CORRELATION ASYMPTOTICS FOR A PLANE ISING LATTICE

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We derive a relation which is an extension of the Szego formula[9] and can be employed for evaluation of the spontaneous magnetic moment of a plane Ising lattice. An expression for correlations at large distances can be obtained by a refinement of the relation. Expressions for correlations near phase-transition points are also derived.

The Ising model is one of the simplest nontrivial many-particle systems. An investigation of this model has by now led to two results, the calculated specific heat[1] and magnetic moment[2] of a plane lattice without an external field. In the present paper we take the next step and calculate the correlation function.

The correlation function depends on the temperature T and on the distance p. By now we know the values of these functions "on the boundary" at $p = \infty$ [2] (for arbitrary T), at the phase transition point T_c [3] (for arbitrary p), and for small p[3] (for arbitrary T). There are also estimates[1] of the behavior of the correlation functions at $T \gg T_c$ and large p. We propose below some methods that make it possible to move away from the "boundary" and obtain the behavior of the correlation functions everywhere except in the vicinity of the line $(T - T_c)/T_c = p^{-1}$ (a more accurate formulation will be given below). We note that the proposed methods do not reduce to expansions in the reciprocal powers of the radius and in powers of the smallness of the distance to the transition point. The expansion is, roughly speaking, in the first case in powers of

$$\exp\{-p(T-T_c)/T_c\},$$

and in the second case in powers of

$$p\frac{T-T_c}{T}\ln\left(p\frac{T-T_c}{T_c}\right).$$

The present article has a bearing on the series of papers[4-8] connected with the use of the Szego-Kac theorem to calculate the spontaneous magnetic moment of the Ising lattice. We formulate the generalization needed to apply this theorem to an exact calculation. Section 1 contains information necessary for the calculation, especially the formulas of Szego (7) and of Kac (9). Two theorems in Sec. 2—(16) and (19)—provide the required generalization of (7) and (9). In Sec. 4 we describe a procedure that enables us to reduce the problem of calculating the correlations away from the transition point to a solution of a simple integral equation (29). In the direct vicinity of the transition point we use a different method of calculation (Sec. 5).

We disregard the dependence of the correlations on the angles, and calculate only the correlations along the diagonal of a square lattice.

1. FORMULATION OF PROBLEM

We consider a plane square lattice. The lattice sites are numbered by means of two indices k and l. To each site we set in correspondence a quantity σ_{kl}, which can assume two values, ± 1. In the Ising model we choose for the energy of such a system the expression

$$J\sum_{k,l}\sigma_{kl}(\sigma_{k+1\,l}+\sigma_{k\,l+1}),$$

where J is a constant that specifies the intensity of the interaction. The correlation of the values of σ_{rs} at two sites (r, s and r′s′) is given by the expression

$$\langle\sigma_{rs};\,\sigma_{r's'}\rangle = \sum_{\{\sigma_{kl}\}}\sigma_{rs}\sigma_{r's'}\exp\left\{-\frac{J}{kT}\sum_{k,l}\sigma_{kl}(\sigma_{k+1\,l}+\sigma_{k\,l+1})\right\}$$
$$\times\left[\sum_{\{\sigma_{kl}\}}\exp\left\{-\frac{J}{kT}\sum_{k,l}\sigma_{kl}(\sigma_{k\,l+1}+\sigma_{k+1\,l})\right\}\right]^{-1}, \qquad (1)$$

where k is Boltzmann's constant and T the temperature.

Before we go over to a formulation of the problem, we introduce the notation*

*th ≡ tanh.

