ON THE AXIOMATIC CONSTRUCTION OF THE SCATTERING MATRIX
4. LAGRANGIAN FORM OF THE THEORY

B. V. MEDVEDEV

V. A. Steklov Mathematical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor May 31, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) 49, 1518-1525 (November, 1965)

It is shown that, at least formally, the axiomatic theory can be cast into Lagrangian form. Conditions are formulated which if imposed on the Lagrangian guarantee the causality and unitarity of the scattering matrix.

1. INTRODUCTION

IN the "axiomatic" approach to the construction of the scattering matrix, which has been intensely developed during the past few years, the existence of a Lagrangian is not assumed from the beginning. Thus the very interesting question arises whether such an approach to the construction of the theory is nevertheless equivalent to the Lagrangian approach, or will lead to some wider class of theories. In the form of the theory which is based on the set of axioms proposed by N. N. Bogolyubov, M. K. Polivanov and the author [1] 1), which has been investigated further by the present author [2-4] 2), it has been possible in III to solve this problem for the very special case of nonrenormalized theories without derivative couplings. It turned out that for such theories the axiomatic formulation could be, albeit formally, reduced to the Lagrangian form. The purpose of the present work is to extend this result to a more general case.

As in the analysis which was carried out in III, we proceed from the general relation

$$O^{H} = T_W(O^{out}S)S^{+} \tag{1}$$

introduced there between the operators in the Heisenberg picture and the corresponding operators in the out-picture. We shall not attempt however to cast this transformation into a unitary form (in general this seems to be impossible) and will investigate the transformation directly in its form (1).

1) This will be quoted below as PTDR.

2) These papers will be quoted below as I, II, III, respectively. We use without explanation the notations of these and preceding papers.

2. THE LAGRANGIAN REPRESENTATION OF THE SCATTERING MATRIX

We write the transformation (1) for the current operator, making the apparently natural assumption that this operator possesses an out-inverse-image (i.e., an out-operator mapped by (1) into the Heisenberg operator)

$$j(x) = i\frac{\delta S}{\delta\varphi(x)}S^{+} = T_W[j^{out}(x)S]S^{+}. \tag{2}$$

Multiplication with S from the right yields the extremely useful equation

$$\frac{\delta S}{\delta\varphi(x)} = -iT_W[j^{out}(x)S], \tag{3}$$

The importance of this relation stems from the fact that it contains neither the adjoint matrix S^{+} nor T-products. Taking the functional derivative of (2) with respect to $\varphi(y)$

$$\frac{\delta j(x)}{\delta\varphi(y)} = T_W\left[\frac{\delta j^{out}(x)}{\delta\varphi(y)}S\right]S^{+} + T_W\left[j^{out}(x)\frac{\delta S}{\delta\varphi(y)}\right]S^{+} - j(x)S^{(1)}(y),$$

and making use of (3) in order to evaluate the derivative of the S-matrix in the second term (one may omit the internal T-product sign), we have

$$\frac{\delta j(x)}{\delta\varphi(y)} = T_W\left[\frac{\delta j^{out}(x)}{\delta\varphi(y)}S\right]S^{+} - iT_W[j^{out}(x)j^{out}(y)S]S^{+} + ij(x)j(y). \tag{4}$$

Rewriting this equation with interchanged x and y, subtracting it from the initial equation and carrying over the term containing the commutator of the Heisenberg currents into the left hand side, we are led on the basis of the symmetry of the T-product to the relation

$$\frac{\delta j(x)}{\delta\varphi(y)}-\frac{\delta j(y)}{\delta\varphi(x)}-i[j(x),j(y)]_-$$
$$=T_W\left[\left(\frac{\delta j^{out}(x)}{\delta\varphi(y)}-\frac{\delta j^{out}(y)}{\delta\varphi(x)}\right)S\right]S^+, \tag{5}$$

The left hand side of this relation vanishes on the basis of the solvability (compatibility) condition (II, (21)) for the current operators.

We can thus formulate the following lemma:

Lemma 1. The solvability (compatibility) condition for the current operator (cf. II, (21)) implies the integrability condition

$$\frac{\delta j^{out}(x)}{\delta\varphi(y)}-\frac{\delta j^{out}(y)}{\delta\varphi(x)}=0 \tag{6}$$

for the out-current. The converse is also true.

