CONCERNING BREMSSTRAHLUNG

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The angular distribution and degree of polarization of bremsstrahlung are calculated for a relativistic charged particle traversing a layer of matter whose thickness is less than the radiation length. Emission of quanta with frequencies much greater than the optical and energies much lower than that of the particle is considered.

THE intensity of bremsstrahlung of à relativistic particle in a condensed medium was exhaustively investigated in detail in a series of earlier papers[1-7]. When the initial condition is made more precise, questions arise with respect to the degree of polarization and angular distribution of the bremsstrahlung. We consider here the emission of quanta with energy much lower than the particle energy, so that the classical theory can be used.

Assume that a relativistic charged particle has at the initial instant $t = 0$ a velocity $\mathbf{v}_0$ and that it moves in a medium for a certain finite time T. The spectral density of the radiation energy per unit solid angle, averaged all possible trajectories[3], is determined in this case from the formula

$$W_{\mathbf{n}\omega} = \frac{e^2\omega^2}{2\pi^2 c^3}\,\mathrm{Re}\int_0^T dt \int [\mathbf{nv}]\, w_1(\mathbf{v}, \mathbf{r}, t)\, d\mathbf{v}d\mathbf{r} \times \int_0^{T-t} d\tau \int [\mathbf{nv}']\, w_2(\mathbf{v}, \mathbf{r}; \mathbf{v}', \mathbf{r}'; \tau)\, e^{i(\mathbf{k}\boldsymbol{\rho} - \omega\tau)}\, d\mathbf{v}'d\boldsymbol{\rho}, \quad (1)^*$$

where $\mathbf{k}$ is the wave vector of the emitted waves; $\mathbf{n} = \mathbf{k}/k$, $\mathbf{v}$ and $\mathbf{r}$ are the velocity and coordinate of the particle at the instant of time t; $\mathbf{v}'$ and $\mathbf{r}'$ are the velocity and coordinate at a later instant $t + \tau$; $\boldsymbol{\rho} = \mathbf{r}' - \mathbf{r}$; $w_1(\mathbf{v}, \mathbf{r}, t)$ is the distribution function with initial condition $w_1(\mathbf{v}, \mathbf{r}, 0) = \delta(\mathbf{v}_0 - \mathbf{v})\delta(\mathbf{r})$; $w_2(\mathbf{v}, \mathbf{r}; \mathbf{v}', \mathbf{r}'; \tau)$ is the probability of the values of $\mathbf{v}'$ and $\mathbf{r}'$ if their values at the instant t are $\mathbf{v}$ and $\mathbf{r}$. The functions w_1 and w_2 satisfy the usual kinetic equation.

Since the angles between the vectors $\mathbf{v}_0$, $\mathbf{v}$, $\mathbf{v}'$, and $\mathbf{n}$ are small, and the velocity does not change upon scattering, we can go over to the angle vectors

$$\boldsymbol{\vartheta} = \frac{\mathbf{v} - \mathbf{v}_0}{v_0}, \quad \boldsymbol{\vartheta}' = \frac{\mathbf{v}' - \mathbf{v}_0}{v_0}, \quad \boldsymbol{\theta}_0 = \mathbf{n} - \frac{\mathbf{v}_0}{v_0}, \quad \boldsymbol{\theta} = \mathbf{n} - \frac{\mathbf{v}}{v_0}, \quad \boldsymbol{\theta}' = \mathbf{n} - \frac{\mathbf{v}'}{v_0}; \quad (2)$$

$$W_{\mathbf{n}\omega} = \frac{e^2\omega^2}{2\pi^2 c}\,\mathrm{Re}\int_0^T dt \int \boldsymbol{\theta}\, w_1(\boldsymbol{\vartheta}, \mathbf{r}, t)\, d\boldsymbol{\vartheta}\, d\mathbf{r} \times \int_0^{T-t} d\tau \int \boldsymbol{\theta}' w_2(\boldsymbol{\vartheta}, \mathbf{r}; \boldsymbol{\vartheta}', \mathbf{r}'; \tau)\, e^{i(\mathbf{k}\boldsymbol{\rho} - \omega\tau)}\, d\boldsymbol{\vartheta}' d\boldsymbol{\rho}. \quad (3)$$