$$x = \operatorname{th}(J/kT),$$

$$\beta = \begin{cases} \left(\dfrac{1-x^2}{2x}\right)^2 & \text{for } \sqrt{2}-1 < x < 1, \\ \left(\dfrac{2x}{1-x^2}\right)^2 & \text{for } 0 < x < \sqrt{2}-1; \end{cases} \qquad (2)$$

$$f(\omega) = \begin{cases} \dfrac{1-\beta e^{-i\omega}}{(1+\beta^2-2\beta\cos\omega)^{1/2}} & \text{for } \sqrt{2}-1 < x < 1, \\ ie^{-\omega/2} & \text{for } x = \sqrt{2}-1;\ 0 < \omega < \pi, \\ -ie^{-i\omega/2} & \text{for } x = \sqrt{2}-1;\ -\pi < \omega < 0, \\ -\dfrac{e^{-i\omega}(1-\beta e^{i\omega})}{(1+\beta^2-2\beta\cos\omega)^{1/2}} & \text{for } 0 < x < \sqrt{2}-1; \end{cases} \qquad (3)$$

$$K_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln f(\omega)\, e^{-in\omega}\, d\omega;$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega)\, e^{-in\omega}\, d\omega = \langle f\rangle_n, \qquad (4)$$

$$D_p = \det c_{k-l} = \begin{vmatrix} c_0 & c_1 & c_2 \ldots c_{p-1} \\ c_{-1} & c_0 & c_1 & \cdot \\ c_{-2} & c_{-1} & c_0 & \cdot \\ \vdots & & & c_0 \; c_1 \\ c_{-p+1} & \ldots & & c_{-1} \; c_0 \end{vmatrix}. \qquad (5)$$

As shown by several authors[3-8], the different correlations are expressed in terms of determinants of the type (5). In particular, the correlation of two types which lie on one diagonal spaced p cells apart is equal to the determinant (5) with c_k given by relations (3)—(4):

$$\langle \sigma_{kl};\ \sigma_{k+p\ l+p}\rangle = D_p. \qquad (6)$$

The matrix c_{k-l} in the right side of (5) depends only on the difference of the indices, and is called a Toeplitz matrix[9]. The determinant (5) was investigated in connection with different problems of the theory of Toeplitz forms. Szego proved for this determinant the following theorem (we present it in somewhat different formulation). Let $f(\omega) = |\delta(e^{i\omega})|^{-2}$ where $\delta(e^{i\omega})$ is a polynomial of order p in $e^{i\omega}$. Then (see the notation in (4) and (5))

$$D_p(f) = \exp\left(pK_0 + \sum_1^\infty n|K_n|^2\right). \qquad (7)$$

Assume further that $f(\omega)$ is an arbitrary positive function, for which $f'(\omega)$ exists and satisfies the condition

$$\sum_1^\infty n|K_n|^2 < \infty. \qquad (8)$$

Then

$$\lim_{p\to\infty} \frac{D_p(f)}{e^{pK_0}} = \exp\left\{\sum_1^\infty n|K_n|^2\right\}.$$

A similar theorem was proved by a different method by Kac[10] for $f(\omega)$ which is not necessarily positive or real, but satisfies as before the condition (8). Namely, for $|f-1| < 1$

$$\lim_{p\to\infty} \frac{D_p(f)}{e^{pK_0}} = \exp\left\{\sum_1^\infty nK_nK_{-n}\right\}. \qquad (9)$$

According to (3), $|f| = 1$ and $\operatorname{Re} f > 0$, so that there always exists a number M such that $f = M[1 + (f-M)/M]$ and $|(f-M)/M| < 1$. $M = \infty$ only at the transition point. In the general case, however, the condition $|f-1| < 1$ may impose some limitations on f, so that it is of interest to derive (9) without this condition (see Sec. 2 below).

These results cannot be used to calculate (6), since $f(\omega)$ (see (3)) does not satisfy condition (8) above the transition point. Nor can we use the exact relation (7), since $f(\omega)$ is not positive. We shall prove, however, the existence of relations similar to (7)—(9) but more general and more suitable, in particular, for the calculation of correlations in the Ising model.

