in a volume with unity value before deformation. The quantity $j_{ei}^{(a)}$ is similarly defined. The Lagrangian and Eulerian quantities are connected by a certain transformation which we shall obtain here. The following relation, analogous to (1.2), holds for the electron density:

$$n_e\Delta = n_e^{(a)}. \quad (1.8)$$

Substituting (1.8) and (1.7), and making use of the identity (1.4) and also of the relation

$$\left(\frac{\partial f}{\partial t}\right)_a = \left(\frac{\partial f}{\partial t}\right)_x + \left(\frac{\partial u_i}{\partial t}\right)_a \frac{\partial f}{\partial x_i},$$

$$\frac{\partial f}{\partial a_i} = \frac{\partial f}{\partial x_k}\left(\delta_{ik} + \frac{\partial u_k}{\partial a_i}\right), \quad (1.9)$$

we obtain

$$e\Delta\frac{\partial n_e}{\partial t} + e\Delta\frac{\partial}{\partial x_i}\left[n_e\left(\frac{\partial u_i}{\partial t}\right)_a\right] + \frac{\partial j_{ei}^{(a)}}{\partial x_k}\left(\delta_{ik} + \frac{\partial u_k}{\partial a_i}\right) = 0. \quad (1.9a)$$

[1)]An impurity semiconductor is considered, with carriers of one sign, which for definiteness are assumed to be electrons. (or a photoconductor with a short lifetime for holes, such as CdS).

The latter equation must be identical to (1.6). Consequently,

$$\frac{\partial}{\partial x_i}\left[j_{ei} - e n_e\left(\frac{\partial u_i}{\partial t}\right)_a\right] = \frac{1}{\Delta}\left(\delta_{ik} + \frac{\partial u_k}{\partial a_i}\right)\frac{\partial j_{ei}^{(a)}}{\partial x_k}. \quad (1.9b)$$

We now make use of the Euler-Piol-Jacobi identity ([5], p. 246):

$$\frac{\partial}{\partial x_k}\frac{1}{\Delta}\left(\delta_{ik} + \frac{\partial u_k}{\partial a_i}\right) = 0. \quad (1.10)$$

Then

$$\frac{\partial}{\partial x_i}\left[j_{ei} - e n_e\left(\frac{\partial u_i}{\partial t}\right)_a\right] = \frac{\partial}{\partial x_i}\left[\frac{1}{\Delta}\left(\delta_{ik} + \frac{\partial u_i}{\partial a_k}\right) j_{ek}^{(a)}\right]. \quad (1.10a)$$

In the absence of current in the Lagrangian system, the current in the Eulerian system is equal to $e n_e(\partial u_i/\partial t)_a$. Consequently, the transformation law for current has the form

$$j_{ei} = \frac{1}{\Delta}\left(\delta_{ik} + \frac{\partial u_i}{\partial a_k}\right) j_{ek}^{a} + e n_e\left(\frac{\partial u_i}{\partial t}\right)_a. \quad (1.11)$$

Poisson's equation (1.4) contains the net concentration of electrons, n. Therefore, it is convenient to write down the equation of continuity for this quantity. Taking into account that the ion density in the Eulerian system satisfies an equation similar to (1.3),

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x_k}\left[n_i\left(\frac{\partial u_k}{\partial t}\right)_a\right] = 0, \quad (1.12)$$

we get

$$e\frac{\partial n}{\partial t} + \frac{\partial j_k}{\partial x_k} = 0, \quad (1.13)$$

$$j_k = \frac{1}{\Delta}\left(\delta_{ik} + \frac{\partial u_k}{\partial a_i}\right) j_{ei}^{(a)} + e n\left(\frac{\partial u_k}{\partial t}\right)_a. \quad (1.14)$$

The current density in the Lagrangian system is defined by the equation

$$j_{ei}^{(a)} = -n_e^{(a)}\mu_{ik}\frac{\partial \zeta}{\partial a_k}, \quad (1.15)$$

where μ_{ik} is the mobility, which generally depends on the deformation; ζ is the electrochemical potential. The problem as to when such a law is valid was discussed previously.[3] In Eulerian variables, (1.15) can be written in the form

$$j_{ei}^{(a)} = e(n_0 + n\Delta)\mu_{ik}\left(\delta_{kl} + \frac{\partial u_l}{\partial a_k}\right)E_l - T\mu_{ik}\left(\delta_{ik} + \frac{\partial u_l}{\partial a_k}\right)\frac{\partial}{\partial x_l}(n\Delta), \quad (1.15a)$$

where n_0 is the ion concentration in the absence of deformation, E_l is the electric field intensity (as shown by White,[1] the contribution from vortex fields can be neglected with an accuracy up to quantities of the order of the ratio of the sound velocity to the light velocity), and T is the temperature in energy units. We then have for the total current

$$j_i = \frac{1}{\Delta} e (n_0 + n\Delta) \widetilde{\mu}_{ik} E_k - \frac{1}{\Delta} T \widetilde{\mu}_{ik} \frac{\partial}{\partial x_k} (n\Delta) + en \left(\frac{\partial u_i}{\partial t}\right)_a;$$

$$\widetilde{\mu}_{ik} = \left(\delta_{il} + \frac{\partial u_i}{\partial a_l}\right)\left(\delta_{km} + \frac{\partial u_k}{\partial a_m}\right) \mu_{lm}. \qquad (1.16)$$

