NONUNIQUENESS OF THE SOLUTION OF THE SCATTERING PROBLEM

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A comparison is made between the dynamical, axiomatic, and dispersion methods as applied to the model of scattering of nonrelativistic particles with a point interaction. The number of solutions of the corresponding equations is determined, along with the analytic properties of the scattering amplitude and the reasons for the appearance of "extra" solutions. A short summary of the results is given in a table at the end of the article.

1. INTRODUCTION

WITH respect to the extent to which they employ unobservable quantities (matrix elements off the mass shell), existing trends in the local theory of the scattering of elementary particles can be conventionally classified into the following three groups. First there is the dynamical (Lagrangian) method, which in its essentials copies nonrelativistic quantum mechanics and gives a detailed space-time description of the scattering process. Next there is the axiomatic method, based on a definite system of axioms; one of them—the axiom of causality—involves departure from the mass shell. Finally, there is the dispersion method (method of the scattering matrix), which has been developed in recent years and deals only with observable quantities.

It is scarcely possible at present to make a reliable estimate of the depth of the difficulties and the comparative promise of these various methods. Indeed we cannot regard as clear even such most important questions as the degree of uniqueness of the solutions of the various equations (if they in fact are equations, and not relations), the existence of nontrivial solutions—i.e., solutions which describe actual scattering—and so on.

The present paper gives a comparison of the various methods from the point of view of their uniqueness. At the same time we ascertain the nature of the analytic restrictions that appear in each of these methods, the causes of the appearance of "extra" solutions, and so on. The treatment is given for the simplest model of the scattering of nonrelativistic particles with a point interaction, which admits of exact solution.[1)]

[1)]Other models can be treated in a similar way, for example the scalar model in the one-particle approximation.

Instead of using the usual asymptotic formulations (see the lectures in the collection [1]), it is more convenient to go over to equations which contain differentiation with respect to the charge, which in their simplest form have been used previously in nonlocal theory.[2] These equations are based on the introduction of a fictitious "intensity of interaction."[3,4] Not to speak of the comparative simplicity of the resulting equations, we can assert that this approach is the one best suited to the problem in question. This can be seen merely from the fact that the new dispersion relations that arise and the analytic properties of their solutions cannot be formulated without introducing derivatives of the scattering amplitude with respect to charge.

The derivation of the equations which are differential in the charge is given in Sec. 2; the detailed calculations are omitted, since they are formally the same as in other axiomatic schemes (see the lectures of Medvedev, of Polivanov, and of Faĭnberg in [1]). The most complicated question of the axiomatic method—the determination of the quasilocal terms—is considered in Sec. 3. Section 5 is devoted to the solution of the resulting equations, and Secs. 4 and 7 to the treatment of the same problem with the dynamical and dispersion methods. In Sec. 6 we investigate an additional solution which arises in the axiomatic method. A brief summary of the results is given in Sec. 8.

The reduced mass of the colliding particles is taken equal to unity. It is assumed that there are no bound states.

2. FORMULATION OF THE AXIOMATIC METHOD WITH DIFFERENTIATION WITH RESPECT TO THE MASS

Following the method expounded in the book of Bogolyubov and Shirkov,[3] we shall regard the scattering matrix S as a functional of the "intensity of interaction" $g(x)$. This function is purely auxiliary, and is replaced in the final equations by a number g, which has the meaning of a renormalized coupling constant (charge). The introduction of the function $g(x)$ allows us to formulate the causality condition and construct the perturbation-theory series.[3] Actually, as will be seen in what follows, in the framework of this approach we can also take the next step—derive "exact" equations which do not involve an expansion in powers of the coupling constant.

The foundation is the usual system of axioms: the correct invariance properties; unitarity of the scattering matrix; completeness and spectral character of the system of in states, in whose space the scattering matrix acts; stability of the vacuum and of one-particle states. Also included here [3] are the postulate that expresses the correspondence principle,

$$S \to 1 + i \int dx\, g(x) \mathcal{L}_0^{in}(x) + \ldots; \quad g(x) \to 0, \tag{1}$$

where $\mathcal{L}_0^{in}$ is the analog (divided by the charge) of the interaction Lagrangian in the in-representation,[2)] and the condition of microscopic causality

$$\delta\mathcal{L}_0(x) / \delta g(y) = 0, \quad x_0 < y_0, \quad (x-y)^2 < 0, \tag{2}$$

