SOVIET PHYSICS JETP VOLUME 22, NUMBER 4 APRIL, 1966

THE ABSOLUTE STABILITY OF SOME PLANE PARALLEL FLOWS AT HIGH REYNOLDS NUMBERS

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Submitted to JETP editor May 21, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **49**, 1326-1331 (October, 1965)

Localized disturbances in the plane parallel flow of a viscous fluid are considered. It is shown that localized disturbances decay with time in any finite region of the flow. High Reynolds numbers are assumed, and the wave numbers approach zero on the corresponding branches of the neutral curves.

THE stability of plane parallel flows has been investigated by numerous authors, and the subject has been surveyed thoroughly by C. C. Lin.[1] Stream functions having the form

$$\psi = e^{ikx}e^{-i\omega t}f(y), \qquad (1)$$

were considered, where the x axis is parallel to the undisturbed flow and the y axis is in the direction of change of the undisturbed velocity. Poiseuille flow (between two fixed planes) has been considered,[2] as well as different cases of boundary layer flows regarded as approximate plane parallel flows.[3,4] For real values of k, $\omega(k)$ was derived at different Reynolds numbers. It was found that instability corresponding to $\operatorname{Im}\omega > 0$ occurs in the region of the k-R plane bounded by the neutral curve $\operatorname{Im}\omega = 0$ in the form of a loop with branches along which $R \to \infty$.

The instability problem usually involves an investigation of the temporal behavior of a disturbance occurring at a given initial time. In cases where, as for plane parallel flows, x varies from $-\infty$ to $+\infty$, localized flows (vanishing outside a finite segment of the x axis) can also be considered as bounded initial nonlocalized disturbances like those represented by Eq. (1), for example. The growth of nonlocalized disturbances with time depends on the existence of $\omega(k)$ with a positive imaginary part. The behavior of the localized disturbances does not depend only on $\omega(k)$ in the upper half-plane. Here, as previously, two possibilities exist in general; disturbances in a fixed region of x (for any values of y) can either grow unlimitedly with t, or can approach zero (or remain bounded). In the first case the flow is called absolutely unstable. In the second case the flow is called absolutely stable (or it is stated that convective instability exists when $\operatorname{Im}\omega(k) > 0$). This problem has been formulated by Landau and Lifshitz.[5] Some general discussion of convective and absolute instabilities with illustrations can be found in[6]. However, the criteria obtained in this article pertain to nondissipative systems (see also[7]) and cannot be applied directly to more complex cases (see[8], for example).

In the present work we investigate the character of instability in certain plane parallel flows at high Reynolds numbers. The localized disturbance of the stream function can be represented by means of a Fourier integral with respect to k:

$$\psi(x, y, t) = \sum_j \int_{-\infty}^{\infty} e^{-i\omega_j(k)t}e^{ikx}a_j(k)\varphi_j(k, y)\,dk. \qquad (2)$$

This equation cannot describe an arbitrary disturbance of the stream function [if the system $\varphi_j(k, y)$ is not complete in the space of the functions depending on y], but it can represent the portion of the disturbance that grows with $t \to \infty$ (for fixed k).

It is easily shown that $\Sigma_j a_j(k)\varphi_j(k, y)$ for localized disturbances is an entire analytic function of k. Individual terms of the sum have singularities at points in the complex k plane where branches for different values of j meet.

We shall now consider the behavior of an individual term of the sum for $t \to \infty$:

$$\psi_1(x, y, t) = \int_{-\infty}^{\infty} e^{-i\omega_1(k)}e^{ikx}a_1(k)\varphi_1(k, y)\,dk, \qquad (3)$$

assuming that the instability interval on the real k axis ($k_1 < k < k_2$) is such that everywhere therein $\operatorname{Im}\omega_1(k) > 0$.

We shall assume that all other branches of $\omega_j(k)$ for real k lie in the lower half-plane and therefore do not contribute to the instability. For plane parallel flows this follows from the fact that we have only two branches of the neutral curve with an instability between them. (Detailed calculations of different branches of $\omega_j(k)$ for some plane parallel flows are given in[9].)

If in the integral (3) the integration contour can be shifted in the complex k plane so that the condition $\operatorname{Im} \omega_1(k) \leq 0$ holds true everywhere on the new contour, then $\psi_1 = 0$ for $t \to \infty$; if not, we have $\psi \to \infty$. Thus the behavior of (3) depends essentially on the analytic properties of $\omega_1(k)$.

