ON THE THEORY OF THE GAS LASER IN A WEAK LONGITUDINAL MAGNETIC FIELD

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The emission from a gas laser in a longitudinal magnetic field is treated assuming that the Zeeman splitting is much smaller than the Doppler line width. The frequency of the resonator is assumed to be the same as the atomic frequency in the absence of a magnetic field. The polarizability of the gas in a magnetic field is evaluated with an accuracy to terms quadratic in the electric field strength. The threshold regime is investigated for the case that the Q's for oscillations polarized along the x axis and along the y axis are different. It is shown that as long as the magnetic field strength is smaller than some critical value H_0 the radiation is linearly polarized and the frequency is constant. The polarization direction rotates in the magnetic field from 0 to 45°. In magnetic field strengths exceeding the critical value two modes with different frequencies arise and are right and left elliptically polarized. The dependence of the critical field, rotation of direction of polarization and frequency shifts on excitation above the threshold value are investigated qualitatively. For sufficiently intense excitation the beat frequency for $H > H_0$ depends nonmonotonically on the magnetic field strength and a second region of linear polarization appears. The results of the theory agree with experiment.

1. INTRODUCTION

In recent experiments Culshaw and Kannelaud[1,2] observed a number of interesting effects occurring when a He-Ne gas laser ($\lambda = 1.153\ \mu$) was placed in a weak magnetic field. In particular, in a longitudinal magnetic field when the cavity frequency ω coincided with the center of the atomic resonance line ω_0 the following was observed. In the absence of a magnetic field the laser emission was linearly polarized in a definite direction (the x axis). When the magnetic field was increased from zero to a certain critical value, H_0, of the order of some tens of oersteds the emission remained linearly polarized but the direction of polarization was rotated $\sim 45°$ from the x axis. The laser frequency did not change when this occurred but remained equal to ω. The rate of rotation of the direction of polarization as a function of magnetic field and the critical value H_0 depended strongly on the excitation. Larger excitation corresponded to greater rotation rates. In a magnetic field larger than the critical value low frequency beats were observed between two left and right circularly polarized modes at different frequencies. The dependence of the beat frequency on the magnetic field was distinctive. With increasing magnetic field the beat frequency increased linearly at first, then attained a maximum, and for a magnetic field H_1 of the order of 10 Oe became zero again. Beginning at this point there was a new narrow region of linearly polarized emission at the frequency ω. Rotation of the polarization direction was also observed in this region. With further increase in field beats were again observed between circularly polarized modes. The beat frequency increased monotonically.

The existence of beats between circularly polarized modes in a magnetic field had been observed previously[3] and was explained by the fact that due to the Zeeman splitting of the working levels in a magnetic field the gain coefficients for right and left circularly polarized modes differ by an amount $2\Omega \sim 2\mu_0 H/\hbar$, where μ_0 is the Bohr magneton. Because of the well known laser frequency pulling towards the maximum gain coefficient[4] the frequencies of the left and right polarized modes were shifted in opposite directions from the frequency ω by an amount $\sim(\Delta\omega/ku)\Omega$ where $\Delta\omega$ is the cavity bandwidth and ku is the Doppler line width.

The nonmonotonic nature of the beat frequency variation is of special interest; the existence of regions where beats are not observed but where the direction of polarization of the emission at the unshifted frequency ω rotates with increasing magnetic field are of particular interest. Culshaw and

Kannelaud[2] made an attempt to explain the direction of polarization at threshold by analogy with the Hanle effect in the scattering of resonance radiation. However, this explanation is unsatisfactory. The authors used the concept of coherent excitation of the Zeeman sub-level by the pump source. They obtained the polarization of the output from a calculation of the probabilities for stimulated transitions between the Zeeman sub-levels of the upper and lower states. This procedure is incorrect since in the theory of lasers it is necessary to solve simultaneously Maxwell's equations and the equations of motion for the density matrix of the gas. The dependence on excitation of the angle of rotation and the critical magnetic field as well as the reversion of the beat frequency to zero for $H = H_1$ were not explained by Culshaw and Kannelaud.

In the present article we treat the emission from a gas laser placed in a longitudinal magnetic field. The excitation is assumed to be isotropic. The magnetic field is assumed to be so weak that the Zeeman splitting of the levels is much less than the Doppler line width. Moreover it is assumed that the Doppler line width is much larger than the natural line width of both the upper (γ_1) and the lower (γ_0) levels. The quantity γ used in various estimates later on in the article is a certain average of the γ_1 and γ_0.