We emphasize the fact that in this derivation no use has been made of the causality condition, i.e., we have operated within the broad Heisenberg scheme, as outlined by the general requirements of section I in Sec. 2 of PTDR.

As soon as the out-current satisfies the condition (6) one can assert that it can be represented as the functional derivative of some functional say $-\Sigma[\varphi]$:

$$j^{out}(x)=-\delta\Sigma/\delta\varphi(x). \tag{7}$$

Then Eq. (3) takes the form of an equation of motion for the scattering matrix

$$\frac{\delta S}{\delta\varphi(x)}=iT_W\left[\frac{\delta\Sigma}{\delta\varphi(x)}S\right], \tag{8}$$

with the obvious solution

$$S=T_W[e^{i\Sigma[\varphi]}]. \tag{9}$$

If we wish to write the functional in the form (which does not restrict the generality of the functional)

$$\Sigma[\varphi]=\int dy\,L(y;\,[\varphi]) \tag{10}$$

and call the operator L the Lagrangian, the traditional representation of the scattering matrix as a chronological (time-ordered) exponential is obtained:

$$S=T_W\left[\exp\left(i\int L(y;\,[\varphi])\,dy\right)\right]. \tag{11}$$

Thus a scattering matrix which satisfies only the general conditions (Sec. I) in Sec. 2 of PTDR can be represented as a chronological (time-ordered) exponential with some Lagrangian functional $L(y;[\varphi])$.

This result, which at first seems paradoxical, can be better understood if one does not lose sight of the fact that we make no assumptions whatsoever about the locality of the Lagrangian; in other words, the expression (11) includes also all possible nonlocal theories which are compatible with a Heisenberg treatment of the S-matrix. Moreover, the form of the transformation from the out-picture to the Heisenberg-picture, as already mentioned in III, does not automatically imply even conservation of hermiticity. Therefore we cannot even assert that the Lagrangian is hermitian.

3. CAUSALITY

Attempts to specialize the expression (11) to theories which satisfy the causality requirement, yield less exhaustive results. One can establish the following proposition.

Lemma 2. If the out-current satisfies the causality requirement

$$\delta j^{out}(x)\,/\,\delta\varphi(y)=0 \quad \text{for} \quad y\lesssim x, \tag{12}$$

then it also satisfies the locality condition

$$\delta j^{out}(x)\,/\,\delta\varphi(y)=0 \quad \text{for all} \quad y\neq x. \tag{13}$$

The proof follows immediately from the fact that the out-current satisfies the integrability condition (6).

Lemma 3. If the out-current satisfies the causality condition (12), then the Heisenberg-current satisfies the causality condition II, (20)

Proof. We write down the generalized Yang-Feldman relation (III, (10)) for the current operator

$$j(x)=T_W(j^{out}(x)S)S^+$$
$$=j^{out}(x)-i\int dy_1\,dz_1\frac{\delta j^{out}(x)}{\delta\varphi(y_1)}D^{adv}(y_1-z_1)S^{(1)}(z_1)$$
$$+\ldots+\frac{(-i)^s}{s!}\int(dy)_s(dz)_s\frac{\delta^s j^{out}(x)}{(\delta\varphi(y))_s}$$
$$\times[D^{adv}(y-z)]_s S^{(s)}((z)_s)+\ldots \tag{14}$$

and carry out in both sides of the equality the functional differentiation with respect to $\varphi(y)$ for $y \lesssim x$ provided by the causality condition II, (20). All the derivatives of the out-current vanish then, due to condition (12). According to Lemma 5 in I, the functional derivatives with respect to $\varphi(y)$ of the radiative operators $S^{(s)}$ can be nonvanishing only if $y \geq$ than at least one of the z_j, $1 \geq j \leq s$. But every z_j is necessarily smaller than the corresponding y_j, otherwise the function $D^{adv}(y_j - z_j)$ would vanish. Consequently only the derivatives of $S^{(s)}$ with respect to $\varphi(y)$ at the points $y \geq$ at least for one y_j $1 \leq j \leq s$ can contribute to the integral. However, the causality condition for the out-current (12) implies that the first factor in the integrand differs from zero only if all $y_j \geq y$, $1 \leq j \leq s$. Thus the functional derivatives of the radiative operators $S^{(s)}$ can contribute to the functional derivative of the current only if the differentiation is carried out in a point $y \geq 8$, which completes the proof of the lemma.