In the following, we shall carry out the calculation in the approximation of small deformations, that is, we shall assume that $|\partial u_i/\partial a_k| \ll 1$. We limit ourselves to processes that are periodic in time. It is convenient to transform the equations of the problem so that only departures of quantities from their mean time values enter into them. For this purpose, we average all quantities with respect to time and subtract the resultant average equations from the initial equations. We note that the spatial derivatives of the averaged quantities are small (see Secs. 2, 3). This allows us to neglect everywhere the quantities $\langle n \rangle$ (the symbol $\langle \ldots \rangle$ denotes the time average) and $\partial\langle E_k\rangle/\partial x_i$. Then, introducing the drift velocity of the electrons V_i and the potential of the variable field φ:

$$V_i = \mu_{ik}\langle E_k\rangle, \quad E_k = \langle E_k\rangle - \partial\varphi / \partial x_k, \qquad (1.17)$$

we get

$$\langle j_i\rangle = en_0 V_i - e\mu_{ik}\langle n\partial\varphi / \partial x_k\rangle, \qquad (1.18)$$

$$\partial\langle j_i\rangle / \partial x_i = 0. \qquad (1.19)$$

The first component on the right side is the conductivity current, and the second

$$j_i^{ac} = -e\mu_{ik}\langle n\partial\varphi / \partial x_k\rangle \qquad (1.20)$$

is the sound-electric current, that is, the additional time-independent current which arises under the action of the sound wave.

Below we shall denote by the symbols u_i and j_i the difference between the corresponding quantity and its mean value. Then, the set of equations of the problem takes the form (see[3]):

$$\rho \frac{\partial^2 u_i}{\partial t^2} = c_{iklm} \frac{\partial u_{lm}}{\partial x_k} - \beta_{l,ki} \frac{\partial^2 \varphi}{\partial x_k \partial x_l} + \eta_{iklm} \frac{\partial^2 u_{lm}}{\partial x_k\, \partial t}, \qquad (1.21)$$

$$\varepsilon_{ik} \frac{\partial^2 \varphi}{\partial x_i\, \partial x_k} + 4\pi\beta_{i,kl} \frac{\partial^2 u_k}{\partial x_i\, \partial x_l} + 4\pi en = 0, \qquad (1.22)$$

$$e\frac{\partial n}{\partial t} + \frac{\partial j_i}{\partial x_i} = 0, \qquad (1.23)$$

$$j_i = enV_i - e\mu_{ik}(n + n_0)\frac{\partial\varphi}{\partial x_k} - eD_{ik}\frac{\partial n}{\partial x_k} + e\mu_{ik}\left\langle n\frac{\partial\varphi}{\partial x_k}\right\rangle. \qquad (1.24)$$

Then c_{iklm}, $\beta_{i,kl}$, and η_{iklm} are the tensors of the elastic moduli, the piezoelectric moduli and the viscosity coefficients, respectively, ϵ_{ik} is the tensor of dielectric susceptibility, and D_{ik} is the tensor of the diffusion coefficients.

2. ITERATION METHOD

Let the displacement, which varies harmonically in time with frequency ω, be given on the boundary of the crystal. We assume that the boundary is perpendicular to a crystal symmetry axis of order higher than second (the x axis) and consider the propagation of traveling sound waves along this axis. We shall neglect effects associated with lattice absorption. The corresponding set of equations has the form (see[1,3])

$$\rho\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^2\varphi}{\partial x^2} = 0, \qquad (2.1)$$

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{4\pi\beta}{\varepsilon}\frac{\partial^2 u}{\partial x^2} + \frac{4\pi e}{\varepsilon}n = 0, \qquad (2.2)$$

$$e\frac{\partial n}{\partial t} + \frac{\partial j}{\partial x} = 0, \qquad (2.3)$$

$$-j + enV - e\mu n_0\frac{\partial\varphi}{\partial x} - eD\frac{\partial n}{\partial x} = e\mu\left(n\frac{\partial\varphi}{\partial x} - \left\langle n\frac{\partial\varphi}{\partial x}\right\rangle\right); \qquad (2.4)$$

Here

$$u = u_i i_i, \quad \beta = \beta_{x,ix} i_i, \quad \varepsilon = \varepsilon_{xx}, \quad c = c_{ixlx} i_i i_l, \quad j = j_x, \quad \mu = \mu_{xx}, \quad V = V_x, \qquad (2.4a)$$

i is the vector of sound polarization.

It is convenient to represent the solution of the set (2.1)—(2.4) in complex form. We assume the measured physical quantities u, $\partial\varphi/\partial x$, n, and j to be the real parts of the calculated complex quantities. In the calculation of the nonlinear term, we shall make use of relations of the type

$$\mathrm{Re}\, A \cdot \mathrm{Re}\, B = {}^1\!/_2 \mathrm{Re}\, (AB + AB^*). \qquad (2.4b)$$

In these cases the nonlinear term is expressed in complex notation; we shall include it in square brackets. We shall so choose the exponents that the term which is proportional to time enters with a minus sign.

In the linear approximation, one can neglect terms on the right hand side of (2.4) and set $V = V_0 \equiv U/L$, where U is the difference in potentials on the ends of the specimen, and L is its length. Then the solution of the set (2.1)—(2.4) is represented by a sum of terms of the form $e^{i(qx - \omega t)}$, while the wave vector q and the frequency are connected by the dispersion relation (see[1])

$$[\rho\omega^2 - (c + 4\pi\beta^2/\varepsilon)q^2][1 + q^2/\varkappa^2 + i(V_0 q - \omega)\tau_M] = -4\pi\beta^2 q^2/\varepsilon. \quad (2.5)$$

Here $\kappa = (4\pi e^2 n_0/\epsilon T)^{1/2}$ is the inverse of the Debye-Hückel radius, $\tau_M = \epsilon/4\pi e n_0 \mu$ is the Maxwell relaxation time.