The function $\omega(k)$ is obtained from the conditions for the existence of a nontrivial solution of the Orr-Sommerfeld equation:

$$\left(\frac{d^2}{dy^2} - k^2\right)^2 \varphi(k, y) = ikR\left\{\left(u - \frac{\omega}{k}\right)\left(\frac{d^2}{dy^2} - k^2\right)\varphi(k, y) - \frac{d^2u}{dy^2}\varphi(k, y)\right\} \tag{4}$$

subject to the boundary conditions

$$\varphi|_{y=-1} = \partial\varphi / \partial y|_{y=-1} = 0 \tag{5}$$

(zero fluid velocity at the wall), and in addition:

a) for symmetric velocity profiles and symmetric solutions of (4)

$$\partial\varphi / \partial y|_{y=0} = \partial^3\varphi / \partial y^3|_{y=0} = 0; \tag{6}$$

b) for the boundary layer with (6) replaced by

$$\varphi \to 0, \quad \partial\varphi / \partial y \to 0 \quad (y \to \infty). \tag{7}$$

In (4), $u(y) = v(y)/v_{max}$ is the dimensionless undisturbed velocity of plane parallel flow that is being investigated with regard to stability, $R = v_{max}L/\nu$ is the Reynolds number, ν is the kinematic viscosity, and L is one-half of the interplanar separation for a symmetric velocity profile or the boundary layer thickness.

In the case of symmetric profiles odd solutions of (4) also exist. These are apparently damped solutions, although the literature contains no complete proof of this fact. However, all available results regarding instability pertain only to the even solutions; we shall confine ourselves to the latter.

Instability of plane parallel flows occurs when $kR \gg 1$.[1] In this case (4) has linearly independent solutions of two types, viscous and nonviscous. The nonviscous solutions satisfy an equation obtained by a formal passage to the limit $R \to \infty$ in (4). One of these solutions, φ_1, is an analytic function of y; another solution, φ_2, has a logarithmic branch at the point where $u = u_c = \omega/k$. In the vicinity of this point the derivatives of φ_2 are large, and the division into viscous and nonviscous solutions becomes arbitrary. For viscous solutions the derivatives are large, and we may retain in (4) only terms containing the fourth and second derivatives. One of the viscous solutions, φ_3, approaches zero; another solution, φ_4, approaches infinity for $y \to \infty$. These solutions can be represented by means of the Hankel function $H_{1/3}$.

The general solution of (4) is

$$\varphi = C_1\varphi_1 + C_2\varphi_2 + C_3\varphi_3 + C_4\varphi_4. \tag{8}$$

The dispersion equation for determining $\omega(k)$ is obtained by equating to zero the determinant of the system of homogeneous linear equations for C_i that are obtained by substituting (8) into the boundary conditions (5) and (6) or (5) and (7). This dispersion equation shows that in the case of undisturbed velocity profiles without inflection points ($u'' \neq 0$), $k \to 0$ for $R \to \infty$ on both branches of the neutral curve. On these branches we have[10] when $R \to \infty$, for symmetric profiles

$$R \sim k^{-11} \quad \text{(upper branch)},$$
$$R \sim k^{-7} \quad \text{(lower branch)} \tag{9}$$

and for flow in the boundary layer

$$R \sim k^{-6} \quad \text{(upper branch)},$$
$$R \sim k^{-4} \quad \text{(lower branch)}. \tag{10}$$

These laws are somewhat modified if the inflection point is located exactly at the wall. When inflection points exist, k approaches zero along one of the branches, while along the other branch it approaches some finite limit k_∞, which in turn approaches zero as the inflection point approaches the wall.[1]