We did not succeed in proving the converse of Lemma 3. The difficulty consists in the fact that if (12) is not valid there appear two kinds of terms which violate the causality condition II, (20) for the Heisenberg current: terms which are obtained through differentiation of $j^{out}(x)$ in (2), and terms which come from the differentiation of the S-matrix. Indeed, it is easy to see that the functional differentiation of (14) can be brought to the form

$$\frac{\delta j(x)}{\delta\varphi(y)} = \frac{\delta T_W [j^{out}(x) S] S^+}{\delta\varphi(y)} = T_W\left[\frac{\delta j^{out}(x)}{\delta\varphi(y)} S\right] S^+ + \sum_{s=1}^{\infty} \frac{(-i)^s}{s!} \int (dy)_s (dz)_s \frac{\delta^s j^{out}(x)}{(\delta\varphi(y))_s} \times [D^{adv}(y-z)]_s \frac{\delta S^{(s)}((z)_s)}{\delta\varphi(y)}. \tag{15}$$

If (12) is not satisfied, then the terms which violate the causality condition can come from both terms in the right hand side of (15), terms which have completely different characters. It remains then only to show that these terms cannot cancel mutually. However such cancellations can in fact take place, as demonstrated by some well known examples of local theories with formally integrable Lagrangians.

4. HERMITICITY

Even more curious is the situation with hermiticity. As indicated, a direct check, based on the Yang-Feldman relations, shows that the transformation (1), or in particular (2), can in general map a hermitian operator into a non-hermitian one. Indeed, taking the hermitian adjoint of the Yang-Feldman relation (14), we obtain

$$j^+(x) = j^{+out}(x) + \sum_{s=1}^{\infty} \frac{i^s}{s!} \int (dy)_s (dz)_s S^{+(s)}((z)_s) \times [D^{adv}(y-z)]_s \frac{\delta^s j^{+out}(x)}{(\delta\varphi(y))_s}. \tag{16}$$

Assuming that the out-current is hermitian $j^{+out}(x) = j^{out}(x)$, and subtracting (16) from (14) we obtain the anti-hermitian part of the Heisenberg current in the form

$$j(x) - j^+(x) = \sum_{s=1}^{\infty} \frac{(-i)^s}{s!} \int (dy)_s (dz)_s [D^{adv}(y-z)]_s \times \left[\frac{\delta^s j^{out}(x)}{(\delta\varphi(y))_s} S^{(s)}((z)_s) - (-1)^s S^{+(s)}((z)_s) \frac{\delta^s j^{out}(x)}{(\delta\varphi(y))_s}\right]. \tag{17}$$

In order to transform the second term to a form closer to that of the first term, we note that the function $iD(x_1 - x_2)$ plays the role of a contraction if one changes the order of factors in the product of fields:

$$\varphi(x_2)\varphi(x_1) = \varphi(x_1)\varphi(x_2) + iD(x_1 - x_2). \tag{18}$$

At the same time, according to Wick's theorem we have the expansion of the inversely ordered product in terms of the directly ordered product of operators

$$O_2 O_1 = O_1 O_2 + \sum_{s=1}^{\infty} \frac{i^s}{s!} \int (dy)_s (dz)_s \frac{\delta^s O_1}{(\delta\varphi(y))_s} \times [D(y-z)]_s \frac{\delta^s O_2}{(\delta\varphi(z))_s} \tag{19}$$

(cf. the notes on III), where as usual it is implied that the operators O_1 and O_2 are represented as series of normal (Wick) products. Making use of (19) in order to rearrange the second term in (17) we obtain

$$S^{+(s)}((z)_s) \frac{\delta^s j^{out}(x)}{(\delta\varphi(y))_s} = \frac{\delta^s j^{out}(x)}{(\delta\varphi(y))_s} S^{+(s)}((z)_s) + \sum_{s'=1}^{\infty} \frac{i^{s'}}{s'!} \int (dy')_{s'} (dz')_{s'} \frac{\delta^{s+s'} j^{out}(x)}{(\delta\varphi(y))_s (\delta\varphi(y'))_{s'}} \times [D(y'-z')]_{s'} \frac{\delta^{s'} S^{+(s)}((z)_s)}{(\delta\varphi(z'))_{s'}}.$$