where the Hermitian operator $\mathcal{L}_0$ is defined by the expression

$$\mathcal{L}_0(x) = -iS^+ \delta S / \delta g(x). \tag{3}$$

Using the symmetry of $\delta^2 S/\delta g(x)\delta g(y)$ in x and y, we find by using (3) that

$$\delta\mathcal{L}_0(x) / \delta g(y) - \delta\mathcal{L}_0(y) / \delta g(x) = i[\mathcal{L}_0(x), \mathcal{L}_0(y)]$$

and on using (2) we get the equation

$$\delta\mathcal{L}_0(x) / \delta g(y) = i\theta(x-y)[\mathcal{L}_0(x), \mathcal{L}_0(y)] + \mathcal{L}_1(x, y). \tag{4}$$

Here $\mathcal{L}_1(x, y)$ is a set of quasilocal terms (abbreviated QLT), proportional to the function $\delta(x-y)$ and its derivatives of finite order. Using the symmetry of the second derivative of $\mathcal{L}_0$ and a condition obtained by repeated differentiation of (2), we arrive at the equation for $\mathcal{L}_1$:

$$\frac{\delta\mathcal{L}_1(x, y)}{\delta g(z)} = i\theta(x-z)[\mathcal{L}_1(x, y), \mathcal{L}_0(z)] + \mathcal{L}_2(x, y, z),$$

where $\mathcal{L}_2$ is the new QLT. There are equations of similar structure for the higher QLT.

These equations can be combined by introducing the generating functional

$$\Lambda(x; g') = \sum_{n=0}^{\infty} \frac{1}{n!} \int dy_1 \ldots dy_n g'(y_1) \ldots g'(y_n) \mathcal{L}_n(x, y_1, \ldots, y_n),$$

where $g'(x)$ is a new functional argument. Then

$$\frac{\delta\Lambda(x; g')}{\delta g(y)} = i\theta(x-y)[\Lambda(x; g'), \Lambda(x; 0)] + \frac{\delta\Lambda(x; g')}{\delta g'(y)}. \tag{4'}$$

The causality condition (2) and the conditions obtained from it by repeated differentiation require that the right member of (4′) vanish outside the light cone. Allowing for the quasilocality of $\mathcal{L}_n$ we get

$$[\Lambda(x; g'), \Lambda(y, 0)] = 0, \quad (x-y)^2 < 0. \tag{5}$$

To get rid of the functional arguments, we let $g(x)$ and $g'(x)$ approach constants g and g′, and take $\delta g(x)$ and $\delta g'(x)$ to mean variations of these constants. Using the general relation

$$\delta A = \int dx \delta g(x) B(x) \to \delta g \int dx B(x),$$

we get from the relations already given

$$\frac{dS}{dg} = iS \int dx \Lambda(x, 0), \tag{6}$$

$$\left(\frac{d}{dg} - \frac{d}{dg'}\right) \Lambda(x, g') = i \int dy \theta(x-y)[\Lambda(x, g'), \Lambda(y, 0)]. \tag{7}$$

We must add to these equations the initial conditions. Comparing (3) with (1), we get for $g \to 0$

$$S \to 1, \quad \mathcal{L}_0(x) \to \mathcal{L}_0^{in}(x). \tag{8}$$

For the QLT we let

$$\Lambda(x, g') \to \Lambda^{in}(x, g'), \tag{9}$$

where the operator Λ^{in} must be determined by both the renormalization conditions (including the conditions of stability of the vacuum and the one-particle states) and also the requirement that there must be no remaining ambiguity of the right member of (7) coming from multiplication of the discontinuous function by the commutator. We emphasize that the conditions (8) and (9) assume that there are no strong singularities of the quantities in question at the point g = 0; this sort of singularity, obviously characteristic of nonrenormalizable theories, will be considered separately. We note that this method, involving differentiations with respect to charge, is extremely convenient precisely for the investigation of the analytic properties of the scattering amplitude in the g plane.[5]

The operator Λ^{in} which appears in the condition (9) must have a quite definite structure, determined by the causality condition (5). This condition, together with (8) and (9), gives for g = 0

2)This quantity specializes the type of interaction, by specifying what fields appear in the interaction and in what way (three-particle, four-particle, etc.).