Equation (2.5) is of fourth order relative to q. Consequently, for a given ω, there are four possible values of the wave vector, two of which have positive real part (direct wave) and two negative (inverse wave). For $\beta = 0$, Eq. (2.5) describes two sound waves and two electron density waves. The latter are intensively damped: the damping decrement of the direct wave (that is, the wave traveling in the direction of the carrier drift) is greater than $1/V_0\tau_M + \kappa$, while that of the inverse wave is greater than κ. If $\beta \neq 0$, but

$$\chi = 4\pi\beta^2/\varepsilon c \ll 1, \quad (2.6)$$

$$\chi \ll 2\omega\tau_M\left(1 - \frac{V_0}{w_0}\right)^2 + \frac{2}{\omega\tau_M}\left(1 + \frac{\omega^2}{\varkappa^2 w_0^2}\right)^2, \quad (2.7)$$

where $w = c/\rho$, then the corresponding four waves differ slightly from those which exist for $\beta = 0$.

Condition (2.6) is satisfied for all known piezoelectric semiconductors[6] ($\chi \approx (1-4) \times 10^{-2}$). The condition (2.7) signifies smallness of the corrections to the wave vectors of the two direct terms, in comparison with the modulus of the difference of these wave vectors. If the inequality (2.7) is not satisfied, then, in spite of (2.6), two direct waves appear as the result of a strong resonance interaction of the mechanical and electron systems and are not similar to the waves which exist at $\beta = 0$ (see also[7]). The inequality (2.7) is satisfied for all frequencies if

$$n_0 \gg 3\cdot 10^{-2}\chi^2 \frac{\varepsilon}{4\pi}\frac{w_0^2}{\mu^2 T}. \quad (2.8)$$

For $\mu \approx 300$ cm²/V-sec and $T \approx 4 \times 10^{-14}$ erg, the expression on the right hand side of (2.8) amounts to 10^{10} cm⁻³. In what follows, we shall assume the condition (2.8) to be satisfied.

If one considers a crystal of sufficient length and neglects the reflected waves, then the solution of the linear problem can be represented in the form of the sum of two terms, one of which is proportional to e^{iqx}, and the other to $e^{iq_e x}$, with the amplitudes of these terms determined from the boundary conditions. Here q is the wave vector of the sound wave, and q_e is the wave vector of the electron density wave. Below, we shall be interested in the amplification of sound waves. The characteristic length, at which the sound amplification is changed by a quantity of the order of its own size, is greater than $1/\chi q$. At the same time, one can neglect the electron density wave at distances greater than V_0 from the boundary. This means that if

$$\chi_1 = \chi q V_0 \tau_M \ll 1, \quad (2.9)$$

then we can neglect the effect of the narrow region near the boundary, and assume that the amplitude of the sound wave is given on the boundary, but there is no electron-density wave at all. We note that when $V \gtrsim w_0$, (2.7) is a consequence of (2.9).

Thus, the solution of the linear problem is written in the form

$$u^{(1)} = ve^{i(q_1 x - \omega t)}, \quad (2.10)$$

$$\frac{\partial\varphi^{(1)}}{\partial x} = -\frac{4\pi\beta}{\varepsilon} i q_0 \frac{a}{1+a} u^{(1)}, \quad (2.11)$$

$$n^{(1)} = \frac{\beta}{e} q_0^2 \frac{1}{1+a} u^{(1)}, \quad (2.12)$$

$$a = q_0^2/\varkappa^2 + i\nu, \quad \nu = (V_0 q_0 - \omega)\tau_M. \quad (2.13)$$

The value of q_1 is determined from the dispersion relation:

$$q_1 = q_0\left(1 - \frac{\chi}{2}\frac{a}{1+a}\right), \quad q_0 = \frac{\omega}{w_0}, \quad (2.14)$$

$$\operatorname{Re} q_1 = q_0\left[1 - \frac{\chi}{2}\frac{q_0^2\varkappa^{-2}(1 + q_0^2/\varkappa^2) + \nu^2}{(1 + q_0^2/\varkappa^2)^2 + \nu^2}\right], \quad (2.15)$$

$$\operatorname{Im} q_1 = -\frac{\chi}{2} q_0 \frac{\nu}{(1 + q_0^2/\varkappa^2)^2 + \nu^2} = i\frac{\chi}{4} q_0 \frac{a - a^*}{|1+a|^2}, \quad (2.16)$$

v is determined from the condition

$$u|_{x=0} = v\cos\omega t. \quad (2.17)$$

Here and below we shall neglect terms of the order of χ_1 in comparison with unity, and it will be necessary for us to take into account terms of the order of χ in the argument of the exponential, since they describe the amplification or the attenuation of sound (the parameter χ_1 could enter the argument of the exponential only in the form of the product $\chi\chi_1$, which we shall neglect).

Formulas (2.10)–(2.12) can serve as an excellent approximation to the solution of the set (2.1)–(2.4) so long as $ve^{-\operatorname{Im} q_1 x}$ is sufficiently small. In order to take into account effects associated with the nonlinear terms, it is necessary to make a more accurate approximation. This can be done by considering in the solution the higher terms of the expansion in a small parameter proportional to $ve^{-\operatorname{Im} q_1 x}$.