In substituting this expression into (17) it is convenient to regroup the sums in such a manner as to combine the terms with identical functional derivatives of the out-current:

$$j(x) - j^+(x) = \sum_{s=1}^{\infty} \frac{i^s}{s!} \int (dy)_s (dz)_s \frac{\delta^s j^{out}(x)}{(\delta\varphi(y))_s} \times \Big\{(-1)^s [D^{adv}(y-z)]_s S^{(s)}((z)_s) - \sum_{m=0}^{s-1} P\begin{pmatrix} y_1, \ldots, y_{s-m} \\ y_{s-m+1}, \ldots, y_s \end{pmatrix} [D^{adv}(y-z)]_{s-m} \times [D(y-z)]_m \frac{\delta^m S^{+(s-m)}((z)_{s-m})}{(\delta\varphi(z))_m}\Big\}. \tag{20}$$

It remains to get rid of the adjoint radiative operators $S^{+(n)}$ by making use of the unitarity condition

$$S^{(n)}(x_1,\ldots,x_n)+\sum_{m=1}^{n-1}P\left(\frac{(x)_m}{(x)_{n-m}}\right)S^{(m)}((x)_m)S^{+(n-m)}((x)_{n-m})$$
$$+S^{+(n)}((x)_n)=0 \qquad (21)$$

(for $n>0$). For this purpose it is convenient first to transform the sum of products of radiative operators S and S^+ into a sum of functional derivatives of S operators only. Further it is necessary to represent the commutation functions D as a difference of retarded and advanced functions. After these operations we obtain the following expression for the anti-hermitian part of the Heisenberg current

$$j(x)-j^+(x)=\frac{i^2}{2!}\int(dy)_2(dz)_2\frac{\delta^2 j^{out}(x)}{\delta\varphi(y_1)\delta\varphi(y_2)}$$
$$\times P\left(\frac{y_1}{y_2}\right)D^{adv}(y_1-z_1)D^{ret}(y_2-z_2)\frac{\delta S^{(1)}(z_1)}{\delta\varphi(z_2)}$$
$$+\frac{i^3}{3!}\int(dy)_3(dz)_3\frac{\delta^3 j^{out}(x)}{(\delta\varphi(y))_3}$$
$$\times\Big\{P\left(\frac{y_1}{y_2,\,y_3}\right)D^{adv}(y_1-z_1)[D^{ret}(y-z)]_{3-1}\frac{\delta^2 S^{(1)}(z_1)}{\delta\varphi(z_2)\delta\varphi(z_3)}$$
$$-P\left(\frac{y_1,\,y_2}{y_3}\right)[D^{adv}(y-z)]_2$$
$$\times D^{ret}(y_3-z_3)\frac{\delta S^{(2)}(z_1,z_2)}{\delta\varphi(z_3)}\Big\}+\ldots, \qquad (22)$$

the construction principle of which is fairly obvious.

The first striking peculiarity is the disappearance of the term containing the first functional derivative of the out-current. It is this circumstance which guarantees the conservation of hermiticity when the transformation (1) is applied to the fields themselves.

The following terms are constructed uniformly. They contain all possible functional derivatives of the radiative operators connected with the arguments with respect to which the differentiations of the out-current are carried out in such a manner as to form chains of ϑ-functions for which the beginning and the end are in the points y in which the out-current is differentiated. This can be seen, remembering the causality condition II, (20) and, the retarded and advanced properties of D^{ret} and D^{adv} functions. Thus, in the first term we obtain the chain

$$\vartheta(y_2-z_2)\vartheta(z_2-z_1)\vartheta(z_1-y_1). \qquad (23)$$

It is clear that the locality condition (13) for the out-current is a necessary condition for the vanishing of such a chain. If this condition is not satisfied, the transformation (2) will obviously map a hermitian out-current into a non-hermitian Heisenberg current. The locality condition is however not sufficient to guarantee the conservation of hermiticity. It is easy to see that if one does not restrict the order of the derivatives in the quasilocal operators

$$L_\nu(x_1,\ldots,x_\nu)=\frac{\delta^{\nu-1}j^{out}(x_1)}{\delta\varphi(x_2)\ldots\delta\varphi(x_\nu)} \qquad (24)$$

with $\nu\geq 3$, a chain of type (23) can lead to a non-vanishing result even if the locality condition is satisfied.