The existence of a region of linearly polarized emission at the unshifted frequency ω is explained by introducing different Q's, Q_x and Q_y, for modes polarized along the x and y axes respectively. At this point right and left circularly polarized modes are not eigenmodes of the cavity. It turns out that for magnetic fields lower than the critical field the right and left circularly polarized modes occur at the same frequency and with equal intensity. Hence the emission is linearly polarized. The phase difference of the right and left circularly polarized modes depends on the magnetic field and determines the direction of polarization. An estimate shows that to explain the experimental observations the values of Q_x and Q_y must differ by 0.1% at most. In Sec. 3 we treat laser operation at threshold in a longitudinal magnetic field and obtain the threshold, the frequency shifts, the critical magnetic field H_0, and the dependence of the rotation angle φ on the magnetic field. In Sec. 4, a qualitative study is made of the dependence of these quantities on the degree to which the excitation exceeds threshold. It is shown that for sufficiently high excitation it is possible to have a second region of linearly polarized output as is observed experimentally. In this case the beat frequency does not depend monotonically on the magnetic field. Roughly speaking, these phenomena are related to the formation of a hole at the center of the gain curve[4,5] and to the corresponding distortion of the dispersion curve of the gas as a function of the magnetic field for $\mu_0 H \lesssim \hbar\gamma$.

2. FUNDAMENTAL EQUATIONS

In the present paper we treat single mode operation, which may be obtained experimentally by using a relatively short laser having a large mode separation[2]. In this case the field inside the cavity may be written

$$\tfrac{1}{2}[\mathbf{E}(t)e^{i\omega t} + \mathbf{E}^*(t)e^{-i\omega t}]\sin kz,$$

where $\omega = kc$ is the resonance frequency of the mode under consideration and $\mathbf{E}(t)$ is a slowly varying function of time. The following equation may be obtained for the amplitude $\mathbf{E}(t)$ [5]

$$\frac{2i}{\omega}\frac{d\mathbf{E}}{dt} + \frac{i}{Q}\mathbf{E} = 4\pi\hat{\alpha}\mathbf{E}. \qquad (1)$$

Here Q is the cavity Q, and $\hat{\alpha}$ is the polarizability tensor, which depends in general on the field.

In the absence of a magnetic field the emission from a plane parallel gas laser is linearly polarized in a certain direction determined, it seems, by small anisotropies in the reflecting coatings[1]. In other words, the energy loss for emission polarized, say, in the direction x is less than the loss for emission polarized in the direction y. Therefore Eq. (1) must be amended by the introduction of different Q's, Q_x and Q_y ($Q_x > Q_y$). In the presence of a magnetic field the tensor $\hat{\alpha}$ is diagonalized by introducing components of circular polarization

$$E_+ = -2^{-1/2}(E_x + iE_y),$$
$$E_- = 2^{-1/2}(E_x - iE_y).$$

For the circular field components we obtain the equations

$$\frac{2i}{\omega}\frac{dE_+}{dt} + i\beta E_+ + i\varkappa E_- = 4\pi\alpha_+ E_+,$$
$$\frac{2i}{\omega}\frac{dE_-}{dt} + i\beta E_- + i\varkappa E_+ = 4\pi\alpha_- E_-, \qquad (2)$$

where $\beta = (1/2)(Q_y^{-1} + Q_x^{-1})$, $\kappa = (1/2)(Q_y^{-1} - Q_x^{-1})$; and α_+ and α_- are the diagonal elements of the polarizability tensor in circular components.

We assume that the working transition occurs between two levels which are characterized by total momenta j_1 and j_0, g factors g_1 and g_0 and lifetimes γ_1^{-1} and γ_0^{-1} ($\gamma_1 < \gamma_0$) respectively. The transition frequency is ω_0. Moreover we introduce the following notation

$$\delta = \omega_0 - \omega, \quad \gamma_{10} = {}^1/_2(\gamma_1 + \gamma_0), \quad \Omega_1 = \mu_0 g_1 H,$$

$$\Omega_0 = \mu_0 g_0 H, \quad \Omega_{m\mu} = m\Omega_1 - \mu\Omega_0, \quad \delta_{m\mu} = \delta + \Omega_{m\mu}.$$

The subscripts m and μ denote the Zeeman sublevels of the upper and lower working levels. The population inversion is denoted by N and is the value which would exist in the absence of an electric field in the laser.