2. GENERALIZATION OF THE SZEGO THEOREM

We shall prove first the correctness of the following relation, which generalizes (7):

If: a) $f(\omega) = \Delta^{-1}(e^{i\omega})$, where Δ is a trigonometric polynomial of order p:

$$\Delta(e^{i\omega}) = \Delta_0 + \sum_{k=1}^{p} (\Delta_k e^{ik\omega} + \Delta_{-k}e^{-ik\omega});$$

and b) in the expansion of $\Delta(e^{i\omega})$ into elementary factors

$$\Delta(e^{i\omega}) = \gamma_0 \prod_{\nu,\mu=1}^{p} (x_\nu - e^{i\omega})(y_\mu - e^{-i\omega}) \qquad (10)$$

the roots x and y_μ lie outside the unit circle, then

$$D_p(f) = \exp\left\{pK_0 + \sum_1^p nK_nK_{-n}\right\}. \qquad (11)$$

The proof duplicates almost verbatim similar deductions by Szego,[9] which lead to relation (7). Indeed, let us expand $\Delta(e^{i\omega})$ into partial fractions:

$$\frac{1}{\Delta(e^{i\omega})} = \sum_{\nu=1}^{p} \frac{1}{\delta_\nu(x_\nu - e^{i\omega})} \sum_{\mu=1}^{p} \frac{1}{\delta_\mu'(y_\mu - e^{-i\omega})}.$$

$$\delta_\nu = \prod_{\alpha\neq\nu}(x_\alpha - x_\nu), \qquad \delta_\mu' = \prod_{\beta\neq\mu}(y_\beta - y_\mu).$$

For the Fourier coefficients of $f(\omega)$ we have

$$c_{kl} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega)\, e^{-i\omega(k-l)}\, d\omega$$

$$= \frac{1}{2\pi}\sum_{\nu,\,\mu=1}^{p}\frac{1}{\delta_\nu\delta_\mu'}\int_{-\pi}^{\pi}\frac{e^{-i\omega(k-l)}\,d\omega}{(x_\nu - e^{i\omega})\,(y_\mu - e^{-i\omega})}. \qquad (12)$$

The integrand in (12) can be expanded in powers of $e^{i\omega}/x_\nu$ and $e^{-i\omega}/y_\mu$. We obtain

$$c_{kl} = \frac{1}{2\pi}\sum_{\nu,\,\mu=1}^{p}\frac{1}{\delta_\nu\delta_\mu'}\frac{(x_\nu y_\mu)^r x_\nu^{-k} y_\mu^{-l}}{x_\nu y_\mu - 1}, \quad r = \begin{cases} l, & k \geqslant l, \\ k, & k \leqslant l. \end{cases}$$

In this expression the results of summation over ν and μ does not depend on the choice of r if $p - 1 \geq r \geq k$, or if $p - 1 \geq r \geq l$. Indeed, if we replace

$$(x_\nu y_\mu)^r / (x_\nu y_\mu - 1)$$

by

$$(x_\nu y_\mu)^{r-1} / (x_\nu y_\mu - 1) + (x_\nu y_\mu)^{r-1},$$

under the summation sign then the sum of the second term will be equal to zero, since the expression $\sum_{\nu=1}^{p} x_\nu^s/\delta_\nu$ can be regarded as a sum of residues outside the unit circle for the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(s+1)\varphi}\prod_{\nu=1}^{p}(x_\nu - e^{i\varphi})^{-1}\, d\varphi,$$

which vanishes as a result of the analyticity of the integrand inside the unit circle. We can therefore put $r = p - 1$. We obtain

$$c_{kl} = \frac{1}{2\pi}\sum_{\nu,\,\mu=1}^{p}\frac{x_\nu^{\,p-1-k}}{\delta_\nu}\,\frac{1}{x_\nu y_\mu - 1}\,\frac{y_\mu^{\,p-1-l}}{\delta_\mu'}.$$

The matrix c_{kl} is equal to the product of three matrices:

$$x_\nu^{p-1-k}/\delta_\nu, \quad y_\mu^{p-1-l}/\delta_\mu' \text{ and } (x_\nu y_\mu - 1)^{-1}.$$

Consequently,

$$\operatorname{Det} c_{kl} = \operatorname{Det}\frac{x_\nu^{\,p-1-k}}{\delta_\nu}\cdot\operatorname{Det}\frac{y_\mu^{\,p-1-l}}{\delta_\mu'}\cdot\operatorname{Det}\frac{y_\mu^{-1}}{x_\nu - y_\mu^{-1}}.$$