Since the Heisenberg current must indeed be Hermitian (otherwise the scattering matrix would be nonunitary), this result means that a theory with derivative couplings requires in general, for its Lagrangian formulation, the introduction of a non-hermitian out-current. This means that a non-hermitian Lagrangian is required in order to preserve the unitarity of the S-matrix.

This circumstance had been known a long time ago[5] for nonlocal theories, in the framework of perturbation theory. Therefore it was natural to expect the same to happen for theories involving an infinite number of derivatives. Our result exhibits the fact that even theories with a finite number of derivatives of sufficiently high order are in this respect closer to nonlocal theories, than to theories with strict locality. As regards those theories which admit a hermitian Lagrangian, i.e., theories for which (22) vanishes, there are reasons to believe that the ensemble of such theories coincides exactly with the totality of renormalizable theories, notwithstanding the fact that up to the present we are not in possession of a complete proof of this circumstance.

5. CURRENT-LIKE OPERATORS

In I and II we established a representation of the radiative operators $S^{(n)}((x)_n)$ in terms of the current-like operators Λ_ν and have derived equations of motion for the latter quantities. We shall now try to relate these operators with the Lagrangian form of the theory. We shall assume that the locality condition has been imposed on the out-current and that the theory is such that a hermitian out-current implies a hermitian Heisenberg current.

We introduce the sequence of operators

$$L_1(x_1)\equiv j^{out}(x_1);\quad L_\nu(x_1,\ldots,x_\nu)=\frac{\delta^{\nu-1}j^{out}(x_1)}{\delta\varphi(x_2)\ldots\delta\varphi(x_\nu)}$$
$$=-\frac{\delta^\nu\Sigma}{(\delta\varphi(x))_\nu}. \qquad (25)$$

In agreement with the assumptions, these will be quasilocal, hermitian operators and the integrability condition implies that they are symmetric in all variables. Returning now to Eq. (3) and taking successive functional derivatives of it with respect to the out-fields, we obtain the following representation for the radiative operators $S^{(n)}$

$$S^{(n)}(x_1, \ldots, x_n) = T_W(S^{(n)out}(x_1, \ldots, x_n)S)S^+, \quad (26)$$

where

$$S^{(n)out}(x_1, \ldots, x_n) = (-i)^n T_W(L_1(x_1)\ldots L_1(x_n)) + \sum_{m, \nu_k} \frac{(-i)^m}{m!} P(x_1, \ldots, x_{\nu_1}|\ldots|\ldots x_n) \times T_W[L_{\nu_1}(x_1, \ldots, x_{\nu_1})\ldots L_{\nu_m}(\ldots, x_n)] - iL_n(x_1, \ldots, x_n). \quad (27)$$

The summation limits are determined by the conditions

$$2 \leqslant m \leqslant n-1, \quad \nu_1 + \nu_2 + \ldots \nu_m = n; \quad \nu_i \geqslant 1.$$

We can now turn to the quasi-Wick product introduced in III, Sec. 3 and which we now generalize to arbitrary Heisenberg operators which possess an out-inverse-image, by setting

$$T_{QW}(O_1{}^H \ldots O_m{}^H) = T_W(O_1{}^{out}\ldots O_m{}^{out}S)S^+. \quad (28)$$

It is clear that with this definition, one should be able to omit the internal T-product in inserting each term of (27) into (26) and that a quasi-Wick product of Heisenberg operators will appear:

$$L_\nu{}^H(x_1, \ldots, x_\nu) = T_W(L_\nu(x_1, \ldots, x_\nu)S)S^+ \quad (29)$$

where L_ν are quasilocal operators (we repeat that the Heisenberg transforms (29) of the quasilocal operators are by no means quasilocal!). Therefore we arrive at the representation of the radiative operators

$$S^{(n)}(x_1, \ldots, x_n) = (-i)^n T_{QW}(L_1{}^H(x_1)\ldots L_1{}^H(x_n)) + \sum_{m, \nu} \frac{(-i)^m}{m!} P(x_1, \ldots, x_{\nu_1}|\ldots|\ldots x_n) \times T_{QW}[L_{\nu_1}{}^H(x_1, \ldots, x_{\nu_1}) \ldots L_{\nu_m}{}^H(\ldots, x_n)] - iL_n{}^H(x_1, \ldots, x_n). \quad (30)$$