The problem is to calculate the integral. To this end we note that in accordance with Eq. (15) of Gol'dman's paper[8] with account taken of the polarization of the medium [Eq. (6) of [9]] we have at frequencies higher than optical ($\omega \gg \omega_0$, $\omega_0^2 = 4\pi n Ze^2/m$), in the Focker-Plank approximation,

$$\int w_2(\boldsymbol{\vartheta}, \mathbf{r}; \boldsymbol{\vartheta}', \mathbf{r}'; \tau)\, e^{i(\mathbf{k}\boldsymbol{\rho} - \omega\tau)} d\boldsymbol{\rho} = \exp\,[\alpha_0(\tau) + (\boldsymbol{\theta}^2 + \boldsymbol{\theta}'^2)\,\alpha_1(\tau) + \boldsymbol{\theta}\boldsymbol{\theta}'\,\alpha_2(\tau)]; \quad (4)$$

$$\alpha_0(\tau) = -i\omega\gamma\tau / 2g^2 - \ln[\mathrm{sh}\,(2i\omega q)^{1/2}\tau];$$
$$\alpha_1(\tau) = -(i\omega / 8q)^{1/2}\,\mathrm{cth}\,[(2i\omega q)^{1/2}\tau],$$
$$\alpha_2(\tau) = (i\omega / 2q)^{1/2}\,\mathrm{sh}^{-1}\,[(2i\omega q)^{1/2}\tau], \quad (5)^*$$

where we use the following notation:

$$g = \frac{E}{mc^2}, \quad q = \frac{\omega_{\mathrm{cr}}^*}{4g^4}, \quad \gamma = 1 + \frac{\omega_{\mathrm{cr}}^2}{\omega^2}, \quad \omega_{\mathrm{cr}} = \omega_0\frac{E}{mc^2},$$
$$\omega_{\mathrm{cr}}^* = \frac{E_s^2E^2}{(mc^2)^4}\frac{c}{L}, \quad \omega_0^2 = 4\pi nZe^2/m,$$
$$E_s = \sqrt{4\pi\, 137}\, mc^2, \quad L^{-1} = \frac{4nZ}{137}\left(\frac{e^2}{mc^2}\right)^2 \ln(191Z^{-1/3}). \quad (6)$$

*$[\mathbf{nv}] \equiv \mathbf{n} \times \mathbf{v}$.

*sh ≡ sinh, cth ≡ coth.

Taking also into account the known relation

$$\int w_1(\vartheta, \mathbf{r}, t)\, d\mathbf{r} = (4\pi q t)^{-1} \exp(-\vartheta^2/4qt), \tag{7}$$

we obtain by integrating in (3) with respect to $\boldsymbol{\rho}$ and $\mathbf{r}$

$$W_{n\omega} = \frac{e^2\omega^2}{8\pi^3 cq} \operatorname{Re} \int_0^T \frac{dt}{t} \int \theta \exp\left(-\frac{\vartheta^2}{4qt}\right) d\vartheta$$

$$\times \int_0^{T-t} d\tau \int \theta' \exp[\alpha_0(\tau)$$

$$+ (\theta^2 + \theta'^2)\,\alpha_1(\tau) + \theta\theta'\,\alpha_2(\tau)]\, d\vartheta'. \tag{8}$$