All three determinants are calculated in the same fashion: the factors which are common to a row or to a column are taken outside the determinant sign, and then the first column is subtracted from all the columns, the common factors are taken out, the first line is subtracted from all the lines in the resultant determinant, and the common factors are again taken out. This results in a determinant similar to the initial one, but of lower order. Repeating all the transformations, we reduce the order of the determinant further by unity, etc. As a result we obtain

$$D_p(f) = (\gamma_0 \Pi\, x_\nu\, \Pi\, y_\mu)^{-p}\, \Pi\, (1 / [1 - 1 / x_\nu y_\mu]). \qquad (13)$$

Substitution of (10) in expression (4) yields

$$K_n = \frac{1}{n}\sum_{\nu=1}^{p}\frac{1}{x_\nu^{\,n}};$$

$$K_{-n} = \frac{1}{n}\sum_{\mu=1}^{p}\frac{1}{y_\mu^{\,n}}; \qquad K_0 = -\ln(\gamma_0\, \Pi\, x_\nu\, \Pi\, y_\mu).$$

Substituting these expressions in (11), we obtain (13).

The theorem (10) can be formulated differently: If there is a polynomial of the type (10) such that

$$\text{a) } \langle\Delta^{-1}\rangle_n = \langle f\rangle_n \text{ for } -p+1 \leqslant n \leqslant p-1, \qquad (14)$$

$$\text{b) } \langle\Delta\rangle_n = 0 \qquad \text{for } \quad n \geqslant p,\ n \leqslant -p, \qquad (15)$$

then

$$D_p(f) = \exp\left\{p\varkappa_0 + \sum_{n=1}^{\infty} n\varkappa_n\varkappa_{-n}\right\},$$

$$\varkappa_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\Delta^{-1}(\omega)\, e^{-in\omega}\, d\omega. \qquad (16)$$

Indeed, it follows from (14) that $D_p(f) = D_p(r)$. For $r(\omega)$ the conditions of theorem (11) are satisfied, and for $D_p(r)$ the expression (11) is valid. As a result we obtain (16) for $D_p(f)$. Thus, to calculate $D_p(f)$, namely the correlation of two sites located a distance p apart, it is necessary to obtain Δ from (14) and (15) and substitute it in (16).

The limitation connected with condition (8) can be greatly relaxed. If $f(\omega)$ satisfies condition (8), then

$$g(\omega) = e^{i\omega} f(\omega) \qquad (17)$$

does not satisfy this condition. However, comparison of $D_p(g)$ and $D_p(f)$

$$D_p(g) = \begin{vmatrix} c_1 & c_2 & c_3 \dots c_p \\ c_0 & c_1 & c_2 \quad \cdot \\ c_{-1} & c_0 & c_1 \quad \vdots \\ \vdots & & \ddots\ c_1\ c_2 \\ c_{-p+2} & \dots & c_0\ c_1 \end{vmatrix}; \quad D_{p+1}(f) = \begin{vmatrix} c_0 & c_1 & c_2 \dots c_p \\ c_{-1} & c_0 & c_1 \\ c_{-2} & c_{-1} & c_0 \\ \vdots & & \ddots\ c_0\ c_1 \\ c_{-p} & \dots & c_{-1}\ c_0 \end{vmatrix} \qquad (18)$$

shows that

$$D_p(g) = (-1)^p \frac{\partial}{\partial c_{-p}} D_{p+1}(f). \qquad (19)$$

Similarly, if

$$g_k(\omega) = e^{-i\omega k} f(\omega), \qquad (20)$$

then

$$D_p(g_k) = (-1)^{k(p-(k-1)/2)}\frac{\partial}{\partial c_{-p-k+1}}\cdots\frac{\partial}{\partial c_{-p-1}}\frac{\partial}{\partial c_{-p}} D_{p+k}(f). \qquad (21)$$

and if

$$g_{-k}(\omega) = e^{i\omega k} f(\omega), \tag{22}$$

then

$$D_p(g_{-k}) = (-1)^{k(p-(k-1)/2)} \frac{\partial}{\partial c_{p+k-1}} \cdots \frac{\partial}{\partial c_{p+1}} \frac{\partial}{\partial c_p} D_{p+k}(f). \tag{23}$$

The determinants $D_p(f)$ satisfy theorem (9), and therefore (21) and (23), together with (9), make it possible to apply the Szego-Kac result to functions of the type (20) and (22), which do not satisfy the condition (8).