$$[\Lambda^{in}(x, g'),\ \mathcal{L}_0{}^{in}(y)] = 0 \qquad (x-y)^2 < 0.$$

Being interested in what follows in the case of a contact interaction, we choose $\mathcal{L}_0^{in}(x)$ in the form of a product of field operators (for definiteness, φ^{in}) taken at the point x. The operator $\mathcal{L}_0^{in}$ commutes with itself and with φ^{in} outside the light cone. Therefore all operators with which we have to deal in this section belong to a single equivalence class, and according to Borchers' theorem [6] commute with each other. In particular, for $(x - y)^2 < 0$ we have

$$[\Lambda^{in}(x, g'), \varphi^{in}(y)] = 0.$$

On the basis of a theorem of Bogolyubov and Vladimirov,[7] Faĭnberg [8] has shown that this type of condition reduces to the requirement that the operator $\delta\Lambda^{in}(x, g')/\delta\varphi^{in}(y)$ be quasilocal. In other words, all matrix elements of Λ^{in} between in states must be polynomials in the appropriate momenta. This restriction is extremely important for the elimination of arbitrariness in the QLT.

3. DETERMINATION OF THE QUASILOCAL TERMS

Let us proceed to the study of the simplest model—the scattering of two nonrelativistic zero-spin particles with a point interaction. Using translational invariance and considering the matrix element between in states in the two-particle channel in the center-of-mass system, we have

$$\langle m|\mathcal{L}_0{}^{in}(x)|m'\rangle = \exp[i(p_m - p_{m'})x]\mathcal{L}_0{}^{in}(\mathbf{k}, \mathbf{k}'), \quad (10)$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the momenta of the particles in the states m and m′. In accordance with the assumption of a point interaction we set $\mathcal{L}_0^{in}(\mathbf{k}, \mathbf{k}') = \text{const}$. It is convenient to take

$$\mathcal{L}_0{}^{in}(\mathbf{k}, \mathbf{k}') = 2\pi. \quad (11)$$

In what follows we shall apply a regularization procedure. In accordance with our assumption of a point interaction we shall use in the intermediate calculations, instead of the expression (11), the form

$$\mathcal{L}_0{}^{in}(\mathbf{k}, \mathbf{k}') = 2\pi\nu^*(k')\nu(k), \quad (12)$$

where ν is a function of k/L (L is the "cut-off" momentum) which decreases with increase of its argument and is equal to unity when the argument is zero. After the calculations we take the limit $L \to \infty$.

In this section we shall determine the QLT for the model considered. Because there are no antiparticles the condition of stability is satisfied identically, which leads to a great simplification of the structure of the QLT. Rejecting solutions which are specific for the nonrelativistic problem and have no relativistic analog, we retain the condition (5), applying it now to the region $x_0 = y_0$, into which the exterior of the light cone degenerates. Accordingly the analysis made at the end of Sec. 2 remains valid, and the quanity $\Lambda^{in}(\mathbf{k}, \mathbf{k}')$—the matrix element of $\Lambda^{in}(x, g')$—must be regarded as a polynomial in $\mathbf{k}$ and $\mathbf{k}'$.

It can be shown that the choice of any other polynomial than a constant leads to the appearance of nonremovable divergences in the terms of the perturbation-theory series for the scattering amplitude.[3)] This can be verified easily by putting (6) and (7) in the momentum representation and integrating these equations (see also Sec. 4). The scattering amplitude will then contain an infinite number of progressively diverging combinations, and for their elimination we have at our disposal only a finite number of counterterms—the coefficients of the polynomial $\Lambda^{in}(\mathbf{k}, \mathbf{k}')$.

This result is unconvincing, since it is known [3] that QLT can be introduced into the interaction Lagrangian, which in the present case corresponds to a nonrenormalizable theory containing higher derivatives. On the other hand, the introduction of QLT which remove all divergences and differ from a constant contradicts condition (5) and essentially leads to a nonlocal theory.[9]

Accordingly, we must set $\Lambda^{in}(\mathbf{k}, \mathbf{k}') = \text{const}$. The fact that the operators $\mathcal{L}_0^{in}(\mathbf{k}, \mathbf{k}')$ and $\Lambda^{in}(\mathbf{k}, \mathbf{k}')$ are equal (apart from a coefficient), together with Eq. (7), leads to the analogous relation between the operators $\Lambda(x, g')$ and $\Lambda(x, 0) = \mathcal{L}_0(x)$. Thus we arrive at the final equations

$$d\mathcal{L}_0(x)/dg = i\int dy\,\theta(x-y)[\mathcal{L}_0(x), \mathcal{L}_0(y)] + \alpha\mathcal{L}_0(x), \quad (13)$$

$$dS/dg = iS\int dx\,\mathcal{L}_0(x), \quad (14)$$

where α is a constant determined by the condition that g is the renormalized charge.