To calculate the degree of polarization of the bremsstrahlung, the vectors $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ must be represented in the form of a sum of two vectors, one parallel and the other perpendicular to the vector $\boldsymbol{\theta}_0$. We designate them by the symbols $\parallel$ and $\perp$ respectively:

$$\theta = \theta_\perp + \theta_\parallel, \quad \theta' = \theta_\perp' + \theta_\parallel', \quad \theta_\perp = \theta - \frac{(\theta_0\theta)}{\theta_0^2}\theta_0,$$

$$\theta_\parallel = \frac{(\theta_0\theta)}{\theta_0^2}\theta_0, \quad \theta_\perp' = \theta' - \frac{(\theta_0\theta')}{\theta_0^2}\theta_0,$$

$$\theta_\parallel' = \frac{(\theta_0\theta')}{\theta_0^2}\theta_0. \tag{9}$$

The scalar product $\boldsymbol{\theta} \cdot \boldsymbol{\theta}'$ in (8) breaks up into a sum of the following scalar products:

$$\theta\theta' = \theta_\perp\theta_\perp' + \theta_\parallel\theta_\parallel'. \tag{10}$$

That part of formula (8) in which $\boldsymbol{\theta}_\parallel\boldsymbol{\theta}_\parallel'$ appears after the substitution (10) describes the angular distribution of the radiation intensity of the waves polarized in the plane of the wave vector and the initial direction of motion of the particle. We denote it by $W_{n\omega\parallel}$. The other part corresponds to a wave polarized in the perpendicular plane. We denote it by $W_{n\omega\perp}$. Integrating with respect to θ', we have

$$W_{n\omega\perp} = \frac{e^2\omega^2}{16\pi^2 cq\,\theta_0^2} \operatorname{Re} \int_0^T \frac{dt}{t} \int_0^{T-t} \frac{\alpha_2}{\alpha_1^2} e^{\alpha_0} d\tau \int [\theta_0\theta]^2$$

$$\times \exp\left[\theta^2\left(\alpha_1 - \frac{\alpha_2^2}{4\alpha_1}\right) - \frac{\vartheta^2}{4qt}\right] d\vartheta,$$

$$W_{n\omega\parallel} = \frac{e^2\omega^2}{16\pi^2 cq\,\theta_0^2} \operatorname{Re} \int_0^T \frac{dt}{t} \int_0^{T-t} \frac{\alpha_2}{\alpha_1^2} e^{\alpha_0} d\tau \int (\theta_0\theta)^2$$

$$\times \exp\left[\theta^2\left(\alpha_1 - \frac{\alpha_2^2}{4\alpha_1}\right) - \frac{\vartheta^2}{4qt}\right] d\vartheta. \tag{11}$$

Integrating now with respect to ϑ, we obtain

$$W_{n\omega\perp} = \frac{e^2\omega^2}{32\pi cq} \operatorname{Re} \int_0^T \frac{dt}{t} \int_0^{T-t} \frac{\alpha_2}{\alpha_1^2 p^2}$$

$$\times \exp\left(\alpha_0 - \frac{\theta_0^2}{4qt} + \frac{\theta_0^2}{16pq^2t^2}\right) d\tau,$$

$$W_{n\omega\parallel} = \frac{e^2\omega^2}{32\pi cq} \operatorname{Re} \int_0^T \frac{dt}{t} \int_0^{T-t} \frac{\alpha_2}{\alpha_1^2 p^2} \left(1 + \frac{\theta_0^2}{8pq^2t^2}\right)$$

$$\times \exp\left(\alpha_0 - \frac{\theta_0^2}{4qt} + \frac{\theta_0^2}{16pq^2t^2}\right) d\tau;$$

$$p = \alpha_2^2/4\alpha_1 + 1/4qt - \alpha_1. \tag{12}$$

Substituting in formulas (12) the functions α_0, α_1, and α_2, which are defined by relations (5), we write out the results in the form:

$$W_{n\omega\perp} = \frac{e^2\omega^2 q}{\pi^2 c} \operatorname{Re} \int_0^T t\,dt$$

$$\times \int_0^{T-t} \exp\left(-\frac{\theta_0^2}{4qt} \frac{\sqrt{2i\omega q}\,t\, \operatorname{th}(\sqrt{2i\omega q}\,\tau)}{1 + \sqrt{2i\omega q}\,t\,\operatorname{th}(\sqrt{2i\omega q}\,\tau)} - \frac{i\omega\gamma\tau}{2g^2}\right)$$

