MASS RENORMALIZATION IN GENERALIZED FIELD THEORY

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The mass renormalization of a boson is obtained on the example of the second order polarization operator for a pseudoscalar theory generalized to a momentum space of constant curvature. It is shown that the mass renormalization leads in this theory to a "smearing out" of the particle mass.

1. INTRODUCTION

GOL'FAND[1,2] and Kadyshevskiĭ[3] have considered a quantum field theory in which the ordinary p-space is replaced by a space of constant curvature. The computation of the S-matrix in the framework of this theory leads to some specific singularities connected with the geometric peculiarities of a p-space of constant curvature. In particular, the renormalization procedure is radically modified in this case. First of all, owing to the finite volume of the p-space, all integrals turn out to be convergent and hence the renormalizations are finite. Second, the renormalization no longer reduces to a simple multiplication of the "bare" quantities by constant factors (and addition of a mass term), as in the ordinary quantum field theory, but is a complicated procedure related to the solution of integral equations, which in the final count leads to a smearing out of the mass. (So far we are unable to make a similar affirmation about the coupling constant.)

In the present note we consider the mass renormalization of the meson in pseudoscalar meson theory, and show that the mass of the meson is "smeared out" by the so called "nondiagonality effect."

The action operator of the pseudoscalar meson theory admits a natural generalization to the case of a p-space of constant curvature:

$$\hat{\Lambda} = g \int \bar{\psi}(p) \langle p | \hat{d}(k) | q \rangle \gamma^5 \psi(q) \varphi(k) \, d\Omega_p d\Omega_q d\Omega_k. \quad (1.1)$$

The phenomenon of "nondiagonality" is related to the properties of the matrix elements of the displacement operator $\langle p | \hat{d}(k) | q \rangle$ (cf. [2]). Strictly speaking there are two such phenomena. One is a direct consequence of the noncommutativity of the displacements in p-space. We shall not investigate it here. We only write down for comparison with the usual

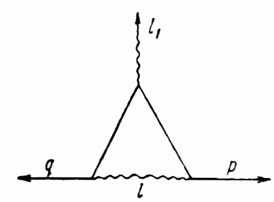

FIG. 1

theory the expression of the matrix element of the displacement operators which appear in the vertex part of Fig. 1. It has the form

$$\langle p | \hat{d}_0(l) \hat{d}_0(l_1) \hat{d}_0^{-1}(l) | q \rangle. \quad (1.2)$$

Since the displacements do not commute with each other, we cannot interchange the places of $\hat{d}_0(l)$ and $\hat{d}_0(l_1)$ and cancel $\hat{d}_0^{-1}(l)$. This gives rise to a complicated dependence of the vertex on the virtual momentum l, which no longer allows, as in the usual theory, to carry out all the integrations and factor out a delta function depending only on the external moments. In the usual theory the matrix element (1.2) is a delta-function:

$$\langle p | \hat{d}_0(l) \hat{d}_0(l_1) \hat{d}_0^{-1}(l) | q \rangle = \delta(p - l_1 - q). \quad (1.3)$$

Let us now consider "nondiagonalities" of the second type. Since the coordinates of a space of constant curvature can be constructed by means of projection from a five-dimensional space as follows:

$$p_\alpha = P_\alpha / P_0 \quad (\alpha = 1, 2, 3, 4),$$
$$P_1^2 + \ldots + P_4^2 + P_0^2 = 1, \quad (1.4)$$

it is easy to rewrite all expressions in five-dimensional form. In particular we obtain the following expression for a displacement:

$$Q = \hat{D}_0(L) P = P - 2L(PL) - 2A(AP) + 4A(AL)(PL). \quad (1.5)$$

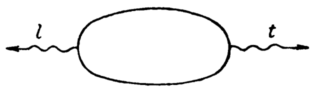

FIG. 2

Regarding this as an equation with respect to L, we have two different solutions

$$L_{PQ} = \frac{P - Q + 2A(AQ)}{[2(1-(PQ)+2(AP)(AQ))]^{1/2}}, \quad (1.6a)$$

$$L_{PQ}' = \frac{P + Q - 2A(AQ)}{[2(1+(PQ)-2(AP)(AQ))]^{1/2}}. \quad (1.6b)$$

This implies that for any P and Q there exist two displacements (motions) $\hat{D}_0(L_{PQ})$ and $\hat{D}_0(L'_{PQ})$ taking P into Q. Therefore the matrix element $\langle q|\hat{d}_0(l)|p\rangle$ is a sum of two terms [1)]:

$$\langle q|\hat{d}_0(l)|p\rangle = \{\delta(l, l_{pq}) + \delta(l, l'_{pq})\}. \quad (1.7)$$