In concluding this section we point out that the structure we have obtained for the operator Λ^{in} corresponds to there being no derivatives of δ functions in the QLT $\mathcal{L}_n(x, y, \ldots)$. Thus we get

[3)]In principle these divergences could vanish when the perturbation-theory series is summed. Then, however, it would be decidedly nonanalytic in the coupling constant, a case which is not considered in the present paper.

the restrictions on the QLT which are expressed in the principle of minimal singularity.[10]

4. THE SCATTERING PROBLEM IN THE DYNAMICAL METHOD

To derive the equations of the axiomatic method it sufficed to subject to the axioms of Sec. 2 the matrix $S[g(x)]$ with $g(x)$ infinitely close to the constant g. The dynamical method is based on a stronger requirement: the axioms in question must be obeyed by the scattering matrix with arbitrary $g(x)$.

Omitting the QLT in Eq. (4) (this corresponds to considering the unrenormalized theory), and introducing a new operator $u(x)$ by the condition $\mathscr{L}_0(x) = S^+T[u(x)S]$, we find from (3) and (4):

$$\delta S / \delta g(x) = iT(u(x)S), \qquad \delta u(x) / \delta g(y) = 0.$$

When we now let $g(x)$ approach not a constant [as in the derivation of (6) and (7)] but the step function $g_0\theta(t - \tilde{t})$,[4)] we find that $S[g(x)]$ goes over into the "half-way" matrix $S(t)$, which satisfies the equation

$$dS(t)/dt = ig_0 \int dx \mathscr{L}_0{}^{in}(x) S(t). \tag{15}$$

In the derivation we have used the unitarity of the S matrix and the condition (1); the tilde has been dropped.

Thus we have arrived at the well known equation of the dynamical method, which in view of the unitarity of $S(t)$ is equivalent to the usual Schrödinger equation. Before proceeding to its solution, we elucidate the physical meanings of the quantities that appear in the axiomatic equations (13), (14).

Writing (15) in the form

$$S(t) = T \exp\left[ig_0 \int dy \theta(x-y) \mathscr{L}_0{}^{in}(y) \right],$$

by differentiating with respect to g we get

$$dS(t)/dg = iS(t) \int dy \theta(x-y) \mathscr{L}_0(y),$$

$$\mathscr{L}_0(x) = \frac{dg_0}{dg} S^+(t) \mathscr{L}_0{}^{in}(x) S(t). \tag{16}$$

Fot $t \to \infty$ (16) goes over into (14). Differentiation of $\mathscr{L}_0$ with respect to g gives Eq. (13) with

$$\alpha = \frac{d^2 g_0}{dg^2} \Big/ \frac{dg_0}{dg}.$$

Thus the meaning of the quantities appearing in (13) and (14) is: $\mathscr{L}_0(x)$ is analogous to the renormalized interaction Lagrangian in the Heisenberg representation, and the constant α describes the charge renormalization.

Let us rewrite (15) in terms of matrix elements in the two-particle channel; in nonrelativistic theory all of the intermediate states will also be of the two-particle type. Translational invariance gives

$$\langle m | S(t) | n \rangle = \delta_{mn} + 2\pi \frac{\exp[i(E_m - E_n)t]}{E_m - E_n - i\varepsilon} \times \delta^3(p_m - p_n) f(m, n). \tag{17}$$

For $t \to \infty$ the last term becomes $2\pi i \delta^4(p_m - p_n) f(m, n)$, and for $p_m = p_n$ the quantity $f(m, n)$ becomes the usual scattering amplitude. Substitution of (17) and (10) in (15) gives[5)]:

$$f(\mathbf{k}, \mathbf{k}') = \frac{g_0}{2\pi} \mathscr{L}_0{}^{in}(\mathbf{k}, \mathbf{k}') + g_0 \int d^3p \frac{\mathscr{L}_0{}^{in}(\mathbf{k}, \mathbf{p}) f(\mathbf{p}, \mathbf{k}')}{p^2 - k'^2 - i\varepsilon}.$$

Using Eq. (12), we must look for the solution of the equation just found in the form

$$f(\mathbf{k}, \mathbf{k}') = \nu(k) \varphi(k') \nu^*(k').$$

This gives

$$f(k) = g_0 |\nu(k)|^2 \left[1 - 2\pi g_0 \int \frac{d^3p |\nu(p)|^2}{p^2 - k^2 - i\varepsilon} \right]^{-1}. \tag{18}$$

In the limit of a point interaction this quantity vanishes, in accordance with the fact that the amplitude for nonresonance scattering of slow particles goes to zero along with the range of the forces. A nonzero result can, however, be achieved by an appropriate renormalization of the charges (in this connection see [11]).

