SOVIET PHYSICS JETP VOLUME 22, NUMBER 4 APRIL, 1966

PROPAGATION SOLUTIONS OF THE BOLTZMANN EQUATION

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Submitted to JETP editor April 30, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) 49, 1248-1260 (October, 1965)

Plane-wave solutions of the Boltzmann equation are discussed. The determination of such solutions is equivalent to setting up the spectrum of the corresponding operator. The spectrum is presented for an initial-value problem and for a stationary problem. A feature of it is the presence, along with the dispersion trajectories, of a continuous spectrum for which no relation between ω and k exists. The continuous spectrum is related to the strongly pronounced lack of equilibrium in the process. The theory developed here is applied to the propagation of ultrasound in a gas.

1. INTRODUCTION

THE propagation solutions (modulated by a plane wave) of the linear Boltzmann equation are of interest for a wide range of problems of modern physics. In this paper we study similar solutions of the Boltzmann equation in the theory of gases for a small deviation from equilibrium. We note some of the papers which have touched upon questions discussed below and related questions. Wang Chang and Uhlenbeck[2] (cf. [1]) were the first to make an attempt to solve the problem of the propagation of ultrasound on the basis of a system of the collision operator eigenfunctions found by them. But the dispersion equation obtained in this approach does not take into account the singularity possessed by the spectrum, and as a result of this the high frequencies are not described by it correctly. For this reason the calculations of Pekeris et al.[3] who solve the Wang Chang equation over the whole range of frequencies turn out to be useful only in the range where $\lambda/l \gtrsim 5$, λ is the wavelength, l is the mean free path.

Below we give a dispersion equation which does take into account the singularity which, as will be seen, is related to the presence of the continuous spectrum. This singularity is similar to the one discussed by Vlasov[4], Landau[5], and others in the problem of plasma oscillations. For the problem of the propagation of sound it was discussed by Gross et al.[6], by Uhlenbeck and Ford[1], and also by Sirovich and Thurber[7]. In the two last papers a limit on the wave number was obtained above which the corresponding dispersion equation had no solutions.[1)] This fact was interpreted in different ways, for example, it was associated with the disappearance of sound[1]. However, within the framework of the special model of the Boltzmann equation with which the authors were dealing the nature of the "pole" was not understood sufficiently clearly, and as a result either conflicting opinions were expressed[1], or an incorrect method of dealing with the high frequencies was utilized[6,7].

The subject of the present paper is close to the paper by Sirovich[8] in which the Wang Chang dispersion equation has been solved for low wave numbers in the case of a problem with given initial conditions. The defect of the initial equation referred to above does not manifest itself in this approach and the picture of the trajectories turns out to be correct. Among the solutions obtained nonacoustic types of propagation may be noted.

The papers mentioned above are based on the dispersion approach to the problem, in other words only the discrete spectrum is considered. However, as a rule, in physically interesting cases this is not sufficient. Thus, Van Kampen has established[9] that the dispersion relationship between ω and k in the problem of plasma oscillations is, generally speaking, absent, since the spectrum in this case is continuous. The continuous spectrum plays an important role in neutron transport theory (Case[10] and others). It is probable that some neutron ex-

[1)]We note that a similar phenomenon was also noted earlier by Vlasov ([14], p. 254).

periments (cf. [11]) can be explained only by taking it into account. In the present paper it is shown that the dispersion method is unable to describe the propagation of high frequency ultrasound (the corresponding range of experiments is given in [12,13]) and that, consequently, the continuous part of the spectrum must be taken into account.

Below we investigate the spectrum for a problem with given initial conditions and for a stationary problem. In the former case which was studied in [1,6,7], unfortunately, the connection between the dispersion expressions $\omega(k)$ obtained above and the quantities characterizing the sound pulse remains unclear. We note, that we are here dealing with "medium" wave numbers, since as has been mentioned previously, for large wave numbers no such relations exist. The stationary problem the spectrum of which enables us to describe the propagation of sound in the case of continuous sound generation (also at "medium" frequencies) turns out to occupy a more favorable position. Therefore, the contents of the corresponding section below can be regarded, in particular, as an investigation of this problem.

2. FORMULATION OF THE PROBLEM

We consider the linearized Boltzmann equation

$$\frac{\partial \Phi}{\partial t} + \mathbf{v}\frac{\partial \Phi}{\partial \mathbf{r}} = J\Phi. \tag{1}$$

Φ is the perturbation of the distribution function taken to be of the form

$$f(\mathbf{r}, \mathbf{v}, t) = f^{(0)}(1 + \Phi(\mathbf{r}, \mathbf{v}, t)),$$

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left(-\frac{mv^2}{2kT}\right) = nf_0.$$

In our future discussion it will be convenient to use the following form of the collision integral ([15], p. 366)

$$J\Phi = -\nu(v)\Phi + \int K(\mathbf{v}, \mathbf{v}_1)\Phi(\mathbf{v}_1)f_{0_1}d\mathbf{v}_1, \tag{2}$$

where

$$\nu(v) = n\int g\sigma(g,\theta)f_{0_1}\,d\mathbf{v}_1\,d\Omega \tag{3}$$

is the collision frequency for a particle of velocity $\mathbf{v}$, $g = |\mathbf{v} - \mathbf{v}_1|$, $\sigma(g, \theta)$ is the differential scattering cross section; $K(\mathbf{v}, \mathbf{v}_1)$ is a symmetric kernel the general expression for which has been given in Waldmann's [15] review paper.