It is the second term in Eq. (1.7) which gives rise to the "nondiagonality."

2. THE VACUUM POLARIZATION OPERATOR

The vacuum polarization operator can serve as a good illustration of these facts. As indicated in [1] the second order vacuum polarization operator of pseudoscalar mesodynamics, corresponding to the Feynman diagram in Fig. 2, has the expression [2)]

$$\Pi(l, t) = \mathrm{Tr}\,\{\hat{G}_0\gamma_5\hat{d}(l)\hat{G}_0\gamma_5\hat{d}(-t)\}. \quad (2.1)$$

Here $\hat{G}_0$ is the fermion propagator which can be expressed in terms of the five-dimensional vector-propagator

$$Q_i = (\delta_{ij} - (1-m)A_iA_j)P_j/(PA)$$
$$(i, j = 1, \ldots, 5) \quad (2.2)$$

in the following form

$$\hat{G}_0\gamma^5 = (\Gamma^i Q_i)^{-1}. \quad (2.3)$$

(The matrices Γ^i have been introduced by Yu. A. Gol'fand for the construction of spinors in p-spaces of constant curvature.) We indicate also the form of the spinor part of the displacement operator

$$\hat{D}_S(L) = \hat{L}\Gamma^5, \quad \hat{L} = L_i\Gamma^i. \quad (2.4)$$

The expression under the Tr sign in (2.1) can be transformed as follows:

$$\{\hat{G}_0\Gamma^5\hat{D}(L)\hat{G}_0\Gamma^5\hat{D}^{-1}(T)\}$$
$$= \{(\Gamma^i Q_i)^{-1}[\hat{D}(L)(\Gamma^j Q_j)^{-1}\hat{D}^{-1}(L)]\hat{D}(L)\hat{D}^{-1}(T)\}$$
$$= \{(\Gamma^i Q_i)^{-1}(\Gamma^j Q_j')^{-1}\hat{D}(L)\hat{D}^{-1}(T)\}_s. \quad (2.5)$$

Here Q_i' denotes the vector-propagator

$$Q_i' = (\delta_{ij} - (1-m)A_i'A_j')P_j'/(PA'), \quad (2.6)$$

where A′ is the "displaced" vacuum vector (A = (0, 0, 0, 0, 1)):

$$A' = \hat{D}_0(L)A. \quad (2.7)$$

The vacuum polarization operator becomes

$$\Pi(l, t) = \int d\Omega_p\, \mathrm{Sp}\left\{\frac{(\Gamma^i Q_i)(\Gamma^j Q_j)\hat{D}_S(L)\hat{D}_S^{-1}(T)}{Q^2Q'^2}\right\}$$
$$\times \langle p|\hat{D}_0(L)\hat{D}_0(T)|p\rangle. \quad (2.8)$$

The matrix element $\langle p|\hat{D}_0(L)\hat{D}_0^{-1}(T)|p\rangle$ can be obtained from (1.7). Replacing there q by $\hat{D}_0^{-1}(T)p$ we obtain

$$\langle p|\hat{D}_0(L)\hat{D}_0^{-1}(T)|p\rangle = \delta(l, t) + \delta(l, t'); \quad (2.9)$$

where t'_μ denotes the quantity $(L'_{PQ})_\mu/(L'_{PQ})_5$ for $Q = \hat{D}_0^{-1}(T)p$. [3)] In five-dimensional notation it has the following form:

$$T' = \frac{P - T(PT)}{[1-(PT)^2]^{1/2}}. \quad (2.10)$$

It follows from Eq. (2.9) that the expression for the vacuum polarization operator (2.1) can be split into two parts: a diagonal part D(l, t) and a nondiagonal one $D_n(l, t)$:

$$\Pi(l, t) = D(l, t) + D_n(l, t). \quad (2.11)$$

As the radius of curvature of the p-space goes to infinity D(l, t) goes into the vacuum polarization operator of the ordinary theory. The quantity D(l, t) has been computed by Gol'fand and is given in [1]. In the present paper we compute the nondiagonal part of the vacuum polarization operator, $D_n(l, t)$, and analyze its properties. The trace appearing in the integrand of (2.8) is easy to evaluate taking into account the commutation relations of the

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1)In the states between which the matrix element is taken we write $\hat{d}_0(l)$ in place of $\hat{d}(l)$, i.e. we consider that the spin part $\hat{d}_s(l)$ of the displacement operator has been separated (cf.[1]).

2)The symbol Tr denotes the total trace of the operator inside the brackets (summation over spin indices and integration over p-space). Summation over spin indices only will be denoted below as Sp.

3)We make use alternatively for the momentum vectors of a four-dimensional notation and of a five-dimensional projective notation. This should not lead to misunderstandings, if one remembers that all computations can be carried out on five-dimensional vectors, and at the end it is necessary to go over to four-dimensional ones by means of Eq. (1.4). Since the integration over virtual momenta is four-dimensional, the delta functions must depend only on four-dimensional vectors.

Γ^i-matrices:

$$\Gamma^i\Gamma^j + \Gamma^j\Gamma^i = 2\delta^{ij} \qquad (2.12)$$

and the relation

$$(LT) = (T'T) = 0, \qquad (2.13)$$

which are consequences of the form of the delta-function $\delta(l, t')$. Neglecting everywhere in the numerators m compared to unity, we obtain

$$\mathrm{Sp}\left\{\frac{\hat{Q}\hat{Q}'\hat{L}\hat{T}}{Q^2Q'^2}\right\} = 8\frac{[(PL)^2(AL)^2-(PT)^2(AT)^2](AT)(AL)}{[1-(1-m^2)((PL)(AL)+(PT)(AT))^2]}$$
$$\times\frac{[-(PL)^2+(PT)^2+(PL)^2(AL)^2-(PT)^2(AT)^2]}{[1-(1-m^2)((PL)(AL)-(PT)(AT))^2]}. \qquad (2.14)$$

For the final calculation of the nondiagonal part it remains to find the mode of operation of the delta function $\delta(l, t')$. For integrations over an elliptic space the function $\delta(p, q)$ is defined as usual according to

$$\int \delta(p,q)f(p)\,d\Omega_p = f(q). \qquad (2.15)$$

In order to find the way in which $\delta(l, t')$ acts it is convenient to choose such a coordinate system in the elliptic space that $\delta(l, t')$ splits into a product of four one-dimensional delta-functions, of which only some depend on p. It turns out that a certain polar coordinate system satisfies this requirement. Before looking for this system, we note that (2.14) implies that the integrand in (2.8) is projectively invariant with respect to the vector P, i.e. is not changed if P is replaced by – P. Therefore we can go over in the expression for $D_n(l, t)$ from an integration over the elliptic space to an integration over a spherical space. The integration will moreover be invariant under five-dimensional rotations of the vector P.

Let us now rotate the five-dimensional projective sphere in such a manner that the 5-vector T be carried into the 5-vector of coordinates (0, 0, 0, 0, 1) and introduce polar coordinates $\varphi, \vartheta, \eta, \rho$ (cf. [2]) with the variation intervals $0 \le \varphi \le 2\pi$, $0 \le \vartheta, \eta, \rho \le \pi$ and such that ρ is measured from the direction of the unit 5-vector T, and the angles φ, ϑ, η are in a hyperplane which is perpendicular to T and are measured from the direction of the vector L.