Theorems (11) and (16) say nothing of the existence of a function $\Delta(\omega)$ with properties (10). Later, when solving (14) and (15), we shall prove by the same token the existence of $\Delta(\omega)$ for finite p. The correctness of the condition (8), that is, the convergence of the series $\sum_1^\infty nK_nK_{-n}$ is verified by substituting the solution in this series.

For the case when $p \to \infty$ it is possible to prove the existence of $\Delta(\omega)$ in a simpler manner, by making use of a theorem (see, for example,[9]), according to which any positive function $g(\omega)$ can be the limit of a polynomial of the type (10) as the number of roots of the polynomial increases without limit. The function

$$\left[\prod_{\mu,\nu} (x_\nu' - e^{i\omega})(y_\mu' - e^{-i\omega})\right]^{-1} \prod_{\mu,\nu} (x_\nu - e^{i\omega})(y_\mu - e^{-i\omega}) g(\omega)$$

$$= \left[\prod_{\mu,\nu} (x_\nu - e^{i\omega})(x_\nu' - e^{i\omega})(y_\mu - e^{-i\omega})(y_\mu' - e^{-i\omega})\right]^{-1}$$

$$\times \left[\prod_{\mu,\nu} (x_\nu - e^{i\omega})(y_\mu - e^{-i\omega})\right]^2 g(\omega)$$

with a finite number of x_ν, y_μ, x_ν', y_μ', whose moduli are larger than unity, can also be the limit of a polynomial of the type (10). We note that any rational function $r(e^{i\omega})$ can be represented in the form

$$e^{ik\omega} \left[\prod_{\mu,\nu} (x_\nu' - e^{i\omega})\right]^{-1} \prod_{\mu,\nu} (x_\nu - e^{i\omega})(y_\mu - e^{-i\omega}),$$

where the moduli of x_ν, y_μ, x_ν', and y_μ', are larger than unity and k is the difference in the number of zeroes of the numerator and the denominator of $r(e^{i\omega})$ inside the unit circle.

We have thus proved the following theorem:

a) The Szego-Kac theorem is valid for a function of the type $r(e^{i\omega})g(\omega)$, where $r(e^{i\omega})$ is a rational function of $e^{i\omega}$ and $g(\omega)$ is a positive function, if the number of zeroes inside the unit circle is the same for the numerator and denominator of $r(e^{i\omega})$ (that is, if the condition (8) is satisfied for $r(e^{i\omega})g(\omega)$).

b) If k, the difference of the number of zeroes of the numerator and the denominator of $r(e^{i\omega})$ inside the unit circle, is not equal to zero, then for $k < 0$ we have

$$D_p(f) = \frac{\partial}{\partial c_{-p-k+1}} \cdots \frac{\partial}{\partial c_{-p}} D_{p+k}(e^{ik\omega} f),$$

and for $k > 0$

$$D_p(f) = \frac{\partial}{\partial c_{p+k-1}} \cdots \frac{\partial}{\partial c_p} D_{p+k}(e^{-ik\omega} f).$$

The Szego-Kac theorem in this form is applicable to the Ising-lattice determinant defined by expressions (3)—(5).

3. ASYMPTOTIC VALUES OF THE FOURIER COEFFICIENTS OF THE FUNCTION $f(\omega)$

Using the identity

$$(1-\beta e^{i\omega})^{-1/2} = \frac{1}{\pi}\int_0^\infty \frac{t^{-1/2}\,dt}{1+(1-\beta e^{i\omega})t},$$

we obtain

$$\langle (1-\beta e^{i\omega})^{-1/2}\rangle_n = \frac{\beta^n}{\pi}\int_0^\infty \frac{t^{n-1/2}}{(1+t)^{n+1}}\,dt.$$