$$\times [1 + \sqrt{2i\omega q}\,t\,\operatorname{th}(\sqrt{2i\omega q}\,\tau)]^{-2} \operatorname{ch}^{-2}(\sqrt{2i\omega q}\,\tau)\, d\tau,$$

$$W_{n\omega\parallel} = \frac{e^2\omega^2 q}{\pi^2 c} \operatorname{Re} \int_0^T t\,dt \int_0^{T-t} \Bigg[1$$

$$+ \frac{\theta_0^2}{2qt\,[1 + \sqrt{2i\omega q}\,t\,\operatorname{th}(\sqrt{2i\omega q}\,\tau)]}\Bigg]$$

$$\times \exp\left(-\frac{\theta_0^2}{4qt} \frac{\sqrt{2i\omega q}\,t\,\operatorname{th}(\sqrt{2i\omega q}\,\tau)}{1 + \sqrt{2i\omega q}\,t\,\operatorname{th}(\sqrt{2i\omega q}\,\tau)} - \frac{i\omega\gamma\tau}{2g^2}\right)$$

$$\times [1 + \sqrt{2i\omega q}\,t\,\operatorname{th}(\sqrt{2i\omega q}\,\tau)]^{-2} \operatorname{ch}^{-2}(\sqrt{2i\omega q}\,\tau)\, d\tau. \tag{13}^*$$

In the derivation of (13) we used the assumption that the energy losses are relatively small. It is known that the charged particle loses a small fraction of its energy on passing through a layer of condensed matter whose thickness is much smaller than the radiation unit of length ($cT \ll L$). The sections of the trajectories outside this layer are straight lines. It must be noted that in some frequency and angle regions the radiation intensity given by (13) may turn out to be smaller than the intensity of the radiation produced when the particle is suddenly stopped at the instant $t = 0$ or is ejected into the vacuum at the instant $t = T$. In these regions an important contribution is made to the radiation by the straight-line sections of the trajectory.

This contribution is described by the expression

$$W_{n\omega}' = \frac{e^2}{\pi^2 c} \frac{\theta_0^2}{(1 - \beta^2 + \theta_0^2)^2}$$

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*th ≡ tanh; ch = cosh.

$$+\frac{e^2\omega}{\pi^2 c}\frac{\theta_0}{1-\beta^2+\theta_0^2}\,\mathrm{Im}\int\limits_0^T dt\int\theta\exp[\alpha_0(t)$$

$$+(\theta_0^2+\theta^2)\alpha_1(t)+\theta_0\theta\alpha_2(t)]\,d\vartheta$$

$$+\frac{e^2\omega}{\pi^2 c}\frac{\theta_0}{1-\beta^2+\theta_0^2}\,\mathrm{Im}\int\limits_0^\infty dt\int\theta\exp[\alpha_0(T)$$

$$+(\theta_0^2+\theta^2)\alpha_1(T)+\theta_0\theta\alpha_2(T)$$

$$-\frac{i\omega t}{2}(1-\beta^2+\theta^2)]\,d\vartheta$$

$$-\frac{e^2\omega^2}{8\pi^3 cqT}\,\mathrm{Re}\int\limits_0^T dt\int\limits_0^\infty dt'\int\theta\exp\Big[\frac{i\omega t'}{2}(1-\beta^2$$

$$+\theta^2)-\frac{\vartheta^2}{4qt}\Big]\,d\vartheta\int\theta'\exp[\alpha_0(t)+(\theta^2+\theta'^2)\alpha_1(t)$$

$$+\theta\theta'\alpha_2(t)]d\vartheta'+\frac{e^2\omega^2}{16\pi^3 cqT}\,\mathrm{Re}\int\limits_0^\infty dt\int\limits_0^\infty dt'\int\theta^2$$

$$\times\exp\Big[-\frac{i\omega t}{2}(1-\beta^2+\theta^2)+\frac{i\omega t'}{2}(1-\beta^2+\theta^2)$$

$$-\frac{\vartheta^2}{4qt}\Big]\,d\vartheta. \qquad (14)$$

The first term corresponds to the radiation when the particle is suddenly stopped at the instant t = 0. The second and third describe the interference between the field generated along the path in vacuum prior to the entry of the particle into the medium, and the fields generated along the paths in the medium and in vacuum respectively after passing through a layer of the medium. The fourth describes the interference between the field generated along the path in the medium and the field connected with the straight-line section of the trajectory after the particle moves out of the medium. The fifth and final term describes the radiation produced when the particle is ejected into the vacuum after passing through a layer of the medium. To simplify the notation, we have changed the time origin in the last three terms in such a way that t = 0 corresponds to the instant at which the particle leaves the medium and enters the vacuum.