We now consider the transformation of the delta function into polar coordinates. The integration over the spherical p-space is in fact an integration over a spherical surface of unit radius in five dimensions. Therefore

$$\int f(p)\delta(p,q)\,d\Omega_p = \int f(\rho,\eta,\vartheta,\varphi)A(\rho,\eta,\vartheta,\varphi)\delta(\rho-\rho_q)$$
$$\times\delta(\eta-\eta_q)\delta(\vartheta-\vartheta_q)\delta(\varphi-\varphi_q)$$
$$\times\sin^3\rho\sin^2\eta\sin\vartheta\,d\rho\,d\eta\,d\vartheta\,d\varphi$$
$$= f(\rho_q,\eta_q,\vartheta_q,\varphi_q) = f(q). \qquad (2.16)$$

Here ρ_q, η_q, ϑ_q, φ_q are the polar coordinates of the vector q and the function $A(\rho, \eta, \vartheta, \varphi)$ is determined by the normalization condition for the delta-function. In order that the integral of the delta function be unity, it is necessary to choose

$$A(\rho,\eta,\vartheta,\varphi) = 1/\sin^3\rho\sin^2\eta\sin\vartheta. \qquad (2.17)$$

Then

$$\delta(p,q) = \delta(\rho-\rho_q)\delta(\eta-\eta_q)\delta(\vartheta-\vartheta_q)\delta(\varphi-\varphi_q)$$
$$\times A(\rho,\eta,\vartheta,\varphi). \qquad (2.18)$$

The vector T′ in which we are interested has the coordinates:

$$\eta_{T'} = \eta_P, \quad \vartheta_{T'} = \vartheta_P, \quad \varphi_{T'} = \varphi_P;$$

$\rho_{T'}$ does not depend on the vector p, which is easy to see from the expansion of P into T and T′ = L following from (2.10):

$$P = T'(PT') + T(PT).$$

There $\delta(l, t')$ can be written in the form

$$\delta(l,t') = \frac{\delta(\rho_{T'}-\rho_L)\delta(\eta_P-\eta_L)\delta(\vartheta_P-\vartheta_L)\delta(\varphi_P-\varphi_L)}{\sin^3\rho_{T'}\sin^2\eta_P\sin\vartheta_P}. \qquad (2.19)$$

In order to bring the expression to a definitive form we note that

$$\rho_{T'} = \arccos\,(T'T) = \pi/2, \qquad (2.20)$$

therefore

$$\delta(\rho_{T'}-\rho_L) = \delta(\rho_L-\pi/2) = \delta(\cos\rho_L) = \delta(LT)$$
$$= \delta(1+lt)\,/\,(AL)(AT). \qquad (2.21)$$

We now have

$$\delta(l,t') = \frac{\delta(\eta_P)\delta(\vartheta_P)\delta(\varphi_P)\delta(1+lt)}{\sin^2\eta_{T'}\sin\vartheta_{T'}(AL)(AT)}. \qquad (2.22)$$

The integrations with respect to η_P, ϑ_P and φ_P disappear due to the delta-functions; the remaining integral with respect to ρ_P can be computed. The final expression for the kernel of the non-diagonal part is as follows:

$$D_n(l,t) = \frac{2}{e(d_+-d_-)}[F(d_-)-F(d_+)]\delta(1+lt),$$
$$F(d) = (e_1 - de_2 + (1+d)^2e_3)J(d) - 2de_3, \qquad (2.23)$$

$$J(d)=\frac{2(1-d)}{(1-2d)^{1/2}}\ln\frac{(1-2d)^{1/2}-1}{(1-2d)^{1/2}+1}$$

$$+\frac{2(1+d)}{(1+2d)^{1/2}}\ln\frac{(1+2d)^{1/2}+1}{(1+2d)^{1/2}-1}. \quad (2.24)$$

The following notations have been introduced:

$$d_{\pm}=\frac{(\nu^2-\mu^2)[-2+\beta(\mu^2+\nu^2)]\pm 4\mu\nu[\beta^2(\mu^2+\nu^2)-1]^{1/2}}{2\beta(\mu^2+\nu^2)^2},$$

$$e_1=(\nu^2-\mu^2)^2, \qquad e_2=4(\nu^2-\mu^2)(\nu^2+\mu^2-1),$$

$$e_3=4(\nu^2+\mu^2)(\nu^2+\mu^2-2), \qquad e=\beta^2(\nu^2+\mu^2)^2,$$

$$\mu=(AT), \qquad \nu=(AL), \qquad \beta=(1-m^2). \quad (2.25)$$

We remark that the vectors T and L can be interpreted physically as five-dimensional particle momenta only in the special coordinate system where A = (0, 0, 0, 0, 1). Therefore in the final result we have to return to this coordinate system.