In order to consider the nonlinear effects with accuracy of second order, it is necessary to compute the right hand side of (2.4) with the aid of (2.10)—(2.12) and to solve the resultant linear inhomogeneous system (2.1)—(2.4). We have

$$\left[e\mu\left(n\frac{\partial\varphi}{\partial x}-\left\langle n\frac{\partial\varphi}{\partial x}\right\rangle\right)\right]=-i\frac{4\pi\beta^2}{\varepsilon}\frac{\mu}{2}q_0^3\frac{a}{(1+a)^2}[u^{(1)}]^2. \tag{2.18}$$

The desired solution is the sum of the particular solution of the inhomogeneous system and the solution of the homogeneous system. The particular solution has the form

$$u^{(2)}=\frac{4\pi\beta}{\varepsilon}\frac{\mu\tau_M}{4}q_0^2\frac{a}{(1+a)(2q_0^2/\varkappa^2+a)}[u^{(1)}]^2, \tag{2.19}$$

$$\frac{\partial\varphi^{(2)}}{\partial x}=-i\frac{4\pi\beta^2}{\varepsilon}\frac{4\pi}{\varepsilon}\frac{\mu\tau_M}{2}q_0^3\frac{a^2}{(1+a)(2q_0^2/\varkappa^2+a)}[u^{(1)}]^2, \tag{2.20}$$

$$n^{(2)}=\frac{4\pi\beta^2}{\varepsilon}\frac{\mu\tau_M}{e}q_0^4\frac{a}{(1+a)(2q_0^2/\varkappa^2+a)}[u^{(1)}]^2. \tag{2.21}$$

The moduli of the denominators in Eqs. (2.19)—(2.21) are always different from zero. This means that the harmonic, with the wave vector $2q_1$ and frequency 2ω is not a resonant harmonic; that is, the system possesses a dispersion which is due to the interaction of the sound with the electrons. Since the dispersion is proportional to the small parameter χ (2.14), the harmonic $e^{2i(q_1x-\omega t)}$ is almost resonant, as a consequence of which $u^{(2)}$ is proportional not to the third but to the first power of β. It can be shown that the ratio of $u^{(2)}$ to $u^{(1)}$ is of the order of the ratio of the "nonlinearity" to the "dispersion" (more exactly, to the sum of the "dispersion" and "attenuation").

The solution of the inhomogeneous system is represented by the sum of the electron density wave and the sound wave propagating in the positive direction along the x axis. In this case, and also in all the subsequent approximations, we shall neglect the reflected waves. The electron-density wave, as noted above, can be neglected at distances larger than V_0 from the boundary. This neglect pertains also to all higher approximations. The remaining sound wave is described by equations of the type (2.10)—(2.12), that is, by a solution of the homogeneous set of equations (we shall therefore denote the corresponding quantities by the index h - homogeneous):

$$u_h^{(2)}=v_2\,e^{i(q_2x-2\omega t)}, \tag{2.22}$$

$$\frac{\partial\varphi_h^{(2)}}{\partial x}=-\frac{4\pi\beta}{\varepsilon}2iq_0\frac{a_2}{1+a_2}u_h^{(2)}, \tag{2.23}$$

$$n_h^{(2)}=\frac{\beta}{e}4q_0^2\frac{1}{1+a_2}u_h^{(2)}, \tag{2.24}$$

$$a_2=4q_0^2/\varkappa^2+2i\nu, \tag{2.25}$$

$$q_2=2q_0\left(1-\frac{\chi}{2}\frac{a_2}{1+a_2}\right). \tag{2.26}$$

The amplitude v_2 is determined from the boundary conditions. It can be shown that, with accuracy up to small quantities proportional to χ and χ_1, these conditions have the same form as on the boundary of two nonpiezoelectric continuous media. It turns out that in a piezoelectric medium two waves are propagated with frequency 2ω: a free wave with wave vector q_2 [(2.22)—(2.24)], and a forced wave with wave vector $2q_1$ [(2.19)—(2.21)]. In a nonpiezoelectric medium, a free wave is radiated from the boundary with frequency 2ω (see the work of Bloembergen and Pershan[8] and also that of Tell[9]). The amplitudes of these waves are generally of the same order.

In order to make the next step and find the third approximation, it is necessary to calculate the contribution to the right side of (2.4), proportional to v^3. To this purpose, it is necessary to use expressions for n and $\partial\varphi/\partial x$ with accuracy to second order. As a result, terms are obtained which contain the first and third harmonics. We shall see below that the third harmonics are of no interest to us. Terms of third order containing the first harmonic arise not only from the nonlinearity of the right side of (2.4) but also because of corrections to V associated with the sound-electric current. Equations (1.18), (1.19) give

$$\langle j\rangle=en_0V-e\mu\left\langle n\frac{\partial\varphi}{\partial x}\right\rangle. \tag{2.27}$$

We then find

$$V=\mu\frac{U}{L}-\frac{\mu}{n_0}\left\{\frac{1}{L}\int_0^L\left\langle n\frac{\partial\varphi}{\partial x}\right\rangle dx-\left\langle n\frac{\partial\varphi}{\partial x}\right\rangle\right\}. \tag{2.28}$$

The second component in this equation does not depend on x and therefore it is most convenient to take it into account by replacing V_0 in the equations of first approximation by

$$\tilde{V}=\mu\frac{U}{L}-\frac{\mu}{n_0}\frac{1}{L}\int_0^L\left\langle n\frac{\partial\varphi}{\partial x}\right\rangle dx. \tag{2.29}$$

We call attention to the fact that $\tilde{V}$ is in general a nonlocal quantity, that is, it depends on the values of the sound amplitude over the whole specimen. So far as the term $\mu n_0^{-1}\langle n\partial\varphi/\partial x\rangle$ is concerned, it is convenient to transfer it to the right side of (2.4) and include it in the perturbation.