The following properties of the collision operator (2) are well known.

1) It turns out to be a self-conjugate negative operator, i.e.,

$$\int \varphi J\psi f_0\,d\mathbf{v} = \int \psi J\varphi f_0\,d\mathbf{v}, \tag{4}$$

$$\int \varphi J\varphi f_0\,d\mathbf{v} \leqslant 0. \tag{5}$$

The latter is related to the fact that the Boltzmann equation is irreversible and dissipative.

2) There exists a five-fold degenerate zero eigenvalue corresponding to the eigenfunctions 1, $\mathbf{v}$, and $\mathbf{v}^2$ (conservation in collisions of the number of particles, of momentum and of energy).

3) The eigenfunctions φ_{rlm}

$$J\varphi_{rlm} = \lambda_{rl}\varphi_{rlm} \tag{6}$$

can be represented in the case of spherically symmetric potentials in the form

$$\varphi_{rlm} = R_{rl}(v)Y_{lm}(\theta, \varphi), \tag{7}$$

where $R_{rl}(v)$ is the "radial" part, Y_{lm} is a spherical harmonic. The eigenvalues λ_{rl} define a spectrum of relaxation frequencies (reciprocal relaxation times). The system of functions φ_{rlm} can be chosen to be orthogonal and normalized, i.e.,

$$\int \varphi_{N'}^{*}\varphi_N f_0\,d\mathbf{v} = \delta_{NN'}.$$

We return to the representation of the collision integral in the form (2). It is convenient because it explicitly separates out the regular (operator K) and the singular (multiplication by frequency) parts. The operator K turns out to be completely continuous (Fredholm) [2)] for a wide class of molecular potentials. It will be of such type, for example, in the case of hard spheres, and also in the case of potentials of the form α/r^s ($\alpha > 0$, $s \geq 4$) of finite range. [3)] The first term in (2) is responsible, as we shall see below, for the continuous spectrum.

We record the expression for the collision frequency and some of its properties in the case of the potentials mentioned above. Since for these potentials we have

$$g\sigma(g, \theta) = F(\theta)g^{\gamma}, \quad \gamma = (s-4)/s, \tag{8}$$

[2)] The properties of such operators are well-known (cf., for example,[16]). We here note only the basic property: a fully continuous operator can be represented in the form of a sum of an operator of finite dimensions (with a degenerate kernel) and of another operator having an arbitrarily small norm.

[3)] This has been established by Carleman[17] and by Grad[18]. Grad obtains the following condition for complete continuity

$$g\sigma(g, \theta) \leqslant \text{const}\,(g + 1/g^{1-\varepsilon}).$$

then in accordance with (3)

$$\nu(v) = nF_0 \int f_{0_1} g^{\gamma} \, d\mathbf{v}_1,$$

$$F_0 = 2\pi \int_0 F(\theta) \sin\theta \, d\theta. \quad (9)$$

We shall need to know the value of the frequency at zero

$$\nu(0) = \frac{2}{\sqrt{\pi}} \left(\frac{2}{a}\right)^{\gamma/2} \Gamma\left(\frac{\gamma+3}{2}\right) nF_0 = 2^{-\gamma/2} \nu_{av}, \quad (10)$$

where a = m/kT, and ν_{av} is the average frequency, and also its behavior at infinity

$$\nu(v) \approx nF_0 v^{\gamma}. \quad (11)$$

The following important limiting cases should be noted: $\gamma = 0$—Maxwellian molecules, $\nu = 1$—hard spheres.

We go over to dimensionless variables

$$t' = \nu_0 t, \quad \mathbf{v}' = \sqrt{a}\,\mathbf{v}, \quad \mathbf{r}' = \sqrt{a}\,\nu_0 \mathbf{r},$$

where for the characteristic frequency ν_0 we take $\nu(0)$. In future we shall use only dimensionless quantities, for example,

$$n' = n / n_0, \quad f_0' = (2\pi)^{-3/2} e^{-v'^2/2}, \quad \nu' = \nu / \nu_0 (\nu'(0) = 1),$$

and the primes will be omitted.