We shall consider the case of Reynolds numbers that are high enough so that k can be regarded as small for velocity profiles without inflection points. When an inflection point exists we shall assume, in addition, that this point is close to the wall, so that the instability interval lies entirely in the region of small k. These limitations are based on the difficulty of solving the nonviscous equation for large k, so that the dispersion equation cannot be written in closed form. In the case of small k the expansions of φ_1 and φ_2 in terms of k^2 may be cut off at their principal terms. Accordingly, we have the simplified dispersion equation[1]

$$\frac{u_c''}{u_1'}\frac{\omega}{k}\left(\ln\frac{\omega}{k} - i\pi\right) + \frac{u_1'\omega}{k^r H_1} - F(z) = 0,$$

$$z = \omega\chi/k^{2/3}, \quad \chi = (R/u_1'^2)^{1/3}, \tag{11}$$

$$F(z) = -z\int_{\infty}^{-z} \zeta^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right] d\zeta \Big/ \int_{\infty}^{-z} \zeta^{3/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right] d\zeta,$$

$$H_1 = \int_{-1}^{0} u^2 dy, \quad u_1' = du/dy\,|_{y=-1},$$

$$u_c'' = d^2u/dy^2\,|_{y=y_c}, \quad y_c = \omega/ku_1'; \qquad (12)$$

r = 3 for case (a) and r = 2 for case (b).

In (9) it is assumed that

$$-3\pi/2 < \arg k < \pi/2, \quad 0 \leqslant \arg z \leqslant \pi,$$

so that the arithmetic branch $k^{2/3}$ is taken.

In (9) and (10) it is assumed that ω/k is small (which is true for small k). We note that for small ω/k the branch point of the nonviscous solution φ_2 approaches the wall and the division of the solutions into viscous and nonviscous types becomes arbitrary in the vicinity of the lower branch of the neutral curve where $z \sim 1$. Although the first term in (11) contains a relative error of the order of unity,[4] near the lower branch of the neutral curve this term can be neglected by comparison with the other terms of the equation. The first term is small everywhere and is important only at large z, where it determines $z \gg 1$ on the second branch of the neutral curve. We shall therefore first consider the equation obtained from (11) when its first term is neglected:

$$k^{r-2/3} = zu_1'/F(z)\chi H_1, \quad \omega = zk^{2/3}/\chi. \qquad (13)$$

In accordance with (10), F(z) is a ratio of entire analytic functions, with poles as its only possible singularities. We also know that $F(z) \to 1$ for $z \to \infty$ in the upper half-plane. On the real axis F(z) has real values at the points z = 2.294 and ∞, corresponding to the two branches of the neutral curve (9). We shall show that by assuming the existence of zeros or poles of F(z) we arrive at a definite conclusion regarding the character of ω(k) defined by means of (11). Let F(z) have a pole at the point z_0 in the upper half-plane. In the vicinity of this point we have

$$k^{r-2/3} = (z - z_0)^n C(z_0)/\chi, \quad \omega = z_0 k^{2/3}/\chi \quad (n > 0). \quad (14)$$

It follows that at least one z(k) curve with real k passes through z_0. Since Im $z_0 > 0$, near this point we have Im $\omega > 0$. When we move along the z(k) curve in the direction of increasing k, then either at some point z we cross the real axis at a point where Im $\omega = 0$ (which corresponds to some neutral curve in the k-R plane), or Im $\omega > 0$ for all $k > 0$, including $0 < k \leq k_1$ (k_1 is the value of k at z = 2.294). We note that the z(k) curve can terminate nowhere because F(z) is analytic. Similarly, if F(z) has a zero at z_0 in the upper half-plane, then through this point there passes a curve z(k) with real k, described by (14) with $n < 0$; Im $\omega > 0$ on this curve also. When we move along this curve in the direction of decreasing k, we can reach values of k corresponding to the upper branch of the neutral curve of (11). Using (9), (10), and the definition of r in (12), along with (14), it is easily shown that for these values of k we have $z \to z_0$ for $\chi \to \infty$. Thus, when a pole of F(z) exists in the upper half-plane we have an unstable branch of ω(k) for (13) with Im $\omega > 0$ and $k > k_2$.

We shall show that all the foregoing possibilities conflict with the fact that instability of the considered flows occurs only in the region between the two branches of the neutral curve on the k-R plane as determined from (11) for high Reynolds numbers. Indeed, the ratio of the first and second terms in (11) will be

$$(u_c''/u_1'^2)\,k^{r-1} \ln (z/\chi k^{1/3}).$$

If k approaches zero and z approaches infinity according to a power law for $\chi \to \infty$, then this ratio approaches zero with an increase of the Reynolds number. In the case of profiles with an inflection point this ratio is also everywhere small if the inflection point is very close to the wall. It is easily shown by means of the implicit function theorem that in this case the solutions of (11) approach the solutions of (13). Therefore, if for very small values of k, Eq. (11) gives solutions of z(k) lying in the upper half-plane at a finite distance from the real axis, then for sufficiently large χ Eq. (11) gives a solution of ω(k) lying in the upper half-plane. It follows that F(z) cannot have zeros or poles in the upper half-plane, because otherwise instability would exist outside of the neutral curve on the k-R plane, i.e., for $k \geq k_2$ and $k \leq k_1$.