The entire third-order term with the first harmonic on the right hand side of (2.4) has the form

$$\frac{e\mu}{n_0}n^{(1)}\left\langle n^{(1)}\frac{\partial\varphi^{(1)}}{\partial x}\right\rangle+\left[e\mu\left(n\frac{\partial\varphi}{\partial x}\right)-e\mu\left\langle n\frac{\partial\varphi}{\partial x}\right\rangle\right]_1^{(3)}$$

$$=i\frac{4\pi\beta^2}{\varepsilon}\frac{4\pi\beta}{\varepsilon}\frac{\mu^2\tau_M}{4}q_0{}^5\frac{R'+R''}{(1+a)\,|1+a|^2}v^3$$
$$\times\exp\left[-2\,\mathrm{Im}\,q_1x-i(q_1x-\omega t)\right]. \qquad (2.30)$$

Here the lower index denotes the number of the harmonic,

$$R'=a\frac{2a^*-a}{2q_0{}^2/\varkappa^2+a}, \qquad (2.31a)$$

$$R''=a-a^*, \qquad (2.31b)$$

a^* denotes the complex conjugate of the quantity, while R'' is associated with the sound-electric current.

The term with the first harmonic, brought by the interaction of the fundamental wave with the free wave of frequency 2ω, has the form

$$\left[e\mu n\frac{\partial\varphi}{\partial x}\right]_{1h}^{(3)}=\frac{4\pi\beta^2}{\varepsilon}i\mu q_0{}^3\frac{2a^*-a}{(1+a_2)\,(1+a^*)}$$
$$\times v_2v\exp\left[i(q_2-q_1{}^*)x-i\omega t\right]. \qquad (2.32)$$

Thus the third approximation can be represented in the form of the sum of two terms

$$u_1{}^{(3)}=u_{10}{}^{(3)}+u_{1h}{}^{(3)}; \qquad (2.33)$$

$$u_{10}{}^{(3)}=i\frac{4\pi\beta^2}{\varepsilon c}\frac{q_0}{16}\frac{R'+R''}{(1+a)^2|1+a|^2}\left|\frac{\beta}{en_0}q_0{}^2v\right|^2$$
$$\times\frac{\exp(-2\,\mathrm{Im}\,q_1x)-1}{\mathrm{Im}\,q_1}u^{(1)}, \qquad (2.34)$$

$$u_{1h}{}^{(3)}=-\frac{4\pi\beta^2}{\varepsilon c}\frac{4\pi\beta}{\varepsilon}\frac{\mu\tau_M}{2}q_0{}^3\frac{2a^*-a_2}{(1+a_2)\,|1+a_2|^2}$$
$$\times\frac{\exp\left[i(q_2-q_1-q_1{}^*)x\right]-1}{q_2-q_1-q_1{}^*}v_2u^{(1)}. \qquad (2.35)$$

$$q_2-q_1-q_1{}^*=-\chi q_0\left\{3\left[\frac{q_0{}^2}{\varkappa^2}\left(1+\frac{q_0{}^2}{\varkappa^2}\right)\left(1+4\frac{q_0{}^2}{\varkappa^2}\right)+\nu^2\right]\right.$$
$$+2i\nu\left[\left(1+\frac{q_0{}^2}{\varkappa^2}\right)^2+\nu^2\right]\Bigg\}\left\{\left[\left(1+4\frac{q_0{}^2}{\varkappa^2}\right)^2+4\nu^2\right]\right.$$
$$\times\left.\left[\left(1+\frac{q_0{}^2}{\varkappa^2}\right)^2+\nu^2\right]\right\}^{-1}. \qquad (2.36)$$

The field intensities and the concentrations corresponding to these components are related to the displacements by formulas completely analogous to (2.11)–(2.12), which we shall not write down here.

Let us now consider the problem of the convergence of this method of successive approximations For this purpose, we should estimate the ratio of the second approximation to the first and the third to the second. For definiteness, we consider the case $q_0 \approx \kappa$; the other cases can be considered in similar fashion.

For the estimates, it is convenient to introduce the dimensionless square of the potential amplitude

$$h=2\left|\frac{1+a}{a}\right|^2\left\langle\left(\frac{e\varphi}{T}\right)^2\right\rangle. \qquad (2.37)$$

The factor $|1+a|^2/|a|^2$ is introduced for convenience. The quantity h is proportional to the energy density of the sound wave

$$W^{ac}=\frac{1}{2}\rho\left\langle\left(\frac{\partial u}{\partial t}\right)^2\right\rangle+\frac{1}{2}c\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle=\frac{1}{2}\frac{Tn_0}{\chi}\frac{q_0{}^2}{\varkappa^2}h. \qquad (2.38)$$

In this formula we have neglected the electric energy of the sound wave which is seen to be of the order of χ in comparison with the mechanical energy.

The following order of magnitude estimates are obtained:

$$|u^{(1)}|\sim ve^{-\mathrm{Im}\,q_1x}, \qquad (2.39)$$

$$|u^{(2)}|\sim\frac{\sqrt{h}}{1+|\nu|}|u^{(1)}|, \qquad (2.40)$$

$$|u_{10}{}^{(3)}|\sim\frac{h}{1+|\nu|}\frac{1-e^{2\,\mathrm{Im}\,q_1x}}{|\nu|}|u^{(1)}|, \qquad (2.41)$$

$$|u_{1h}{}^{(3)}|\sim\frac{h}{1+\nu^2}e^{2\,\mathrm{Im}\,q_1x}\left|\exp\left[i(q_2-q_1-q_1{}^*)x\right]-1\right||u^{(1)}|. \qquad (2.42)$$

Here the expression $1+|\nu|$ denotes a quantity of the order of unity for $|\nu|\lesssim$) and a quantity of the order of $|\nu|$ for $|\nu|\gtrsim 1$.