The tensor polarizability of the gas in a magnetic field is obtained by the method of successive approximations developed by Lamb [5] and is calculated to terms quadratic in the electric field. We will assume that: 1) the excitation is uniform and isotropic; 2) the cavity detuning, the Zeeman splitting, and the natural line widths are small in comparison to the Doppler width of the line, i.e., δ, Ω_1, Ω_0, γ_{10} $\ll$ ku, where u is the thermal velocity of the gas atoms; 3) the electric field in the cavity is small in the sense that $\mathbf{E}\cdot\mathbf{d} < \gamma$ where $\mathbf{d}$ is the dipole moment of the transition. Under these conditions we obtain the following expression for the diagonal elements of the polarizability tensor occurring in Eq. (2):

$$4\pi\alpha_+ = a_+ + b_+|E_+|^2 + c_+|E_-|^2,$$

$$4\pi\alpha_- = a_- + b_-|E_-|^2 + c_-|E_+|^2, \tag{3}$$

where

$$a_+ = 4\pi^{3/2} i \frac{N}{ku} \sum_{m\mu} |d_{m\mu}^+|^2 \left(1 + i\frac{2}{\sqrt{\pi}}\frac{\delta_{m\mu}}{ku}\right), \tag{4}$$

$$b_+ = -\frac{\pi^{3/2}}{4} i \frac{N}{ku}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_0}\right) \times \sum_{m\mu} |d_{m\mu}^+|^4 \left(\frac{1}{\gamma_{10}} + \frac{1}{\gamma_{10} - i\delta_{m\mu}}\right), \tag{5}$$

$$c_+ = -\frac{\pi^{3/2}}{4} i \frac{N}{ku} \sum_{m\mu} |d_{m\mu}^+|^2 \Big\{ |d_{m,\,\mu+2}^-|^2 \Big[\frac{1}{\gamma_1}\Big(\frac{1}{\gamma_{10} - i\Omega_0} + \frac{1}{\gamma_{10} - i\delta_{m\mu} + i\Omega_0}\Big) + \frac{1}{\gamma_0 - 2i\Omega_0}\Big(\frac{1}{\gamma_{10} - i\Omega_0} + \frac{1}{\gamma_{10} - i\delta_{m\mu}}\Big)\Big] + |d_{m-2,\,\mu}^-|^2 \Big[\frac{1}{\gamma_1 - 2i\Omega_1}\Big(\frac{1}{\gamma_{10} - i\Omega_1} + \frac{1}{\gamma_{10} - i\delta_{m\mu}}\Big) + \frac{1}{\gamma_0}\Big(\frac{1}{\gamma_{10} - i\Omega_1} + \frac{1}{\gamma_{10} - i\delta_{m\mu} + i\Omega_1}\Big)\Big]\Big\}. \tag{6}$$

Here $d_{m\mu}^+$ and $d_{m\mu}^-$ are the matrix elements of the circular components of the dipole moment operator. The coefficients a_-, b_-, and c_- are obtained from the corresponding terms in Eqs. (4)—(6) by changing the sign of the magnetic field, i.e., by making the substitutions:

$$\Omega_0 \to -\Omega_0, \quad \Omega_1 \to -\Omega_1, \quad \delta_{m\mu} \to \delta - \Omega_{m\mu}.$$

We recall that the assumption $\delta_{m\mu} \ll$ ku is made in obtaining Eqs. (3)—(6).

3. THE LINEAR THEORY

In the present section we treat the threshold operation of the laser in a longitudinal magnetic field. We will determine the threshold excitation as a function of the magnetic field and will also find the laser frequency shift and the polarization of the laser emission at threshold. It is sufficient to use the linear approximation for the polarization in obtaining these quantities. We insert the first approximation for the polarizability determined from Eq. (4) in Eq. (2). In Eq. (4) we put $\delta = 0$, $\delta_{m\mu} = \Omega_{m\mu}$ i.e., we assume that the resonance frequency ω coincides with the center of the atomic line ω_0. Then

$$a_+ = -a_-^* = a = a' + ia''.$$

Equation (2) takes the form

$$\frac{2i}{\omega}\frac{dE_+}{dt} + (i\beta - a)E_+ + i\varkappa E_- = 0,$$

$$\frac{2i}{\omega}\frac{dE_-}{dt} + (i\beta + a^*)E_- + i\varkappa E_+ = 0. \tag{7}$$

We seek a solution of Eq. (7) in the form

$$E_+ = E_+^0 e^{i\Delta t}, \quad E_- = E_-^0 e^{i\Delta t},$$

where Δ is the shift of the lasing frequency from cavity frequency ω. We then obtain the following system of algebraic equations

$$(i\beta - a - 2\Delta/\omega)E_+ + i\varkappa E_- = 0,$$

$$i\varkappa E_+ + (i\beta + a^* - 2\Delta/\omega)E_- = 0. \tag{8}$$

Setting the determinant of this system equal to zero and separating real and imaginary parts, we obtain two equations which determine the frequency shift and the threshold excitation

$$(\beta - a'')\Delta = 0, \quad (\beta - a'')^2 = \varkappa^2 - (a')^2 + (2\Delta/\omega)^2. \tag{9}$$

It is clear that there are two mutually exclusive possibilities

$$1)\ \Delta = 0, \quad (\beta - a'')^2 = \varkappa^2 - (a')^2; \tag{10}$$

$$2)\ a'' = \beta, \quad (2\Delta/\omega)^2 = (a')^2 - \varkappa^2. \tag{11}$$

It follows from Eq. (4) that the quantity $a' = \mathrm{Re}\,(a_+)$ is proportional to the magnetic field.