Thus, we consider the "propagation" solutions of the Boltzmann equation

$$\Phi = e^{i(\omega t - \mathbf{kr})} \varphi(\mathbf{v}), \quad (12)$$

ω is the frequency, **k** is the propagation vector. Substituting this expression into (1) we obtain the integral equation for φ

$$i(\omega - \mathbf{kv})\varphi = -\nu(v)\varphi + \int K(\mathbf{v}, \mathbf{v}_1)\varphi(\mathbf{v}_1) f_{0_1} \, d\mathbf{v}_1. \quad (13)$$

Thus, the problem of obtaining similar solutions reduces to the construction of the spectrum of (13).

We note that the frequency and the wave number can both be complex. The following cases are of interest: 1) ω is complex, k is real, 2) k is complex, ω is real. The former case is related to the problem of the evolution of a given initial distribution (propagation of an ultrasound pulse, of a shock wave, the relaxation spectrum of neutrons, etc.). The latter case corresponds to the stationary regime for a given frequency of perturbation (the working regime in a long line, the propagation of ultrasound in the case of continuous generation, the evolution of a freely falling jet, etc.).[4)]

Usually in obtaining solutions of the plane wave type (12) one deals with a dispersion equation. We obtain it from equation (13) with the aid of the "modified" method of moments. Utilizing the orthonormal system of functions φ_n satisfying the completeness conditions

$$\sum_n \varphi_n^*(\mathbf{v}) \varphi_n(\mathbf{v}_1) = \delta(\mathbf{v} - \mathbf{v}_1),$$

we bring equation (13) to the form [5)]

$$i(\omega - \mathbf{kv})\varphi = -\nu(v)\varphi + \sum_{m,n} a_m K_{mn} \varphi_n,$$

$$a_m = \int \varphi_m^* \varphi f_0 \, d\mathbf{v} = \langle \varphi_m, \varphi \rangle, \quad K_{mn} = \langle \varphi_m, K\varphi_n \rangle. \quad (14)$$

Here for the sake of convenience we have introduced the notation $\langle \ldots, \ldots \rangle$ for the scalar product of weight f_0.

From (14) we obtain the following expression for φ

$$\varphi = \sum_{m,n} a_m K_{mn} \varphi_n \Big/ [\nu(v) + i(\omega - \mathbf{kv})]. \quad (15)$$

Multiplying it by φ_p^* and integrating we obtain a system of linear equations whose condition of solubility yields the dispersion equations

$$D(\omega, k) = \left\| \delta_{mp} - \sum_n K_{mn} T_{pn}(\omega, k) \right\| = 0, \quad (16)$$

$$T_{pn}(\omega, k) = \int \frac{\varphi_p^* \varphi_n f_0 \, d\mathbf{v}}{\nu + i(\omega - \mathbf{kv})}.$$

The solutions of (16) are the dispersion relations $\omega(k)$ ($k(\omega)$) which are usually associated with the collective motions of the medium—the modes. These relations determine the trajectories in the complex plane.

In (16) we have taken into account the singularity of the distribution function (15) (the denominator can vanish) which, as we shall see, is associated with the continuous spectrum. This leads to the result that it turns out to be soluble only up to certain definite values of ω (or k), while the dispersion equation of Wang Chang [2] where this singularity is not taken into account has solutions for all frequencies. (One of them has been evaluated by Pekeris et al [3].)

3. PROBLEM WITH INITIAL CONDITIONS

In the case under consideration the relaxation

4)Although such a classification is convenient, nevertheless it is not invariant with respect to a transformation to a moving system of coordinates.

5)The expansion of the operator K has been written taking into account the property of complete continuity. If in this equation we restrict ourselves to a finite number of terms, we shall obtain the models for the collision term which have been studied in a particular case by Gross and Jackson[19]. The simplest of these is the well known BGK model[6].

frequency is complex ($\omega = \omega_1 + i\omega_2$), while the wave number is real. In this case Eq. (13) assumes the form

$$(-\omega_2 + i\omega_1)\varphi = (iku - \nu(v))\varphi + K\varphi, \qquad (17)$$

k is directed along the z axis, $u = v_3$.

We investigate the spectrum of the values of ω as a function of k. First of all the transformation properties of (13) enable us to verify that the spectrum is invariant with respect to transformations such as

$$\omega,\ k \rightleftarrows \omega,\ -k, \qquad (18)$$

$$\omega,\ k \rightleftarrows -\omega^*,\ k^*. \qquad (19)$$

For the dispersion equation (16) this means that

$$D(\omega, k) = D(\omega, -k) = D(-\omega^*, k^*) = 0. \qquad (20)$$

The latter is, naturally, related to the symmetry of the left hand and the right hand waves (if solutions (12) are of a wave nature, i.e., $k \neq 0$ and $\omega_1 \neq 0$). Further, it turns out that because of the dissipative property (5) the spectrum can lie only in the half-plane $\omega_2 \geq 0$. Indeed, multiplying equation (17) by φ^* and integrating we obtain after separating the real and the imaginary parts [6)]

$$-\omega_2\langle\varphi, \varphi\rangle = \langle\varphi, J\varphi\rangle, \qquad (21)$$

$$\omega_1\langle\varphi, \varphi\rangle = k\langle\varphi, u\varphi\rangle. \qquad (22)$$