On the other hand, by means of functional differentiation of (29) taking into account (3), the unitarity and the definition of (29), we arrive at an equation of motion for the current-like operators L_ν^H:

$$\frac{\delta L_\nu{}^H(x_1, \ldots, x_\nu)}{\delta\varphi(y)} = -iT_{QW}[L_\nu{}^H(x_1, \ldots, x_\nu)L_1{}^H(y)] + iL_\nu{}^H(x_1, \ldots, x_\nu)L_1{}^H(y) + L^H_{\nu+1}(x_1, \ldots, x_\nu, y). \quad (31)$$

Comparing the representation (30) and the equation of motion (31) with I, (28) and II, (18), respectively, which were written in terms of the old current-like operators $\Lambda_\nu(x_1, \ldots, x_\nu)$ we see that these are completely identical in form and differ only by a simultaneous replacement of the current-like operators Λ_ν by new current-like operators L_ν^H and of all Dyson chronological products by quasi-Wick products. If Dyson and quasi-Wick products would coincide then the comparison of the two expressions would imply also that the corresponding current-like operators are pairwise equal. This is the situation we encountered in III, when we considered a nonrenormalized theory without derivative coupling.

In the presence of derivative couplings this equality no longer holds (this also happened for a special class of theories indicated in III, for which our axiomatics coincided with the Lehmann-Symanzik-Zimmermann axiomatics[6], since a characteristic feature of this class is that not all Dyson and quasi-Wick products coincide, but only a selected class). We then encounter a situation very similar to that occuring in perturbation theory with the traditional representation of the S-matrix of the form (11). This representation is written in the two well-known forms

$$S = T_W \exp\left(i\int L(x)dx\right) \quad (32.1)$$

or

$$S = T_D \exp\left(-i\int H(x)dx\right). \quad (32.2)$$

In the presence of derivative couplings $L(x) \neq -H(x)$, and the equality of the left hand sides of the two equations (32) is attained due to the fact that the first contains a quasi-Wick product and the second a Dyson T-product (cf. the detailed discussion of this in Sukhanov's papers [7,8]). Similarly, one might think that in our case too the current-like operators L_ν^H are expressed in terms of the Lagrangian, whereas the old radiative operators Λ_ν are closely related to the Hamiltonian.

In conclusion I should like to thank M. K. Polivanov with whom I had frequent conversations while carrying out this investigation and also I. I. Todorov, B. M. Stepanov, and A. D. Sukhanov for a discussion of the results obtained here. I should also like to thank the rector of the Winter School of 1964 in Dubna, A. N. Tavkhelidze for an invitation to lecture at this school, during the preparation for which the first version of this paper took shape.

[1] N. N. Bogolyubov, B. V. Medvedev and M. K. Polivanov, Voprosy teorii dispersionnykh sootnosheniĭ (Problem of the Theory of Dispersion Relations) Fizmatgiz, M. 1958.

[2] B. V. Medvedev, JETP 40, 826 (1961), Soviet Phys. JETP **13**, 580 (1961).

[3] B. V. Medvedev, JETP 47, 147 (1964), Soviet Phys. JETP **20**, 99 (1965).

[4] B. V. Medvedev, JETP 48, 1479 (1965), Soviet Phys. JETP **21**, 989 (1965).

[5] B. V. Medvedev, JETP **32**, 87 (1957), Soviet Phys. JETP **5**, 48 (1957).

[6] Lehmann, Symanzik and Zimmermann, Nuovo Cimento **1**, 205 (1955), **6**, 319 (1957).

[7] A. D. Sukhanov, JETP 41, 1915 (1961), Soviet Phys. JETP **14**, 1361.

[8] A. D. Sukhanov, JETP 47, 1303 (1964), Soviet Phys. JETP **20**, 881 (1965).

Translated by M. E. Mayer
194