Hence for small magnetic fields the expression $\kappa^2 - (a')^2$ is positive and we have the first case, Eq. (10), i.e., the oscillation occurs at the unshifted frequency ω. There is a critical value of the magnetic field H_0 such that for $H > H_0$ the oscillations are described by Eq. (11). In this case there are two modes with frequencies shifted in opposite directions from ω. The shift is

$$\Delta = \pm {}^1\!/_2\omega[(a')^2 - \varkappa^2]^{1/2}.$$

(We note that the existence of a region where $\Delta = 0$ is related to the difference between Q_x and Q_y, since for $\kappa = 0$ Eq. (10) has no solution.)

We now consider the polarization of the emission at threshold in both of these cases. The first of Eqs. (8) gives a relationship between the amplitudes E_+ and E_-:

$$E_+ = \varkappa[a'' - \beta - i(a' + 2\Delta/\omega)]^{-1}E_-. \qquad (12)$$

In the first case, using Eq. (10) and Eq. (12) and introducing the angle φ,

$$\sin 2\varphi = -a'/\varkappa, \qquad (13)$$

we obtain

$$E_+ = \pm e^{\mp 2i\varphi}E_-,$$

which corresponds to a linear polarization. Here the upper signs correspond to modes polarized in zero magnetic field along the y axis. These modes have a larger threshold and will not be considered further. The lower signs describe polarization along the x axis in zero magnetic field. With increasing magnetic field the emission remains linearly polarized, but the direction of polarization is rotated from the x axis by an angle φ. As can be seen from Eq. (13) the maximum value of the rotation is 45°.

In the second case ($H > H_0$) from Eq. (11) and Eq. (12) we find*

$$E_+ = iE_- \operatorname{tg}\theta \quad \text{for} \quad \Delta > 0,$$
$$E_+ = iE_- \operatorname{ctg}\theta \quad \text{for} \quad \Delta < 0,$$

where $\sin 2\theta = -\kappa/a'$. These relations determine emissions which are right and left elliptically polarized, respectively. Both ellipses are inclined to the x axis at an angle of 45° and their semi-axes have the following ratio

$$(1 - \operatorname{tg}\theta)/(1 + \operatorname{tg}\theta). \qquad (14)$$

As the strength of the magnetic field is increased, for $|a'| \gg \kappa$, the elliptical polarization changes to circular polarization. The relative intensities of the right and left rotating fields can not be determined from a linear theory.

*tg ≡ tan, ctg ≡ cot.

To determine the dependence of the angles φ and θ on magnetic field and to determine the thresholds we calculate the quantities a' and a'' explicitly. For the transition at 1.153 μ in Ne ($j_1 = 1$, $j_0 = 2$) Eq. (4) gives

$$a' = -\xi(\beta - \varkappa)N/N_0, \qquad a'' = (\beta - \varkappa)N/N_0, \quad (15)$$

where

$$\xi = \frac{3g_0 - g_1}{\sqrt{\pi}}\frac{\mu_0 H}{\hbar k u}, \qquad N_0 = \frac{3}{4\pi^{3/2}}\frac{\hbar k u}{|(1\|d\|2)|^2}(\beta - \varkappa),$$

where N_0 is the threshold inversion for laser action in the high Q mode (polarized along the x axis) in the absence of a magnetic field, $(1 \| d \| 2)$ is the matrix element of the dipole moment transition $j_1 = 1 \to j_0 = 2$. It is convenient to express the quantity N_0 in terms of the transition probability W:

$$N_0 = \frac{1}{3\pi^{3/2}}\frac{k^3}{Q_x}\frac{ku}{W}.$$

Putting Eq. (15) in Eq. (10) we obtain a dependence of the threshold N_H on magnetic field for the region of linear polarization

$$\frac{N_H}{N_0} = \frac{1 \pm (\varepsilon^2 - \xi^2)^{1/2}}{1 - \varepsilon}, \qquad \varepsilon = \frac{\varkappa}{\beta} = \frac{Q_x - Q_y}{Q_x + Q_y}. \quad (16)$$

In deriving Eq. (16) we have assumed that $\epsilon \ll 1$ and we have discarded terms of order ϵ^2 and ξ^2 compared to unity. From the two solutions it is necessary to choose the one corresponding to oscillation in the higher-Q x-polarization in the absence of a magnetic field [the minus sign in (16)]. It is clear that with increasing magnetic field the threshold for both modes increases. Physically this corresponds to the fact that with increasing magnetic field there is an increase in the admixture of the lower Q oscillation into the eigenmode.