We substituted this expression in the relation

$$\langle f\rangle_n = \sum_{k=0}^\infty \langle (1-\beta e^{i\omega})^{-1/2}\rangle_{n+k} \langle (1-\beta e^{-i\omega})^{1/2}\rangle_{-k}$$

$$= \sum_{k=0}^\infty \frac{1}{\pi}\int_0^\infty \langle (1-\beta e^{-i\omega})^{1/2}\rangle_k \left(\frac{\beta t}{1+t}\right)^{k+n} \frac{t^{-1/2}dt}{1+t}.$$

The result of the summation over k will obviously be the replacement of $e^{-i\omega}$ by $\beta t/(1+t)$. Therefore when $n \to \infty$ ($z = n(1-\beta^2)/\beta^2$) we have

$$\langle f\rangle_n = \frac{\beta^n}{\pi}\int_0^\infty \left(\frac{t}{1+t}\right)^n \frac{(1-\beta^2 t/(1+t))^{1/2}}{(1+t)\sqrt{t}}\,dt \to$$

$$\xrightarrow[n\to\infty]{} \frac{\beta^n(1-\beta^2)^{1/2}}{\pi n}\int_0^\infty \frac{e^{-y}(1+y^2/2n)(y+z)^{1/2}}{y^{1/2}}\,dy.$$

For large z

$$c_n = \frac{\beta^n(1-\beta^2)^{1/2}}{\pi^{1/2} n^{1/2}}\left(1+\frac{1}{4z}+\ldots\right). \tag{24}$$

For small z (near the phase-transition point, when $\beta \to 1$)

$$c_n = \frac{1}{\pi n}\left(1 - \frac{z}{2}\ln z + \ldots\right). \tag{25}$$

Similar calculations yield for negative n

$$c_{-|n|} = -\begin{cases} \dfrac{\beta^n}{2(1-\beta^2)^{1/2}\pi^{1/2}|n|^{3/2}}\left(1-\dfrac{1}{4|z|}\right) & \text{for } \beta\to 0 \\ \dfrac{1}{\pi|n|}\left(1+\dfrac{|z|}{2}\ln|z|\right) & \text{for } \beta\to 1 \end{cases} \tag{26}$$

4. CORRELATION FAR FROM THE PHASE TRANSITION POINT

We seek the solution of (14) and (15) in the form $\Delta^{-1} = f + \rho$. For ρ we obtain the following equations:

$$\langle\rho\rangle_n = 0, \qquad |n| \leqslant p-1; \tag{27}$$

$$\left\langle \frac{1}{f+\rho} \right\rangle_n = 0, \qquad |n| \geqslant p. \tag{28}$$

Below the phase transition, the Fourier series for $f(\omega)$ converges (see (24)), and when $p \to \infty$ we obviously obtain $\rho = 0$. It is natural to assume that when $p \to \infty$ the ratio $|\rho|/|f| \to 0$ and for sufficiently large p we can neglect in (28) the terms that are quadratic in ρ/f. In this approximation we obtain

$$\langle f^{-1}\rangle_n = \left\langle \frac{\rho}{f^2} \right\rangle_n, \qquad |n| \geqslant p. \tag{29}$$

We seek a solution satisfying the conditions (27) in the form

$$\rho = e^{ip\omega}x(\omega) + e^{-ip\omega}y(\omega);$$

$$x(\omega) = \sum_0^\infty x_{p+k}\, e^{ik\omega}, \quad y(\omega) = \sum_0^\infty y_{-p-k}\, e^{-ik\omega}. \tag{30}$$

According to (3),

$$\frac{1}{f^2} = \frac{1-\beta e^{i\omega}}{1-\beta e^{-i\omega}}. \tag{31}$$

Substituting (30) and (31) in (29) we obtain equations for x_k and y_k (in the equations for y_k we neglect the contribution of order β^{3p} from x_k):

$$\langle f^{-1}\rangle_{n\neq p} \equiv -\beta f_{n-1} + (1-\beta^2)\sum_{s=0}^\infty f_{n+s}\beta^s$$

$$= -\beta x_{n-1} + (1-\beta^2)\sum_{s=0}^\infty x_{n+s}\beta^s,$$

$$\langle f^{-1}\rangle_p \equiv -\beta f_{p-1} + (1-\beta^2)\sum_{s=0}^\infty f_{p+s}\beta^s = (1-\beta^2)\sum_{s=0}^\infty x_{p+s}\beta^s,$$