It is seen from Eqs. (2.39)–(2.42) that in this formulation perturbation theory represents a decomposition in terms of two parameters: $\sqrt{h}/(1+|\nu|)$ and $\sqrt{h}/|\nu|$. The first of these parameters has the physical meaning of the ratio of the "nonlinearity" to the "attenuation." The meaning of this statement is that, in the absence of the nonlinear equation describing the propagation of the wave, damping and dispersion, there would be no basis for expecting that the second harmonic would be small in comparison with the first even at small amplitudes. This can easily be established, for example, by attempting to construct the iteration method for the equation

$$\frac{\partial^2u}{\partial t^2}-w^2\frac{\partial^2u}{\partial x^2}=\lambda u^2. \qquad (2.42a)$$

If dispersion or attenuation is included in this equation, then for sufficiently small amplitudes there is a solution that is close to the harmonic; that is, the nonlinearity on the one hand and dispersion and attenuation on the other are competing factors which determine the shape of the wave. If the nonlinearity is small in comparison with the dispersion and attenuation—in our case, this also means the satisfaction of the inequality

$$\sqrt{h}/(1+|\nu|) \ll 1,$$
$$\sqrt{h}/|\nu| \ll 1, \qquad (2.43)$$

—then this guarantees the possibility of the direct construction of the iteration method in terms of the nonlinearity.

If the attenuation is much less than the dispersion, then the situation can arise in which the nonlinearity would be smaller than the dispersion but larger than the attenuation. In our case, this corresponds to the satisfaction of the inequality

$$|\nu| \lesssim \sqrt{h} \ll 1. \qquad (2.44)$$

Here the nonlinear effects accumulate as the wave is propagated; this would not happen for sufficiently large attenuation. In fact, by comparing the third approximation with the second, we see that in the case (2.44), the corrections of the third approximation are larger than or of the order of the corrections of the second. The case $\nu = 0$ is most clear from this viewpoint (we recall that $\mathrm{Im}\, q_1 = 0$ for $\nu = 0$, and consequently $h = \mathrm{const}$), when $u^{(3)}$ is a linear function of the coordinate x. In other words, secular terms appear, similar to those which are encountered in the theory of nonlinear vibrations. Another method of perturbation theory is necessary for the calculation of the accumulating nonlinear effects, to the exposition of which we now turn.

It can be noted that the divergence of perturbation theory is due to the fact that in the third approximation the nonlinear term in (2.4) produces the harmonic

$$\exp[-2\,\mathrm{Im}\, q_1 x + i(q_1 x - \omega t)], \qquad (2.44a)$$

which differs slightly from the solution of the system (2.1)—(2.4) with the discarded nonlinear term in (2.4). In order to eliminate divergence, it suffices to include the resonant or almost resonant harmonics in the first approximation. For this purpose, we write (2.4) in the form

$$-j + en(V+\Delta V) - e\mu n_0 \frac{\partial \varphi}{\partial x} - eD\frac{\partial n}{\partial x}$$
$$= en\Delta V + e\mu\left(n\frac{\partial \varphi}{\partial x} - \left\langle n\frac{\partial \varphi}{\partial x}\right\rangle\right) \qquad (2.4a)$$

In the calculation of each succeeding approximation, it will now be necessary for us to compute both terms on the right hand side of (2.4a) with the help of the previous approximations, selecting ΔV in such a way that the resonant harmonics are eliminated. Since the resonant harmonics first appear in the third approximation, it is then necessary to consider ΔV as a quantity of first order of smallness in h.

Inasmuch as we shall have V + ΔV in place of V_0 in the dispersion relation, the wave vector increases (in first approximation) in proportion to h. Such a renormalization of the wave vector corresponds to the idea that the appearance of secular terms in the solution connected with the expansion of the type

$$\exp[i(q_1+\Delta q)x] = \exp(iq_1 x)(1 + i\Delta q x + \ldots), \qquad (2.44')$$

is invalid for sufficiently large x. We note that for $\mathrm{Im}\, q_1 \neq 0$, the quantity V + ΔV will be a weak function of the coordinate. The same applies to the renormalized wave vector $q = q_1 + \Delta q$. Since the wave vector is defined as the derivative of the exponential with respect to the coordinate, divided by i, we must write in the exponent, in place of qx:

$$\int_0^x q\, dx'. \qquad (2.44'')$$

We shall everywhere neglect the derivatives of q with respect to x as quantities of higher order of smallness.

We call attention to the fact that the quantity ΔV does not have direct physical meaning as a correction to the drift velocity of the electrons due to the sound. This is already evident from the fact, for example, that ΔV is generally a complex quantity. The real change in the drift velocity because of the sound-electric current is determined by Eq. (2.28). The introduction of the quantity ΔV is no more than a convenient mathematical device which permits us to compute the correction to the wave vector Δq, a quantity with a direct physical meaning.

The perturbation-theory method, which is based on the renormalization of the wave vector is similar to a method which is well known in the theory of nonlinear vibrations (see the book of Bogolyubov and Mitropol'skiĭ[10] and also the book of Landau and Lifshitz,[11] p. 109) and was applied to the investigation of nonlinear waves.[12-14] The present paper is in large measure devoted to the development of this method and its application to piezoelectrics.