Equation (16) holds for $\xi \leq \epsilon$. Hence the critical magnetic field is given by the expression $\xi_0 = \epsilon$ or

$$H_0 = \frac{\sqrt{\pi}}{3g_0 - g_1}\frac{\hbar k u}{\mu_0}\varepsilon. \qquad (17)$$

In the experiments of Culshaw and Kannelaud [2] the critical magnetic field was about 0.2 Oe. Since the g factors are of the order of unity and the Doppler linewidth is about 800 Mc, we obtain an estimate of the order 10^{-3} for the value of ϵ obtained in these experiments. Thus a well defined region of unshifted frequencies and linear polarization exists even for very small differences between Q_x and Q_y. With the help of Eqs. (13)—(16) we find the following expression for the angle of rotation of the polarization direction when $H < H_0$

$$\sin 2\varphi = \frac{\xi}{\xi_0}[1-(\xi_0^2-\xi^2)^{1/2}]$$
$$= \frac{H}{H_0}\left[1-\frac{3g_0-g_1}{\sqrt{\pi}}\frac{\mu_0}{\hbar ku}(H_0^2-H^2)^{1/2}\right]. \tag{18}$$

In a very weak magnetic field $H \ll H_0$ the rotation angle increases linearly; $\varphi \approx H/2H_0$. When the critical field is reached $\sin 2\varphi = 1$ and the direction of polarization has been turned by 45°. These conclusions are in agreement with experimental results.[2]

In magnetic fields larger than the critical field, $H > H_0$, we find from (11) and (15) the threshold, the shifted frequencies, and the angle θ that determines the ratio of the semi-axes of the ellipse in Eq. (14):

$$N_H/N_0 = (1-\varepsilon)^{-1},$$
$$\frac{2\Delta}{\omega} = \pm\beta(\xi^2-\xi_0^2)^{1/2} = \pm\beta\frac{3g_0-g_1}{\sqrt{\pi}}\frac{\mu_0}{\hbar ku}(H^2-H_0^2)^{1/2},$$
$$\sin 2\theta = H_0/H. \tag{19}$$

Thus in the present case, when the cavity frequency ω coincides with the center of the atomic line ω_0, we have the following evolution of the threshold regime when Q_x and Q_y are different. As the magnetic field is increased from zero the frequency does not change and the output remains linearly polarized but the direction of polarization is rotated from the x axis by an angle φ (Eq. (18)). The critical value of the field H_0 determined from Eq. (17) corresponds to rotation by 45°. For further increase in the magnetic field ($H > H_0$) two modes appear with the same threshold and with frequencies shifted in opposite directions from ω according to Eq. (19). Close to the critical value of the magnetic field these two modes are polarized in strongly eccentric right and left ellipses inclined at 45° to the x axis. With increasing magnetic field the elliptical polarizations change into right and left circular polarization. For $H > H_0$ beats at frequency 2Δ may be observed between these two modes. According to Eq. (19), in the region $H > H_0$ threshold is independent of the magnetic field and the frequency shift increases linearly for $H \gg H_0$. This is valid only within the limits of the assumptions made, namely that $\mu_0 H \ll \hbar ku$. When the Zeeman splitting is equal to the Doppler line width the threshold begins to increase and the dependence of the frequency shift on magnetic field becomes nonlinear. In the linear approximation it is not difficult to obtain a formula similar to Eq. (19) which is valid for $\mu_0 H/\hbar \sim ku$. However we will not do this here.

To conclude this section we derive the change which occurs if the cavity frequency ω is shifted from the atomic resonance ω_0. However, we will assume that $\delta = \omega_0 - \omega \ll ku$. Then the only changes involve the frequencies. For $H < H_0$ the frequency shift will be towards ω_0:

$$\frac{\Delta}{\omega} = \frac{\beta}{\sqrt{\pi}}\frac{\delta}{ku}[1-(\xi_0^2-\xi^2)^{1/2}].$$

This shift depends very weakly on magnetic field. For $H > H_0$ one must add to the splitting obtained above the usual constant frequency shift towards the atomic resonance frequency

$$\frac{2\Delta}{\omega} = \beta\left[\frac{2}{\sqrt{\pi}}\frac{\delta}{ku} \pm (\xi^2-\xi_0^2)^{1/2}\right].$$

All of the conclusions concerning the thresholds and polarizations remain valid.