The above assertion follows from (21).[7)]

The spectrum of (17) consists of a continuous part and of trajectories. We obtain the continuous spectrum by assuming that it is invariant with respect to a completely continuous perturbation.[8)] In accordance with this assumption it is sufficient to study the equation

$$[-\omega_2 + \nu(v) + i(\omega_1 - ku)]\varphi = 0, \qquad (23)$$

which is obtained as a result of crossing out the operator K in (17). As can be easily seen, the spectrum of (23) is continuous. The region which it fills is determined from the conditions

$$\omega_2 = \nu(v), \qquad (24)$$

$$\omega_1 = ku. \qquad (25)$$

It is clear that to a given k there corresponds a

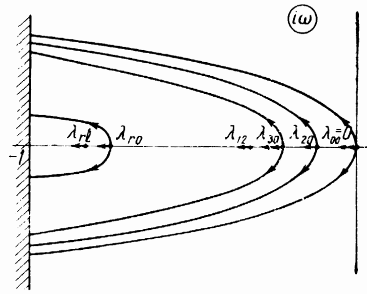

FIG. 1. Spectrum for the problem with initial conditions.

continuum of values of ω, and, consequently, there exists no dispersion relation $\omega(k)$ describing collective excitations. For this reason it is natural to associate the continuous spectrum with single particle motions (particularly since in the collisionless case only a continuous spectrum of frequencies is present). In accordance with the properties of the collision frequency, conditions (24) and (25) single out a region lying to the left of the straight line $\omega_2 = 1$ (cf., Fig. 1). If the frequency is independent of the velocity then the continuous spectrum is concentrated along the line $\omega_2 = 1$, there is no spectrum to the left of it and, consequently, the decrement does not exceed unity. But in the general case the decrement does not have an upper bound,[9)] contrary to the assumption that it is bound which was proposed by Uhlenbeck and Ford [1]. It should also be noted that the existence of a continuous spectrum excludes the dispersion approach to the problem for large decrements ($\omega_2 > 1$).

We now proceed to study the discrete spectrum. We first consider the special molecular model with the potentials α/r^4, the simplicity of which was noted by one of the founders of kinetic theory. For it the quantity $g\sigma(g, \theta)$ and, consequently, the collision frequency do not depend on the velocity (cf., (9)). This circumstance has enabled Wang Chang and Uhlenbeck [2] and others (cf. [16], Sec. 12) to obtain the eigenfunctions and the eigenvalues of the collision operator. The eigenfunctions for Maxwellian molecules are in accordance with (7)

$$\psi_{rlm} = \left[\frac{\pi^{1/2} r!\,(2l+1)\,(l-|m|)!}{2^{l+1}\,\Gamma(r+l+3/2)\,(l+|m|)!}\right]^{1/2} \times v^l L_r^{l+1/2}\left(\frac{v^2}{2}\right) Y_{lm}(\theta, \varphi), \qquad (26)$$

6) Here we have taken into account the fact that the quadratic form of a self-conjugate operator is real.

7) We note that the "quadratic-form method" utilized above enables us in a number of cases to draw general conclusions about the nature of the spectrum.

8) In the special case of a self-conjugate operator this theorem (H. Weyl) is given in [16], p. 395.

9) In the case of hard potentials with which we are dealing in this paper. In the case of "soft" potentials ($2 < s < 4$) which will be considered separately the spectrum of relaxation frequencies turns out to be bounded, but its behaviour in this case is essentially different.

where $L^{l+1/2}$ are the Laguerre polynomials [20], Y_{lm} are the spherical harmonics

$$Y_{lm} = P_l^{|m|}(\cos\theta)\, e^{im\varphi}. \tag{27}$$

The functions ψ_{rlm} are orthonormal

$$\int \psi^*_{r'l'm'}\psi_{rlm} f_0\, d\mathbf{v} = \delta_{rlm}^{r'l'm'} \tag{28}$$

and satisfy the recurrence relation

$$v_3\psi_{rlm} = \left[\frac{(l+1)^2-m^2}{(2l+3)(2l+1)}\right]^{1/2}\{[2(r+l)+3]^{1/2}\psi_{r,\,l+1,\,m} - (2r)^{1/2}\psi_{r-1,\,l+1,\,m}\} + \left[\frac{l^2-m^2}{(2l+1)(2l-1)}\right]^{1/2} \times \{[2(r+l)+1]^{1/2}\psi_{r,l-1,m} - [2(r+1)]^{1/2}\psi_{r+1,l-1,m}\}. \tag{29}$$

We give several of the lower eigenfunctions

$$\psi_{000}=1,\ \psi_{010}=v_3,\ \psi_{01\pm1}=2^{-1/2}(v_1\mp iv_2),$$
$$\psi_{100}=\sqrt{3/2}(1-{}^1/_3v^2),\ \psi_{020}={}^1/_2\sqrt{3}(v_3{}^2-{}^1/_3v^2),$$
$$\psi_{110}=\sqrt{5/2}(1-{}^1/_5v^2)\,v_3. \tag{30}$$

They are directly related to the perturbation of observable quantities such as density, velocity, temperature, pressure and heat flux in the direction of the z axis.