The integration in (14) with respect to ϑ and ϑ' is carried out in the same manner as in the derivation of (13), the formula being first divided into two parts corresponding to the emission of waves of different polarizations. Carrying out this integration and adding the result to (13) we obtain the angular distribution of the spectral density of the bremsstrahlung energy when a charged particle passes through a layer of condensed medium. To study waves of different polarizations, we write out the results in the form

$$W_{n\omega\perp}=\frac{e^2g^2}{2\pi^2c}$$

$$\times\,\mathrm{Re}\,\sigma\Big\{-2\int\limits_0^{x_0}x\,dx\int\limits_0^{x_0-x}\exp\Big[-\sigma\Big(\gamma y+\frac{z\,\mathrm{th}\,y}{1+x\,\mathrm{th}\,y}\Big)\Big]$$

$$\times(1+x\,\mathrm{th}\,y)^{-2}\,\mathrm{ch}^{-2}y\,dy+2x_0\int\limits_0^{x_0}dx\int\limits_0^\infty\exp\Big\{-\sigma\Big[\gamma x-y$$

$$+\frac{z(\mathrm{th}\,x-y)}{1+x_0(\mathrm{th}\,x-y)}\Big]\Big\}[1+x_0(\mathrm{th}\,x-y)]^{-2}\,\mathrm{ch}^{-2}x\,dy$$

$$-x_0\int\limits_0^\infty dx\int\limits_0^\infty\exp\Big\{-\sigma(x-y)\Big[1+\frac{z}{1+x_0(x-y)}\Big]\Big\}$$

$$\times\frac{dy}{[1+x_0(x-y)]^2}\Big\}; \qquad (15)$$

$$W_{n\omega\parallel}=\frac{e^2g^2}{\pi^2c}\,\mathrm{Re}\Big\{-2\sigma\int\limits_0^{x_0}dx\int\limits_0^{x_0-x}\exp\Big[-\sigma\Big(\gamma y+\frac{z\,\mathrm{th}\,y}{1+x\,\mathrm{th}\,y}\Big)\Big]$$

$$\times\Big(\frac{x}{2}+\frac{\sigma z}{1+x\,\mathrm{th}\,y}\Big)\frac{dy}{(1+x\,\mathrm{th}\,y)^2\,\mathrm{ch}^2y}+\frac{z}{(1+z)^2}$$

$$-\frac{2\sigma z}{1+z}\int\limits_0^{x_0}\frac{\exp[-\sigma(\gamma x+z\,\mathrm{th}\,x)]}{\mathrm{ch}^2x}\,dx$$

$$-\frac{2\sigma z}{1+z}\int\limits_0\exp\Big\{-\sigma\Big[\gamma x_0+x+\frac{z(\mathrm{th}\,x_0+x)}{1+x\,\mathrm{th}\,x_0}\Big]\Big\}$$

$$\times\frac{dx}{(1+x\,\mathrm{th}\,x_0)^2\,\mathrm{ch}^2x_0}+2\sigma\int\limits_0^{x_0}dx\int\limits_0^\infty\exp\Big\{-\sigma\Big[\gamma x-y$$

$$+\frac{z(\mathrm{th}\,x-y)}{1+x_0(\mathrm{th}\,x-y)}\Big]\Big\}\Big[\frac{x_0}{2}+\frac{\sigma z}{1+x_0(\mathrm{th}\,x-y)}\Big]$$