4. NONLINEAR THEORY

In what follows we will consider the case where the resonance frequency ω coincides with the atomic frequency ω_0 ($\delta = 0$). Then the coefficients $a_\pm$, $b_\pm$, $c_\pm$ which occur in Eq. (3) are related as follows:

$$a_+ = -a_-^* = a, \quad b_+ = -b_-^* = b, \quad c_+ = -c_-^* = c.$$

Putting Eq. (3) for the polarizability in Eq. (2) we obtain a system of nonlinear equations

$$\frac{2i}{\omega}\frac{dE_+}{dt} + i\beta E_+ + i\varkappa E_- = (a + b|E_+|^2 + c|E_-|^2)E_+,$$
$$\frac{2i}{\omega}\frac{dE_-}{dt} + i\beta E_- + i\varkappa E_+$$
$$= -(a^* + b^*|E_-|^2 + c^*|E_+|^2)E_-. \tag{20}$$

A. We consider the region in which oscillations occur at the unshifted frequency ω [i.e., the region where there are stationary solutions of the system Eq. (20)]. In this region the emission intensity and the polarization state are determined from the system of equations

$$[i\beta - a - b|E_+|^2 - c|E_-|^2]E_+ + i\varkappa E_- = 0,$$
$$[i\beta + a^* + b^*|E_-|^2 + c^*|E_+|^2]E_- + i\varkappa E_+ = 0. \tag{21}$$

Putting E_- from the second equation into the first equation and separating real and imaginary parts we find the condition

$$|E_+|^2 = |E_-|^2 = |E|^2 \tag{22}$$

and an equation for the intensity $|E|^2$ as a function of the excitation and magnetic field:

$$[\beta - a'' - (b'' + c'')|E|^2]^2$$
$$= \varkappa^2 - [a' + (b' + c')|E|^2]^2. \tag{23}$$

Equation (22) shows that the emission is linearly polarized. As in the linear case, we find from Eqs. (21) and (23) that in a magnetic field the polarization direction is rotated with respect to the x axis by an angle determined by the equation

$$\sin 2\varphi = -[a' + (b' + c')|E|^2]\varkappa^{-1}, \tag{24}$$

where $|E|^2$ must be determined from Eq. (23).

In view of the complexity of the coefficients b and c [Eqs. (5) and (6)] we limit ourselves to a qualitative treatment of the dependence of the rotation angle φ and the critical magnetic field $H_0(N)$ on the excitation. We assume that $\kappa/\beta \ll N/N_0 - 1$ and that the magnetic field is small enough so that the Zeeman splitting is considerably less than the natural line width of the levels: $\mu_0 H \ll \hbar\gamma$. It follows from Eqs. (5) and (6) that in such small magnetic fields the real part of the coefficients b_+, $c_+ - b'$, and c' are proportional to the magnetic field, whereas the imaginary parts $-b''$ and c'' are constant. Then the intensity $|E|^2$ will differ little from the intensity in the absence of a magnetic field for the same cavity losses. From Eq. (23) we find

$$|E|^2 = (\beta - a'') / (b'' + c''). \tag{25}$$

This expression is valid (within the limits of the assumptions we have made) as long as the right hand side of Eq. (23) does not become negative. From this we obtain an equation for the determination of the critical field $H_0(N)$ as a function of the excitation

$$\varkappa = \pm\left[a' + \frac{b' + c'}{b'' + c''}(\beta - a'')\right]. \tag{26}$$

Inserting the intensity $|E|^2$ obtained from Eq. (25) into Eq. (24) we obtain an equation for the angle of rotation of the polarization direction

$$\sin 2\varphi = -\left[a' + \frac{b' + c'}{b'' + c''}(\beta - a'')\right]\varkappa^{-1}. \tag{27}$$

We write the ratio $(b' + c')/(b'' + c'')$ in the form

$$\frac{b' + c'}{b'' + c''} = -A(\xi)\xi. \tag{28}$$

For the assumptions we have made, viz., $\xi \ll \gamma/ku$, the quantity $A(\xi)$ is a positive constant of the order of ku/γ. With the help of Eqs. (15) and (28) we find for the critical field the equations

$$H_0(N) = H_0\left[A - \frac{N}{N_0}(A - 1)\right]^{-1} \text{ for } \frac{N}{N_0} - 1 \ll \frac{\gamma}{ku}, \tag{29}$$

$$H_0(N) = H_0\left[\frac{N}{N_0}(A - 1) - A\right]^{-1} \text{ for } \frac{N}{N_0} - 1 \gg \frac{\gamma}{ku}, \tag{30}$$

where H_0 is the critical field at threshold given by Eq. (17). The inequalities in expressions (29) and (30) are necessarily fulfilled, since the critical magnetic field was small in the sense that $\mu_0 H_0(N) \ll \hbar\gamma$. If these inequalities are not satisfied then it is improper to neglect the variation of $A(\xi)$. This case will be treated qualitatively later.