By defining ΔV in the way stated above, we get, with the help of (2.30),

$$\Delta V = -i\frac{4\pi\beta^2}{\varepsilon}\frac{4\pi}{\varepsilon}\frac{\mu^2\tau_M}{4}q_0^3\frac{R'}{|1+a|^2}v^2 e^{-2\,\mathrm{Im}\, qx}$$
$$= -\frac{iq_0^3}{4\tau_M\varkappa^4}\frac{R'}{|1+a|^2}h. \qquad (2.45)$$

Now, using the dispersion relation (2.14), we find the correction to the wave vector

$$\Delta q = -\frac{\chi}{2} q_0 \frac{\partial}{\partial V}\left(\frac{a}{1+a}\right)_{V=V_0}(V+\Delta V - V_0)$$
$$= -\frac{\chi}{8} q_0 \frac{q_0^4}{\varkappa^4} \frac{1}{(1+a)^2|1+a|^2}$$
$$\times \left\{R'h + R''\left[h - \frac{1}{L}\int_0^L h dx\right]\right\}. \qquad (2.46)$$

Thus this perturbation-theory method leads generally to the following result: with accuracy up to a component of order h,

$$u = v \exp\left\{i\int_0^x q\, dx - i\omega t\right\}, \qquad (2.47)$$

$$q = q_1 + \Delta q. \qquad (2.48)$$

It follows from (2.47) that h should satisfy the differential equation

$$dh / dx = -\Gamma h, \qquad (2.49)$$

where the damping coefficient of sound attenuation $\Gamma = -2 \operatorname{Im} q$.

Formulas of the type (2.46) make it possible to obtain the first terms of the expansion of Γ in powers of h:

$$\Gamma = \Gamma^0 + \Gamma' h + \ldots \qquad (2.50)$$

Limiting ourselves in (2.50) to the first two terms, and substituting this expression for Γ in (2.49), we get

$$dh / dx = -(\Gamma^0 + \Gamma' h) h. \qquad (2.51)$$

An equation of the type (2.51), which describes the dependence of the amplitude of the vibrations on the time, has already been investigated previously.[4]

We call attention to the fact that use of Eq. (2.51) in the case of (2.43) is an unnecessary increase in accuracy. Actually, this equation takes into account corrections to h of third order, but does not take into account corrections of second order, which in this case are of the same order or larger.

Let us investigate the opportunities afforded in principle by the solution of Eq. (2.51). By specifying the boundary condition $h = h_0$ at $x = 0$, we have

$$\frac{1}{h} = -\frac{\Gamma'}{\Gamma^0} + \left(\frac{1}{h_0} + \frac{\Gamma'}{\Gamma^0}\right)\exp(\Gamma^0 x). \qquad (2.52)$$

If $\Gamma'/\Gamma^0 < 0$, then a stationary mode is possible, with amplitude $h = h_{st} \equiv -\Gamma^0/\Gamma'$. When $\Gamma^0 < 0$, this mode is stable relative to small changes in amplitude and is established over distances of the order of $-1/\Gamma^0$. In the case $\Gamma^0 > 0$, this mode is unstable: waves with $h < h_{st}$ will be damped while those with $h > h_{st}$ will grow. Thus, for $\Gamma^0 > 0$ and $\Gamma' < 0$ the state of the piezoelectric is stable relative to small amplitude perturbations and unstable relative to perturbations of sufficiently large amplitude (see[15]).

If $\Gamma'/\Gamma^0 > 0$, the stationary mode is generally impossible in the region of applicability of (2.51). For $\Gamma^0 > 0$, all perturbations will be damped, while for the opposite inequality they will all grow. If $\Gamma^0 = 0$, then

$$1/h = 1/h_0 + \Gamma' x; \qquad (2.53)$$

for $\Gamma' > 0$ the wave will be damped, and for $\Gamma' < 0$, it will grow.

From (2.16) and (2.46), we find

$$\Gamma = \Gamma^0\left\{1 - \frac{1}{2}\frac{q_0^4}{(q_0^2+\varkappa^2)^2}\frac{1}{L}\int_0^L h dx + \frac{1}{36}\frac{(11\varkappa^2 + 5q_0^2)q_0^4}{(q_0^2+\varkappa^2)^3} h\right\}, \qquad (2.54)$$

$$\Gamma^0 = -\chi q_0 \nu \frac{\varkappa^4}{(q_0^2+\varkappa^2)^2}. \qquad (2.55)$$

Here it is taken into account that a condition of the type (2.44), for an arbitrary value of q_0^2/κ^2, leads to $|\nu| \ll q_0^2/\kappa^2$. It is seen from these formulas that the wave grows if $\nu = (V_0 q_0 - \omega)\tau_M > 0$, while it is damped if the opposite inequality holds.

Calculating Re Δq with the same accuracy, we get a correction to the phase velocity of the wave,

$$\left.\frac{\partial w}{\partial h}\right|_{h=0} h = \frac{\chi}{24} w_0 \frac{q_0^6 \varkappa^2}{(q_0^2+\varkappa^2)^4} h, \qquad (2.56)$$

which was obtained in the research of one of the authors[2] in the study of stationary waves of small amplitude. We now see that this formula is applicable not only to stationary waves, but also generally to waves of small amplitude for $|\nu| \ll 1$ and $|\nu| \ll q_0^2/\kappa^2$.