The eigenvalues of the model under consideration are given by the expression [1] [10]

$$\lambda_{rl} = \int_0^\pi d\theta\sin\theta F(\theta)\left[\cos^{2r+l}\frac{\theta}{2}P_l\left(\cos\frac{\theta}{2}\right) + \sin^{2r+l}P_l\left(\sin\frac{\theta}{2}\right) - 1 - \delta_{r0}\delta_{rl}\right]. \tag{31}$$

The values of (31) for a large number of values of r and l are given in [21]. In the case of the normalization adopted by us the lowest nonzero eigenvalues are equal to

$$\lambda_{20} = -0.0637,\ \lambda_{30} = -0.0955,\ \lambda_{12} = -0.1101. \tag{32}$$

Noting that in the spatially homogeneous case (k = 0) equation (17) reduces to (6) we obtain the trajectories for small k, on the basis of the eigenfunctions (26) and of the spectrum of relaxation frequencies (31). We carry out the calculations by means of perturbation theory developed in quantum mechanics (cf., for example, [22]).

The eigenvalues of the unperturbed problem are not simple. In addition to the previously noted radial degeneracy with respect to m, $-l \leq m \leq l$, and the degeneracy of the zero eigenvalue

$$\lambda_{00} = \lambda_{01} = \lambda_{10} = 0, \tag{33}$$

there exists also an "accidental" degeneracy characteristic of the model under consideration, viz., a twofold degeneracy

$$\lambda_{r0} = \lambda_{r-1,1} \tag{34}$$

and a threefold degeneracy

$$\lambda_{30} = \lambda_{21} = \lambda_{02}, \tag{35}$$

as follows from (31).

Just as in the theory of the Stark effect ([22], Secs. 75-77), the perturbation matrix

$$U_{rlm}^{r'l'm'} = \int \varphi^*_{r'l'm'}\ u\ \varphi_{rlm} f_0\, d\mathbf{v} \tag{36}$$

has the following properties: a) it is diagonal with respect to m, so that there is no need to take into account the degeneracy with respect to this index; b) the diagonal matrix elements are equal to zero, i.e., there is no first order perturbation of the nondegenerate (with respect to r, l) "levels"; c) only the elements for the "transitions" $l \rightleftarrows l \pm 1$ can differ from zero; d) the matrix elements are even functions of m, and consequently, the degeneracy with respect to the sign of m is not lifted. For Maxwellian molecules these properties follow in an obvious manner from formula (29). From this formula it is also clear that the matrix elements $U_{r-1,1}^{r0}$ between the degenerate "states" (33), (34) and (35) are not equal to zero, and this means that for the corresponding levels a first order correction will exist.

Thus, we consider the behavior of the trajectories up to second order perturbation theory. [11]

The hydrodynamic modes ($\lambda_{00} = \lambda_{01} = \lambda_{10} = 0$). Zero gives rise to two acoustic modes which differ by their directions of propagation,

$$i\omega_{1,2} = \pm ik\sqrt{5/3} + k^2\left(\frac{1}{3\lambda_{11}} + \frac{2}{3\lambda_{02}}\right), \tag{37}$$

and also two diffusion modes: the thermal mode

$$i\omega_3 = k^2/\lambda_{11} \tag{38}$$

and the transverse mode (originates from $\psi_{0,1,\pm1}$)

$$i\omega_4 = k^2/3\lambda_{02}. \tag{39}$$

The phase velocity of the acoustic mode for $k \to 0$, in accordance with its name, is equal to the acoustic sound velocity. But there is no such connection with the velocity of the sound pulse for

[10]An inaccuracy exists in[1], viz., it is asserted that $\lambda_{rl} \to \infty$ for r, $l \to \infty$, if the total cross section is infinite. Actually λ_{rl} has a finite limit.

[11]The expressions given below have also been obtained for m = 0 by a different method by Sirovich[8].

large wave numbers. Moreover, preliminary calculations indicate a transition to negative dispersion.[12] The diffusion modes (38), (39) are not propagated and, evidently, describe relaxation. Of these two the transverse mode is damped out at a lower rate (cf., (32)).