We note that the region of applicability of Eq. (30) is limited by the condition $N/N_0 - 1 \ll 1$, which was required for the valid application of the method of successive approximations in calculating the polarizability. Under the same conditions we find for the angle of rotation

$$\sin 2\varphi = \frac{H}{H_0}\left[A - \frac{N}{N_0}(A - 1)\right]. \tag{31}$$

In actuality this formula is also applicable only for $\mu_0 H \ll \hbar\gamma$. From Eqs. (29)—(31) it is clear that for not too large increases in excitation above threshold the critical fields increase whereas the rate of rotation slows down. For a comparatively large excitation $N/N_0 - 1 \gg \gamma/ku$ (but $N/N_0 - 1 \ll 1$), the critical field decreases and the direction of the rotation of the polarization changes to the opposite sense and with increasing excitation the rotation proceeds more rapidly. This increase in the rate of rotation with increasing excitation was observed experimentally[2]. For $H = H_0(N)$, where $H_0(N)$ is determined from Eq. (29) or Eq. (30), the direction of polarization is rotated by either $+45°$ or $-45°$, respectively. For threshold excitation $N = N_0$ Eqs. (29) and (31) give $H_0(N_0) = H_0$ and $\sin 2\varphi = H/H_0$ which, up to terms or order $\xi \ll \gamma/ku \ll 1$, coincide with the results of the linear theory.

B. We now consider the operating regime in which there are two modes with different frequencies. In this case we assume that the magnetic field is considerably larger than the critical value and that one may therefore neglect the quantity κ in Eq. (20). Thus the region of elliptical polarization is excluded from consideration. Putting $\kappa = 0$ in Eq. (20) we seek a solution of the form

$$E_+ = E_+^0 e^{i\Delta_+ t}, \qquad E_- = E_-^0 e^{i\Delta_- t}.$$

We then obtain

$$(i\beta - 2\Delta_+/\omega - a - b|E_+|^2 - c|E_-|^2)E_+ = 0,$$

$$(i\beta - 2\Delta_-/\omega + a^* + b^*|E_-|^2 + c^*|E_+|^2)E_- = 0. \tag{32}$$

The system (32) has the following non-trivial solutions:

1) $E_- = 0, \quad |E_+|^2 = \dfrac{\beta - a''}{b''}, \quad \dfrac{2\Delta_+}{\omega}$

$$= -\left[a' + \frac{b'}{b''}(\beta - a'')\right];$$

2) $E_+ = 0, \quad |E_-|^2 = \dfrac{\beta - a''}{b''},$

$$\frac{2\Delta_-}{\omega} = \left[a' + \frac{b'}{b''}(\beta - a'') \right];$$

3) $$|E_+|^2 = |E_-|^2 = \frac{\beta - a''}{b'' + c''}, \quad \frac{2\Delta_+}{\omega} = -\frac{2\Delta_-}{\omega}$$
$$= -\left[a' + \frac{b' + c'}{b'' + c''}(\beta - a'') \right]. \tag{33}$$

Thus it is possible that there will exist only right or left circular polarization or else both together with different intensities. Analysis shows that the first and second solutions are unstable with respect to small perturbations under the condition $c''/b'' < 1$, whereas the third is unstable for $c''/b'' > 1$. It follows from Eqs. (5) and (6) that for $j_1 = 1$, $j_0 = 2$ and arbitrary magnetic field the inequality $c''/b'' < 1$ is satisfied. Hence we limit our considerations to the third solution of Eq. (33). We again make use of Eqs. (15) and (28) and find the frequency shift

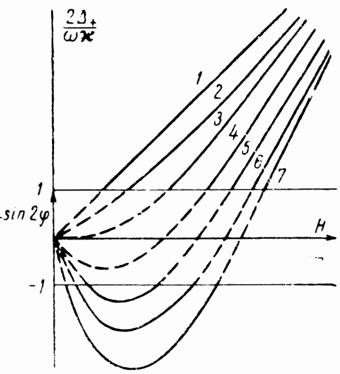
$$\frac{2\Delta_+}{\omega} = \beta\xi \left[\frac{N}{N_0} - A(\xi)\left(\frac{N}{N_0} - 1 \right) \right]. \tag{34}$$

From this expression it is clear that if $\xi \ll \gamma/ku$ and if the quantity $A(\xi)$ may be considered constant, then for sufficiently large excitation $N/N_0 > A/(A - 1)$, the frequency shift Δ_+ becomes negative (and the frequency shift Δ_- positive) which is just the opposite of what happened in the linear theory.