$$\langle f^{-1}\rangle_{-n} \equiv -\beta f_{-n-1} + (1-\beta^2)\sum_{s=0}^\infty f_{-n+s}\beta^s$$

$$= -\beta y_{-n-1} + (1-\beta^2)\sum_{s=0}^{|n|-p} y_{-n+s}\beta^s. \tag{32}$$

If we denote by f_p the remainder of the Fourier series $f(\omega)$:

$$f_p = \sum_{k=p}^\infty [\langle f\rangle_k\, e^{ik\omega} + \langle f\rangle_{-k}\, e^{-ik\omega}],$$

then the solution of (32) takes the form

$$\rho(\omega) = f_p(\omega) + \frac{\beta}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\left[\frac{e^{ip(\varphi-\omega)+i\varphi}}{1-\beta e^{i\varphi}} - \frac{e^{-ip(\varphi-\omega)+i\varphi}}{1-\beta e^{i\omega}}\right] d\varphi. \tag{33}$$

We now turn to the expression (16) for D_p. We expand $\ln\Delta = \ln(f+\rho)$ in powers of ρ/f and confine ourselves to terms that are linear and quadratic in ρ. The value of D_p as $p \to \infty$ will be denoted by D_∞. We obtain

$$D_p = D_\infty \exp\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\rho}{f} - \frac{\rho^2}{2f}\right) \times \left(p - \frac{i\beta\sin\omega}{1+\beta^2-2\beta\cos\omega}\right) d\omega. \tag{34}$$

This expression is valid with accuracy of order $f_p/f \sim \beta^{2p}$. This means that for small β we can choose small p. Expression (35) can be simplified by taking small β. In this case (we can calculate the integrals, for example, in the same manner as in Sec. 3)

$$D_p = (1 + \beta^{2p}/\pi p)(1-\beta^2)^{1/4}. \tag{35}$$

Above the transition point it is necessary to use formula (19), which we rewrite in the form

$$D_p = \frac{\partial}{\partial x} A_{p+1}, \tag{36}$$

$$A_{p+1} = \begin{vmatrix} c_0' & c_1' \ldots & c_p' \\ c_{-1}' & c_0' & \\ \vdots & & \ddots \\ c'_{-p+x} \ldots & & c_0' \end{vmatrix}, \quad c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(\omega)\, e^{-in\omega} d\omega. \tag{37}$$

The determinant A_p is calculated in the same manner as D_p, except that $f(\omega)$ must be replaced by $f(\omega) + xe^{i\omega}$, and (34) and (35) must be rewritten, leaving everywhere only the first term of the expansion in powers of x. Substituting the obtained expression for A_p in (36) we obtain (leaving the principal term in $1/p^{1/2}$)

$$D_p = -\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\, e^{-ip\omega}\frac{\rho}{(f^*)^2} \times \left(p - \frac{i\sin\omega}{1+\beta^2-2\beta\cos\omega}\right) d\omega\, D_\infty(A_p),$$

$$D_\infty(A_p) = (1-\beta^2)^{1/4}. \tag{38}$$

For small β we have

$$D_p = \frac{\beta^p}{\sqrt{\pi p}}. \tag{39}$$

5. CORRELATIONS NEAR THE TRANSITION POINT

Near the phase-transition point the series $\Sigma(\rho/f)^n$, ΣnK_nK_{-n} and $\Sigma\,\mathrm{Sp}A^n$ converge poorly. We can use, however, the relative simplicity of D_p at the transition point. We have here an analytic expression for both $D_p \sim p^{-1/4}$ (see[3]) and its minors. We can therefore use the formula

$$\mathrm{Det}(A+\Delta) = \mathrm{Det}\,A\,\mathrm{Det}(1+\Delta A^{-1})$$
$$= \mathrm{Det}\,A \exp\left\{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\,\mathrm{Sp}(\Delta A^{-1})^n\right\} \quad (40)$$

and put

$$A = \|c_{kl}^{(1)}\| = \left\|\frac{1}{\pi(k-l+1/2)}\right\|,$$

where A is the value of the matrix (5) at the transition point (here $\beta = 1$) and $\Delta = \|c_{k-l}^{(\beta)} - c_{k-l}^{(1)}\|$ is the correction to the matrix (5) near the transition point.