Let us consider the curve of z(k) (for real k) defined by (13), starting at $z = z_1 = 2.294$ and going in the direction of increasing k. This curve is not bounded in the upper half-plane of z: it cannot terminate at a finite point such that $z \to z_0$ for $k \to \infty$, because this would correspond to a zero of F(z) in the upper half-plane. The curve cannot intersect the real z axis, because this would correspond to the existence of a superfluous branch of the neutral curve. Therefore, along this curve $z(k) \to \infty$ for $k \to \infty$. Equation (13) shows that the values of k at which z(k) becomes very large approach zero for $\chi \to \infty$. Since at the same time $F(z) \to 1$, it is easily seen from (13) that z(k) approaches the real axis asymptotically. Therefore even for small k (but large |z|) it becomes essential to include the term containing $i\pi$ in (11) to determine the upper branch of the neutral curve, because a small change of z(k) can reverse the sign of Im z(k) and therefore of Im ω. The appropriate calculations are easily performed by means of the iteration method and lead to the conclusion that the z(k) curve defined in (11) intersects the real axis at z_2 for $k = k_2(R)$ (the upper neutral branch).

It can be shown similarly that the $z(\omega)$ curve for real ω defined in (11) and going from z = 2.294 towards larger k lies in the upper half-plane and intersects the z(k) curve only at z_2 on the real axis.

Since in the integral (3) we have $\mathrm{Im}\,\omega \geq 0$ only in the interval (k_1, k_2), the integral can grow only in this segment. For this portion of the integral we shall make a change of the variables of z(k) in accordance with (11):

$$\psi(x, y, t) = \int_{k_1}^{k_2} e^{-i\omega_1(k)t} e^{ikx} a_1(k) \varphi_1(k, y)\, dk$$

$$= \int_{z_1}^{z_2} e^{-i\omega(z)t} e^{ik(z)x} a_1[k(z)] \varphi_1[k(z), y] \frac{dk}{dz} dz. \qquad (15)$$

The analytic functions k(z) and $\omega(z)$ are close to the analytic functions $\tilde{k}(z)$ and $\tilde{\omega}(z)$ given by (13) everywhere in the region G between the z(k) and $z(\omega)$ curves, since this is the region of small k. It is easily seen from (11) that $\tilde{k}(z)$ and $\tilde{\omega}(z)$ have singularities (branch points) only at the zeros and poles of F(z). It has already been shown that F(z) has neither zeros nor poles in the region G. Thus $\tilde{\omega}(z)$ and $\tilde{k}(z)$, and therefore $\omega(z)$ and k(z), have no singularities in G. Therefore the integration contour in (15) can be shifted so that the integral is taken along the $z(\omega)$ curve with real ω instead of along the z(k) curve. It can easily be seen that this last integral approaches zero for $t \to \infty$.

It has therefore been shown that all plane parallel flows having sufficiently small values of k on the neutral curve are absolutely stable. It was important for the proof that the neutral curve separates the region of stability in the k–R plane from the region of instability and that the region inside the neutral curve corresponds to instability. The existence of inflection points in the velocity profile does not in itself lead to absolute instability if the inflection points are sufficiently close to the wall.

The foregoing results cannot be applied to the cases of Reynolds numbers for which k on the neutral curve becomes of the order of unity, or to velocity profiles where the inflection point is located at a dimensionless distance of the order of unity from the wall. We were unable in these cases to obtain analytic proof of either absolute stability or instability. This question can apparently be decided only by direct calculation of $k(\omega)$ for real ω.

We note that the foregoing result regarding absolute stability can be used in magnetohydrodynamics for plane parallel flow in a transverse magnetic field when the magnetic Reynolds number $R_m \ll 1$, with the Hartmann number $M \ll \sqrt{kR}$, because in this case disturbances are described, as previously, by the Orr-Sommerfeld equation (4).[11]

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Translated by I. Emin
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