Nondegenerate modes. In this case we obtain for the shift of the eigenvalue λ_{rl}

$$i\omega_{rl|m|} = \lambda_{rl} - k^2 \left\{ \frac{(l+1)^2 - m^2}{(2l+3)(2l+1)} \times \left[\frac{2(r+l)+3}{\lambda_{rl} - \lambda_{r,\,l+1}} + \frac{2r}{\lambda_{rl} - \lambda_{r-1,\,l+1}} \right] + \frac{l^2 - m^2}{(2l+1)(2l-1)} \times \left[\frac{2(r+l)+1}{\lambda_{rl} - \lambda_{r,\,l-1}} + \frac{2(r+1)}{\lambda_{rl} - \lambda_{r+1,\,l-1}} \right] \right\}. \quad (40)$$

The modes under consideration are diffusion modes, and are related to the relaxation of the corresponding tensor quantities. The coefficients of k^2 are positive—the trajectories are moving to the left (cf., Fig. 1).

Twofold degeneracy. In this case which corresponds to the degeneracy (34) we have nonacoustic (diffusion) propagation

$$i\omega_{1,2} = \lambda_{r0} \pm ik\left(\frac{2r}{3}\right)^{1/2} - k^2 \left[\frac{2r+3}{6(\lambda_{r0} - \lambda_{r,\,1})} + \frac{2r+1}{6(\lambda_{r0} - \lambda_{r-1,\,0})} + \frac{2(2r+3)}{15(\lambda_{r0} - \lambda_{r-1,\,2})} + \frac{4(r-1)}{15(\lambda_{r0} - \lambda_{r-2,\,2})} \right], \quad (41)$$

and also the diffusion mode (from $\psi_{r-1,1,\pm 1}$)

$$i\omega_3 = \lambda_{r0} - k^2 \left[\frac{2r+3}{5(\lambda_{r0} - \lambda_{r-1,\,2})} + \frac{2(r-1)}{5(\lambda_{r0} - \lambda_{r-2,\,2})} \right]. \quad (42)$$

The trajectories, as before, move to the left (Fig. 1).

With respect to the triple degeneracy (35) we note that it is a particular case of those already considered and is described by formulas (42) and (41) respectively for $\lambda_{30} = \lambda_{21}$ and λ_{02}.

We discuss the discrete spectrum for small values of k in the case of non-Maxwellian molecules. It is clear that the qualitative picture remains the same for the zero eigenvalue (acoustic branch and two diffusion branches), and also for the eigenvalues with such indices r, l, with respect to which there was no degeneracy in the case of the Maxwellian model ($l + 1$ is the diffusion branch). The eigenvalues λ_{r0}, $\lambda_{r-1,1}$ require a more detailed investigation. The "non-Maxwellian property" (which can be regarded as a perturbation) evidently lifts the degeneracy. But the splitting of the levels turns out to be small (this, in particular, can be seen from the table for hard spheres given in [21]).[13] The latter circumstance enables us to apply perturbation theory for closely spaced levels (cf. [22]), which yields the following expression for the shift:

$$i\omega_{1,2} = {}^1/_2(\lambda_{r-1,\,1} + \lambda_{r0}) \pm [{}^1/_4(\lambda_{r-1,\,1} - \lambda_{r0})^2 - k^2\,|\,U_{r,\,0}^{r-1,\,1}\,|^2]^{1/2}. \quad (43)$$

From this expression, in particular, it follows that the propagating trajectories will now begin not from k = 0, but from a value corresponding to the point of confluence[14] (the expression under the square root vanishes). An analysis of expression (43) for the hard sphere model will probably enable one to answer the question of the existence of nonacoustic propagations in the general case.

In concluding the construction of the spectrum we discuss the behaviour of the trajectories for "large" values of k. The fact that the trajectories reach the continuous spectrum for certain values of k (cf., Fig. 1) has been established by means of going over from the degenerate collision operator (cf. footnote [5]) to the exact one within the framework of perturbation theory. For the "kinetic model" the existence of a limiting wave number k_{lim} (or, more accurately, a set of such numbers) is almost obvious and was essentially evident to Sirovich and Thurber [7] but with the one difference that the absence of solutions of the corresponding dispersion equation (the "truncated" equation (16)) was associated with the limit of applicability of this model, and not with the continuous spectrum. But in actual fact k_{lim} turns out to have its own value for each mode and tends to zero for $n \to \infty$ (n is the mode index). A very simple calculation yields for the acoustic mode a value of the limiting wave number for which $\lambda/l \approx 4$.

The description of the spectrum obtained (Fig. 1) can be interpreted as a transition of the system from collective motions (trajectories) to single particle motions (continuous spectrum) under the influence of a perturbation. In this sense modes characterized by a high index which go over into the continuous spectrum for low values of k, or

[12] As is well known, the dispersion of low frequency sound is positive, i.e., $V_{phase} < V_{group}$.