3. SECOND ORDER RADIATIVE CORRECTIONS

Knowing the expression of the vacuum polarization operator up to second order we can compute the Green's operator of the meson in the g^2-approximation, by summing the chain of diagrams represented in Fig. 3. The computation reduces to the usual summation of a geometric progression[4]. As a result we obtain for the Green's operator the expression

$$\hat{G}(l,t)=(l^2+\mu^2/4+\hat{\Pi})^{-1}. \quad (3.1)$$

We note that in distinction from the usual theory, the mass renormalization in field theory in a p-space of constant curvature is not simply a displacement of the mass, due to virtual pairs. The renormalization effect gets considerably more complicated because of the nondiagonal terms.

In order to analyze this effect of "nonlocal" renormalization we remember the Klein-Gordon equation, which in p-space is simply the relativistic relation between the squares of energy-momentum and mass, and can be considered as a definition of the mass. The solution of this equation

$$(p^2+m^2)\tilde{\varphi}(p)=0 \quad (3.2)$$

is unique and expresses the relativistic mass-momentum relation:

$$\tilde{\varphi}(p)=\delta(p^2+m^2)\varphi(\mathbf{p}). \quad (3.3)$$

We note that the transition from the euclidean to pseudoeuclidean (physical) square of a four-vector can be realized by the replacement $p^2 \to -p^2$. All real relations among physical quantities must be written for the pseudoeuclidean vectors. It is however convenient to reason up to the last moment in the euclidean metric, and to go over to the pseudoeuclidean form only in the very final expression. This results from the euclidean formulation of field theory. In this connection we note that the singularities of the theory are situated in the region of negative p^2 ($p^2 = -m^2$).

FIG. 3

In the field theory in a p-space of constant curvature the renormalized meson mass in the g^2-approximation is the quantity

$$\tfrac{1}{4}\mu_1^2=\tfrac{1}{4}\mu^2+\hat{\Pi}, \quad (3.4)$$

which is an integral operator. This generalization of the expression (3.2) describes the effect of smearing out of the meson mass. Introducing a notation which is clear from the context

$$\Pi(l,t)=D(l,t)+D_n(l,t)=g^2\Psi(l^2)\delta(l,t)$$

$$+g^2\Phi(l^2,t^2)\delta(1+lt), \quad (3.5)$$

we can rewrite our expression in the form

$$\left(l^2+\frac{\mu^2}{4}+g^2\Psi(l^2)\right)\varphi(l)$$

$$+g^2\int d\Omega_t\delta(1+lt)\Phi(l^2,t^2)\varphi(t)=0. \quad (3.6)$$

4. ANALYSIS OF THE INTEGRAL EQUATION

The equation (3.6) is very complicated. Therefore, in order to obtain preliminary information about the functions $\varphi(l)$ we consider a model, introducing a series of simplifications into (3.6). We will consider the unknown function φ as a function depending only on the square of the momentum vector: $\varphi = \varphi(l^2)$. We also assume that the effect of the diagonal part $\Psi(l^2)$ reduces to the usual additive mass renormalization. (We recall that in the theory under consideration all renormalizations are finite.) The new meson mass will be denoted by the same letter μ. Since $\Phi(l^2, t^2)$ is a slowly varying function, we assume that the qualitative features are not modified by replacing it by a constant.