With increasing magnetic field, when $\mu_0 H \gtrsim \hbar\gamma$ the function $A(\xi)$ decreases and goes to zero as $(\gamma/ku)\xi^{-2}$, and hence for certain magnetic field $H_1 \sim \hbar\gamma/\mu_0$ the frequency shift Δ_+ becomes zero and thereafter becomes positive and increases linearly with the magnetic field (as long as the quantity ξ is not comparable to unity). A corresponding change in the character of the beat frequency variation is observed experimentally [2]. Close to the magnetic field H_1, when $|\Delta_+|/\omega \sim \kappa$, one may not neglect the difference between Q_x and Q_y and Eq. (34) is not applicable. It is clear that there will be an additional region of elliptical polarization close to the point H_1.

The expression for the relative frequency shift $2\Delta_+/\omega$ in Eq. (33) is no longer applicable for $|2\Delta_+/\omega| \lesssim \kappa$, since for such magnetic fields it is no longer legitimate to omit terms containing κ in Eq. (20). As a consequence of the nonlinear dependence of expression $a' + (b' + c')(b'' + c'')^{-1}(\beta - a'')$ on the magnetic field for sufficiently large excitation $N/N_0 - 1 \gtrsim \gamma/ku$, there may be a second region of linear polarization in the vicinity of the point $H = H_1$.

The above qualitative considerations are illustrated schematically in the figure. The ordinate is the ratio $2\Delta_+/\omega\kappa$ determined from Eq. (34) and the abscissa is the magnetic field. Curves 1–7 are numbered in the order of increasing excitation. The intersection of the curves with the abscissa occurs at magnetic fields $H_1 \sim \hbar\gamma/\mu_0$. The straight line 1 corresponds to the frequency shift at threshold ($N = N_0$) for various Q's. The points of intersection of the curves 1–7 with the straight lines $2\Delta_+/\omega\kappa = \pm 1$ determine approximately the critical fields for the boundaries of the regions of linear polarization. Under the condition that $\mu_0 H \ll \hbar\gamma$ the dashed portions of the curves lying between the horizontal lines ± 1 are approximate graphs of the function $\sin 2\varphi(H)$ for various excitation levels, where φ is the angle of rotation of the polarization direction away from the x axis.



Close to the values $\sin 2\varphi = \pm 1$ the dashed curves incorrectly describe the rotation. This follows since according to Eq. (18) near the field H_0 the function $\sin 2\varphi$ depends nonlinearly on the magnetic field. The actual frequency shifts which occur for various Q's are given by the solid portions of curves 1–7 far from the straight lines ± 1, when $2|\Delta_+|/\omega\kappa \gg 1$. It may be seen from the figure that when the excitation is increased from threshold the critical magnetic field $H_0(N)$ initially increases (curves 1–4), and the rotation of the polarization direction becomes less rapid. Curve 4 corresponds to a non-monotonic dependence of the rotation angle on the magnetic field: the angle φ is originally negative, then goes through zero and increases to 45°. Curves 5–7 correspond to the existence of two regions of linear polarization separated by the region where beats are observed between two circularly polarized modes of different frequency. For these curves the critical magnetic field $H_0(N)$ decreases with increasing excitation, the rotation of the polarization direction becomes more rapid, and the critical field corresponds to rotation by – 45°. It can be seen from the figure that in traversing the second region of linear polarization (curves 5–7) the direction of polarization is rotated by about 90°. Eqs. (29)—(31) describe the motion of the critical point $H_0(N)$ and the function $\varphi(H)$ in small fields, i.e., for curves 1, 2 and 6, 7. The experiments [2] evidently correspond to the case represented by either a curve of type 6 or 7.

In conclusion we note that the existence of regions of linearly polarized laser emission in a longitudinal magnetic field is related to the fact that there is a special direction perpendicular to the laser axis along which the equations for E_+ and E_- are coupled even in the linear approximation. In our case this direction (the x axis) is chosen because $Q_x > Q_y$. In principle a given direction might be singled out for some other reason, e.g., if the excitation were slightly anisotropic in the (xy) plane. This would lead to qualitatively similar results.

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