[13] Unfortunately the eigenvalues have been defined in this reference in a manner different from the customary one, but for the lowest levels this is probably not essential.

[14] The possibility of the confluence of closely spaced levels in contrast to the usual "repulsion" is evidently due to the fact that the perturbation is non-Hermitian.

which turn out to be in the continuous spectrum at the outset, should be called quasicollective modes. The latter case will be encountered in the study of "soft" potentials.

4. THE STATIONARY PROBLEM

In the case under consideration the wave number is complex ($k = k_1 - ik_2$), the frequency is real, and equation (13) can be written in the form:

$$(k_2 + ik_1) u\varphi = (\nu(v) + i\omega)\varphi - K\varphi. \qquad (44)$$

We determine the dependence on ω of the spectrum of the values of k. As can be seen from (18) and (19) it is situated in the k-plane with appropriate symmetry, so that in future it will be sufficient to limit ourselves to the fourth quadrant.

In analogy to the preceding section we can obtain the continuous spectrum by studying the equation

$$[-k_2 u + \nu(v) + i(\omega - k_1 u)]\varphi = 0. \qquad (45)$$

The region which it fills is determined by the conditions

$$k_1 = \omega / u, \qquad (46)$$

$$k_2 = \nu(v) / u, \qquad (47)$$

or by the more convenient combination:

$$k_1 = \omega k_2 / \nu(v). \qquad (48)$$

For the Maxwellian model the expression (48)

$$k_1 = \omega k_2 \qquad (49)$$

specifies a family of straight lines with the angular coefficient ω. In other words, the ray along which the continuous spectrum is concentrated sweeps out the fourth quadrant counterclockwise as the frequency varies in the range $0 \leq \omega \leq \infty$. In the general case $\nu(v) \geq 1$, so that the relation (48) defines the region of the continuous spectrum lying below the ray (49) (cf., Fig. 2),

$$k_1 \leqslant \omega k_2. \qquad (50)$$

The appearance of the continuous spectrum for hard spheres is somewhat different. First of all relation (47) together with (11) gives

$$k_2 \geqslant \nu(v) / v \geqslant b, \quad b = 0.627. \qquad (51)$$

Combining (51) with (46) and (48) we obtain the range for the continuous spectrum in the case of spheres

$$k_1 \leqslant \omega (k_2 - b). \qquad (52)$$

We proceed to investigate the discrete spectrum. First of all, by utilizing the quadratic form method

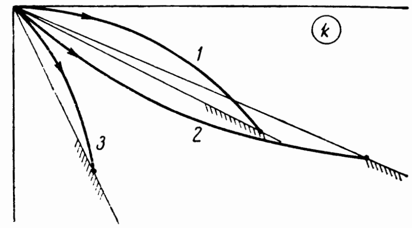

FIG. 2. Spectrum for the stationary problem.

for Eq. (44), we establish that the trajectories all emerge from the origin. In this connection it is puzzling to see the third solution of the problem of the propagation of sound obtained by Moiseev-Ol'khovskiĭ [23] which begins at the point [15] $k_1 = 0$, $k_2 \approx 1$. The problem of the number of branches originating at the origin is not as simple as might appear, since zero is a singular point (the continuous spectrum reaches it). A study of the dispersion equation (16) by means of the theory of implicit functions shows that exactly three trajectories leave the origin (in the fourth quadrant). [16]

We consider these modes in the case of low frequencies, and for the sake of simplicity we restrict ourselves to the Maxwellian model. For this it is sufficient to solve the "truncated" dispersion equation which is obtained from (16) by separating out the upper left hand determinant of the fifth order [17] (it is formed by the first five eigenfunctions (30)). Taking into account the conservation of the number of particles the equation assumes the form

$$D_5(\omega, k) = (T_{44} - 1)(T_{55} - 1) \begin{vmatrix} \omega & -k & 0 \\ T_{21} & T_{22} - 1 & T_{23} \\ T_{31} & T_{32} & T_{33} - 1 \end{vmatrix} = 0. \qquad (53)$$

It yields us the following three modes: the acoustic mode

$$k \approx \sqrt{3/5}\,(\omega - {}^3/_5 i\omega^2), \qquad (54)$$

the thermal mode

$$k \approx \sqrt{\omega / 2}\,[1 + 3\omega - i(1 - 3\omega)] \qquad (55)$$

and the transverse mode

$$k \approx \sqrt{\omega / 2}\,[1 - \omega / 2 - i(1 + \omega / 2)]. \qquad (56)$$

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[15] Probably its appearance can be explained as being the result of using the moment equations outside the domain within which they are valid.

[16] As is well known, the number of roots of an implicit equation is determined by the nonvanishing derivative of the lowest order. In this case this is $d^5D(0, 0)/d\omega^5$, i.e., there are five roots $\omega = \omega(k)$, with one of them being a double root, and one laying outside the fourth quadrant.