We now introduce a spherical coordinate system t, η, ϑ, φ with the polar axis along the vector l and polar angle η. The volume element is

$$d\Omega_t=\frac{1}{2}\frac{t^2\sin^2\eta\sin\vartheta\, d\eta\, d\vartheta\, d\varphi\, dt^2}{(1+t^2)^{5/2}}. \quad (4.1)$$

The delta-function $\delta(1 + lt)$ can be used to reduce the η-integration. The integral with respect to η in (3.6) can be represented in the form

$$\int_0^\pi \delta(1+lt)\sin^2\eta\, d\eta = -\frac{\vartheta(t^2-1/l^2)}{(l^2t^2)^{1/2}}\int_1^{-1}\delta(\cos\vartheta+(l^2t^2)^{-1/2})$$
$$\times\sqrt{1-\cos^2\eta}\, d\cos\eta$$
$$= -\frac{\vartheta(t^2-1/l^2)}{(l^2t^2)^{1/2}}\left(1-\frac{1}{l^2t^2}\right)^{1/2}. \quad (4.2)$$

We simplify the expression $(1 - 1/l^2t^2)^{1/2}$. The presence of the factor $\vartheta(t^2 - 1/l^2)$ shows that the interesting region is only from $1/l^2$ to ∞. At $1/l^2$ its derivative is infinite but the function vanishes; for large t^2 the function converges sufficiently rapidly to 1, therefore we can make the replacement

$$(1-1/l^2t^2)^{1/2}\to\vartheta(t^2-1/l^2). \quad (4.3)$$

After all these simplifications our equation reduces to

$$\frac{c}{\sqrt{s}}\int_{1/s}^\infty \frac{\varphi(t)\sqrt{t}}{(1+t)^{5/2}}dt = \left(\frac{\mu^2}{4}+s\right)\varphi(s). \quad (4.4)$$

We use here the notation

$$4\pi g^2\Phi(l^2,t^2) = \text{const} = c, \qquad l^2 = s, \qquad t^2 = t. \quad (4.5)$$

This integral equation can be reduced to a differential equation. Define the function

$$F(s) = \int_s^\infty \frac{\varphi(t)\sqrt{t}}{(1+t)^{5/2}}dt, \quad (4.6)$$

then

$$\varphi(s) = -\frac{(1+s)^{5/2}}{\sqrt{s}}\frac{dF}{ds} \quad (4.7)$$

and the integral equation (4.4) takes the form

$$\left(\frac{\mu^2}{4}+s\right)(1+s)^{5/2}\frac{dF(s)}{ds} = cF(1/s). \quad (4.8)$$

Making the substitution $s \to 1/s$ and introducing the expression for $F(1/s)$ from (4.8) we obtain

$$\left(\frac{\mu^2}{4}+\frac{1}{s}\right)s(1+1/s)^{5/2}\frac{d}{ds}\left(\left(\frac{\mu^2}{4}+s\right)(1+s)^{5/2}\frac{dF(s)}{ds}\right)$$
$$+c^2F(s)=0. \quad (4.9)$$

We now investigate the behavior of the function $\varphi(s)$ near the point $s = \mu^2/4$, where the "classical" Green's function had a pole. In the system of units under consideration we have the estimate $\mu^2 \ll 1$. Therefore, in a neighborhood of the point $s = -\mu^2/4$ Eq. (4.6) becomes

$$\frac{d^2F}{dx^2}+\frac{1}{x}\frac{dF}{dx}+F(x)=0, \quad (4.10)$$

where the new variable x is defined by

$$s+\frac{\mu^2}{4} = \frac{x^2}{4(\mu^2/4)^{3/2}c^2}. \quad (4.11)$$

Finally, in the neighborhood of the point $s = -\mu^2/4$ we obtain the "distribution"

$$|\varphi(s)|^2 = \frac{1}{2}\frac{c^2\mu^3}{|s+\mu^2/4|}. \quad (4.12)$$

Thus on the hand of a simplified model it has been shown that the mass renormalization leads to a smearing of the mass in the theory under consideration. Since all our computations were estimates, it does not yet make sense to speak about the physical consequences of the generalized renormalization procedure, although one might be tempted to connect the mass smearing with the lifetime of the particle.

In conclusion I would like to express my gratitude to Yu. A. Gol'fand for constant interest in this work and some remarks.

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