$$\times\frac{dy}{[1+x_0(\mathrm{th}\,x-y)]^2\,\mathrm{ch}^2x}$$

$$-\sigma\int\limits_0^\infty dx\int\limits_0^\infty\exp\Big\{-\sigma(x-y)\Big[1+\frac{z}{1+x_0(x-y)}\Big]\Big\}$$

$$\times\Big[\frac{x_0}{2}+\frac{\sigma z}{1+x_0(x-y)}\Big]\frac{dy}{[1+x_0(x-y)]^2}\Big\}, \qquad (16)$$

where x and y are new dimensionless integration variables connected with the time by the factor $\sqrt{2i\omega q}$,

$$x_0=\sqrt{2i\omega q}\,T,\quad z=\theta_0^2g^2,\quad \sigma=(i\omega/2\omega_{\mathrm{cr}}^*)^{1/2} \qquad (17)$$

and where we used the notation (6).

The degree of polarization of the bremsstrahlung is determined from the formula

$$P=(W_{n\omega\parallel}-W_{n\omega\perp})/(W_{n\omega\parallel}+W_{n\omega\perp}). \qquad (18)$$

It is interesting to consider the radiation for a thin layer. Assume that $\omega\gg\omega_{\mathrm{cr}}$. In this region of frequencies the parameter γ is close to unity. Putting $\gamma = 1$ and expanding the results in powers of T, we obtain the first nonvanishing term of the expansion for the angular distribution and the spectral radiation-energy density in the form

$$W_{n\omega} = \frac{4e^2qTg^4}{\pi^2 c}\frac{1+z^2}{(1+z)^4}, \quad W_\omega = \frac{8e^2qTg^2}{3\pi c}. \tag{19}$$

They are much larger than the succeeding terms of the expansion, which are proportional to the square of T, provided $x_0/\sigma \ll 1$, that is, if during the time T the multiple scattering does not take the particle outside the limits of the angle $\sim (1-\beta^2)^{1/2}$. Each of the formulas (15), (16), or (18) yields in this case the following results:

$$W_{n\omega\perp} = \frac{2e^2qTg^4}{\pi^2 c}\frac{1}{(1+z)^2},$$

$$W_{n\omega\parallel} = \frac{2e^2qTg^4}{\pi^2 c}\frac{(1-z)^2}{(1+z)^4}, \quad P = -\frac{2z}{1+z^2}. \tag{20}$$

By calculating the corresponding quantities for single scattering through the angle given by the theory of multiple scattering in the layer of matter under consideration, we can show that the results coincide with (19) and (20). This agreement can be attributed to the interference between the radiation produced in the individual collision acts.

In the case of a thick layer, the first terms in (15) and (16) prevail over all others, and replacement of the upper limit of integration with respect to y in these terms by infinity leads to no greater error than neglecting the remaining terms, but simplifies the formulas. Taking these circumstances into account and choosing tanh y as the new integration variable, we obtain the following results for a layer of large thickness ($\eta = \sqrt{2i\omega q}$):

$$W_{n\omega\perp} = \frac{e^2\omega}{\pi^2 c}\,\mathrm{Im}\int_0^T \eta t\,dt\int_0^1 \exp\left(-\frac{\sigma z x}{1+\eta t x}\right) \times \left(\frac{1-x}{1+x}\right)^{\sigma\gamma/2}\frac{dx}{(1+\eta t x)^2}, \tag{21}$$

$$W_{n\omega\parallel} = \frac{2e^2\omega}{\pi^2 c}\,\mathrm{Im}\int_0^T dt\int_0^1 \exp\left(-\frac{\sigma z x}{1+\eta t x}\right) \times \left(\frac{1-x}{1+x}\right)^{\sigma\gamma/2}\left(\frac{\eta t}{2}+\frac{\sigma z}{1+\eta t x}\right)\frac{dx}{(1+\eta t x)^2}. \tag{22}$$

The results are complicated functions of the layer thickness, and also of parameters σ and γ which characterize the influence of multiple scattering and polarizations of the medium on the radiation. They can be investigated with the aid of numerical integration with an electronic computer.

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