[17] The latter is equivalent to taking the first five terms in the expansion of the operator K[cf. (14)].

As can be seen, they all describe propagation, with the velocity of the two latter modes tending to zero as the frequency tends to zero. It is clear that the acoustic mode is the dominant one. For greater frequencies the damping of the modes becomes comparable, but the transverse mode is damped at an appreciably faster rate (cf. Fig. 2). It should be emphasized that the acoustic mode (54), and the mode from the preceding section (37) analogous to it, in spite of their similarity in the case of weak inhomogeneity, have at high frequencies different phase velocities and decrements.[18)] As estimates have shown this becomes noticeable for $\omega \gtrsim 0.3$ ($\lambda/l \lesssim 20$).

In the consideration of the one-dimensional problem (only the eigenfunctions (7) with m = 0 are taken into account) we have, evidently, two trajectories (acoustic and thermal) so that the discussion of a larger number of solutions in certain papers (cf., for example, [13]) can probably be explained by careless use of moment expansions. We note that in our case we do not obtain the second acoustic mode obtained by Vlasov ([14], p. 255) for neutral gases.

We now go on to "high" frequencies. As the frequency increases the trajectories move as shown in Fig. 2 until in accordance with the criterion (49) limiting values are reached (a separate one for each mode) at which the trajectories merge with the continuous spectrum. The attainment of the continuous spectrum is based on the same considerations as in the preceding section. The limiting frequency for the acoustic mode, for example, according to preliminary calculations is equal to $\varphi_{\lim} \approx 1.8$ ($\lambda/l \approx 4$).

It is clear that the results obtained above have nothing to say about the behavior of sound at frequencies above the limiting frequency. In this domain the dispersion approach is inapplicable, and various extrapolations of this method to high frequencies utilized by different authors[2,6,7] cannot be considered to be correct.

5. DISCUSSION AND CONCLUSIONS

1. First of all we discuss the attainment by the trajectories of the continuous spectrum, which, as has been stated, can be related to the transition of the system from collective motions in the case of weak inhomogeneity to single particle motions in the case of strong gradients. Uhlenbeck and Ford[1] have arrived at a similar picture by analyzing the problem of the propagation of sound on the basis of the simplest model of the kinetic equation. But in their arguments there is a number of inconsistencies. Thus, the existence of a limiting ("critical") frequency was related to the existence of a lower limit to the spectrum of relaxation frequencies, while the spectrum itself was assumed to be discrete in the general case. The results of the present paper show that both these facts do not hold in the general case and are valid only for the Maxwellian model.

It should be emphasized that the limiting frequency is not the "critical" frequency for sound. A transition to the domain of single particle motions, of course, does not exclude the possibility of the existence of "quasicollective" motions, for example, such as zero sound, Langmuir plasma oscillations, and others. In accordance with the criterion (50) one should include among this type of processes also the propagation of sound in the high frequency range of experiments[2,13].

2. The description of the spectrum obtained above permits one to conclude that dispersion theory (which can be schematically represented as an exponential substitution plus a study of the dispersion equation) is insufficient for the description of rapid (quasicollective) processes and the continuous spectrum must be taken into account.

For this reason Landau's procedure[5] which can be regarded as an analytic continuation of the dispersion equation into the domain of high frequencies must be considered to be incorrect.[19)] Weitzner[24] has arrived at a similar conclusion by studying the problem of collisionless damping of plasma waves; the mathematical arguments utilized in this connection are essentially related to the continuous spectrum. Apparently moment equations also can not be used for a discussion of high frequencies.

The segregation of the continuous spectrum has enabled us to clarify the relationships between two types of dispersion equations describing the same problem. Some of them (for example, the Wang Chang equation[2]) were obtained by the method of moments, while others (for example, the equations in Bhatnagar's paper[6]) were obtained similarly to (16). From what has been said earlier it is clear that equations of the second type are preferable since they take into account the existence of the continuous spectrum.

[18)]The damping k_2 is, evidently, associated with the decrement by the expression $k_2 = \omega_2/V_{\text{phase}}$.

[19)]We note that, in spite of the a priori incorrectness of this procedure, calculations show that Eq. (53) continued analytically beyond the limiting frequency gives a good description of experiments[12] right up to the maximum frequencies ($\lambda/l \approx 0.1$).

The direct study of the spectrum carried out in the present paper will probably enable one to clarify the nature of the available "special" solutions of the Boltzmann equation (such as, for example, Ol'khovskiĭ's third solution [23]).

In conclusion the author takes this pleasant opportunity of thanking the members of L. É. Gurevich's seminar for stimulating discussions, S. V. Vallander for critical remarks concerning the paper, and also M. L. Zaĭtsev for his aid in putting the work in final form.

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Translated by G. Volkoff
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