Above, we did not consider the lattice sound absorption. To allow for it, we must add the term $-\eta\,\partial^3 u/\partial x^2 \partial t$ to the left side of Eq. (2.1). Here $\eta = \eta_{ixlx} i_i i_l$. All the previous theory is easily generalized to this case. For specific calculations, use should be made of the inequality

$$\delta = \eta\omega / c \ll 1, \qquad (2.57)$$

for only under this condition is it permissible to neglect the dispersion of the coefficient η. Furthermore, if we neglect terms of order $\chi\delta$ in comparison with unity, then the results of the calculation are altered in the following way.

In the linear approximation, the expression for the imaginary part of q_1 is changed, i.e., in place of (2.16) we have

$$\operatorname{Im} q_1 = -\frac{\chi}{2} q_0 \frac{\nu}{(1+q_0^2/\varkappa^2)^2+\nu^2} + \frac{\delta}{2} q_0, \quad (2.58)$$

while Re q_1 remains unchanged. The expressions for the second harmonic and, in particular, the value of $u^{(2)}$ will be determined by the following relation instead of (2.19):

$$u^{(2)} = \frac{[\pi\beta\mu\tau_M q_0^2 a/\varepsilon(1+a)][u^{(1)}]^2}{2q_0^2/\varkappa^2 + a - i\delta\chi^{-1}(1+a)(1+q_0^2/\varkappa^2+2a)}. \quad (2.59)$$

In the third approximation, the only change will be the replacement of R' in (2.30) by

$$R_\eta' = \frac{[2a^* - a - i\delta\chi^{-1}(2a^*+a)]a}{2q_0^2/\varkappa^2 + a - i\delta\chi^{-1}(1+q_0^2/\varkappa^2+2a)(1+a)}. \quad (2.60)$$

In the previous work of the authors,[3] stationary traveling waves were considered, i.e., such sound waves whose amplification coefficient is equal to zero throughout the entire specimen. The values of Γ' were computed for the two possible modes (for a given δ) with small amplitude. The same formulas can also be obtained from the results here by setting $R'' = 0$, i.e., not taking the sound electric current into account. However, the approach set forth in the present research allows us to consider the effect of the sound electric current. The sound-electric current changes the value of Γ' so that, for example, the first mode can be shown to be unstable: for $\delta/\chi \ll 1$,

$$\Gamma' = -\frac{q_0}{12} \frac{q_0^2\varkappa^2(3\varkappa^2+q_0^2)}{(q_0^2+\varkappa^2)^3}\delta. \quad (2.61)$$

In principle, such an experimental situation is conceivable in which a constant electric field is homogeneous along the crystal, i.e., the inhomogeneity of the field created by the sound electric current is eliminated in some sort of an artificial manner. It must be thought that the inhomogeneity of the field must be appreciably reduced in a sufficiently long and thin crystal, placed between metallic plates. Detailed estimates show that a constant electric field can be regarded as homogeneous with sufficient accuracy if the following two conditions are satisfied:

$$R_M / R_s \ll 1, \qquad (l_t\Gamma)^2 \ll 1,$$

where R_M is the resistance of the metallic plates, R_s is the resistance of the semiconductor, l_t is its transverse dimension. Under these conditions all the formulas previously obtained by the authors are literally applicable.[3]

3. CONCLUSION

We shall make clear when the concentration nonlinear effect considered by us is the principal one, and for this purpose, we enumerate the other possible sources of nonlinearity.

1. Nonlinear effects of elasticity theory. These effects lead to the necessity of considering terms in the elastic strain tensor of higher order in the deformation and electric field; however, it is natural to expect that the second is less than the first in respect to χ. It can be shown that the effect of the first is characterized by the quantity $Q \equiv \langle (\partial u/\partial x)^2 \rangle$ in comparison with

$$\chi^2 \frac{(q^2/\varkappa^2+|\nu|)^2}{(1+q^2/\varkappa^2+|\nu|)^4}.$$

2. Dependence of the mobility on the deformation. In the case of a simple band, the effect of the nonlinear effects is also characterized by a quantity Q in comparison with unity. In the case of a complicated band, it is larger in respect to the parameter Λ/T, where Λ is of the order of the constant of the deformation potential. It can be still greater in the case of semiconductors with many energy minima. However, in the latter case, the picture is more complicated and is not described by Eq. (2.4), even with additional terms which take into account the dependence of the mobility of the deformation (see[16]). In semiconductors of the CdS type, where a sound instability was observed, the band is simple, as existing experimental data testify. Therefore, estimates for this case are of highest interest.

3. Nonlinearity associated with departures from Ohm's law. At low temperatures, when the electron scattering takes place principally from acoustic phonons, this effect must be taken into consideration (see[18]), and it can generally serve as the basic source of nonlinearity. At high temperatures (room and above) when the principal scattering mechanism is provided by the optical phonons, the departures from Ohm's law are small and can be neglected.

We now compare the relative role of the concentration nonlinearity and the nonlinear effects characterized by the quantity

$$\frac{1}{\chi}\left|\frac{du}{dx}\right| \frac{(1+q^2/\varkappa^2+|\nu|)^2}{q^2/\varkappa^2+|\nu|}.$$

We consider a typical case. For example, let $q \approx \kappa$, $|\nu| \ll 1$. Then the ratio of these nonlinearities is characterized by the ratio $Q/h\chi^2$, which is of the order of magnitude n_0T/χ^3c. This ratio is usually very small.

One must mention the work of Abe,[19] who studied stationary waves with consideration of both the nonlinear effects of elasticity theory and the concentration effect. Unfortunately, it is difficult to make a comparison of the concentration and other nonlinearities on the basis of that work, for the correction to the mobility, due to deformation, is not taken into account, nor is it shown that it is possible to neglect it.

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