The minors of the matrix A and its determinant are determined by a matrix with coefficients $(x_k - x_l)^{-1}$ and are calculated in the same manner as the determinant (12). We obtain

$$(A^{-1})_{l,k} = \frac{f(k)g(l)}{\pi(k-l+1/2)};$$

$$f(k) = \frac{\Gamma(k+1/2)\Gamma(p-k+1/2)}{k!\,(p-k)!},$$

$$g(l) = \frac{\Gamma(l-1/2)\Gamma(p-l+3/2)}{l!\,(p-l)!}. \quad (41)$$

If we neglect $\mathrm{Sp}(\Delta A^{-1})^n$ with $n \geq 2$ and use (25), then we obtain ($1-\beta = \tau \approx T - T_c$ and A_0 is a constant)

$$D_p = \frac{A_0}{p^{1/4}} \exp\left\{\sum_{k,l=1}^{p} \frac{|k-l|\tau\ln(|k-l|\tau)}{(k-l+1/2)^2} f(k)g(l)\right\}. \quad (42)$$

Estimates show that $\sum_1^\infty \mathrm{Sp}(\Delta A^{-1})^n$ converges like $\sum_1^\infty (\tau p \ln \tau p)^n$. When $p \gg 1$ we get for D_p the simpler expression

$$D_p = \frac{A_0}{p^{1/4}}\left(1 + \frac{1}{4}\tau p \ln|\tau p|\right). \quad (43)$$

6. CONCLUSION

The calculation shows that there exist three regions, in each of which the correlations behave differently. These are the regions

$$1)\ \tau < -p^{-1},\quad 2)\ -p^{-1} < \tau < p^{-1},\quad 3)\ \tau > p^{-1}.$$

The second region—the vicinity of the phase-transition point, in which the distance to the phase-transition point is smaller than the temperature fluctuations ($\sim 1/\sqrt{N}$, N is the number of particles) in regions with dimensions of order p. The temperature fluctuations require a definition, but their intuitive meaning is clear. On going over through the limit $\tau = \pm p^{-1}$, both the character of the physical phenomena and the form of the correlation functions change. We denote the temperature fluctuation by τ_f. We can expect the correlation to depend only on the ratio $\tau/\tau_f = p\tau$. This circumstance explains the dependence like $e^{-p\tau}$ at large values of $p\tau$ (see (39)).

The equation $D_p \sim 1/p^{1/4}$ at the phase-transition (see (43)) can be related with the temperature dependence $D_\infty \sim \tau^{1/4}$ in the "phase transition region." The average temperature is $\tau \approx 1/p$ and the correlation is of the order of $\tau^{1/4} = 1/p^{1/4}$.

These considerations can be applied to the three dimensional case. The parameter τ/τ_f will have here the form $p^{3/2}\tau$ and the correlations will fall off possibly like $\exp(-p\tau^{2/3})$. If $D_\infty \sim \tau\beta$, then the correlation of the transition point will be $\sim p^{-3\beta/2}$. According to[11], approximate methods yield for the three-dimensional lattice $0.303 \leq \beta/2 \leq 0.318$, whereas experiment yields $\beta/2 = 0.33$—0.36. If we take $\beta = 2/3$, then $D_p \sim 1/p$.

The procedures described make it possible to calculate far from the transition point also the correlation between two spins lying on a single row (or column) of the lattice. This correlation is determined[4-7] by the expressions (4) and (5) with $f(\omega)$ of the type (3). In place of (35) and (39) we obtain (for small β) below the transition point

$$D_p = 1 + \left(\frac{1-x}{2}\right)^{2p},$$

and above the transition point

$$D_p = x^p.$$

The correlation in the vicinity of the transition point was calculated by Vaks et al.[12] and coincides with (43) accurate to terms of order $1/p$. Thus, the correlations along the rows and along the diagonals turn out to be different only in the region above the transition point, where the correlation radius is small and is close to the distance between the nearest neighbors.

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