We introduce the renormalized charge g with the condition $g = f(0)$. Then[6)]

$$g = g_0 \left[1 - g_0 \frac{2}{\pi} \int_0^\infty dp |\nu(p)|^2 \right]^{-1}$$

and

$$f(k) = g |\nu(k)|^2 \left[1 - gk^2 \frac{2}{\pi} \int_0^\infty \frac{dp |\nu(p)|^2}{p^2 - k^2 - i\varepsilon} \right]^{-1}. \tag{18'}$$

For the point interaction

$$f(k) = g(1 - ikg)^{-1}. \tag{18''}$$

4)g_0 is the "bare" charge.

5)An analogous equation, in which we must replace $L_0{}^{in}$ by Λ^{in} and replace g′ by g in the latter quantity, is obtained with the use of QLT of arbitrary form. Iteration of this equation in fact gives the divergences referred to at the end of Sec. 3.

6)In the limit of a point interaction there is something like the "nullification of charge" of relativistic theory. Here, however, the value of g_0 that leads to a given g can be chosen without coming into contradiction with the general requirements.

The scattering amplitude is then independent of the angles and satisfies the unitarity condition $\operatorname{Im} f(k) = k|f(k)|^2$.

In conclusion we enumerate the analytic properties of the amplitude (18″) [in the case of a smeared-out interaction the quantity that has these properties is $f(k)/|\nu(k)|^2$]. We introduce the function of a complex variable $f(z)$, which goes over into the scattering amplitude for $z \to k^2 + i\epsilon$. This function has no singularities on the first sheet of the complex plane ($\operatorname{Im} z^{1/2} > 0$) with a cut along the positive semiaxis,[7)] and along with its derivative with respect to the charge it has no zeroes and is bounded at infinity on both sheets. The absence of zeroes of the scattering amplitude has been discussed earlier in connection with the problem of nonuniqueness of the Low equation.[13]

5. THE SCATTERING PROBLEM IN THE AXIOMATIC METHOD

Proceeding to the solution of Eqs. (13) and (14), we take into account translational invariance and the fact that the scattering amplitude is independent of the angles. Equation (14) takes the form

$$\frac{df(k)}{dg} = \frac{\mathcal{L}_0(k, k)}{2\pi}(1 + 2ikf(k)),$$

from which there follows the usual representation

$$f(k) = k^{-1}e^{i\delta(k)} \sin \delta(k),$$

where the scattering phase shift is

$$\delta(k) = \frac{k}{2\pi}\int_0^g dg\mathcal{L}_0(k, k). \tag{19}$$

Equation (13) reduces to the form

$$\frac{d\mathcal{L}_0(k, k')}{dg} = \frac{1}{\pi^2}\int_0^\infty dpp^2\left[\frac{1}{p^2 - k^2 - i\varepsilon} + \frac{1}{p^2 - k'^2 + i\varepsilon}\right] \times \mathcal{L}_0(k, p)\mathcal{L}_0(p, k') + \alpha\mathcal{L}_0(k, k'). \tag{20}$$

It can be shown that a symbolic solution of this equation can be written in the form

$$\mathcal{L}_0(k, k') \infty \hat{O}_k + \mathcal{L}_0^{in}(k, k')\hat{O}_{k'},$$

where $\hat{O}_k$ is an operator which operates on the indicated variable. By using (8), (11), and (12) we convince ourselves that the solution (20) must be of the "separated" form $\mathcal{L}_0(k, k') = 2\pi\chi^*(k')\chi(k)$. Substitution of this expression in (20) and comparison with (19) gives, in the case of the point interaction,

$$\chi(k) = \left[\frac{\delta'(k)}{k}\right]^{1/2} e^{-i\delta(k)} \tag{21}$$

(the prime denotes differentiation with respect to g). For the phase shift we get the equation

$$\frac{\delta''(k)}{\delta'(k)} = \frac{4}{\pi}k^2\int_0^\infty \frac{dp}{p(p^2 - k^2)}\delta'(p), \tag{22}$$

where we have used the expression

$$\alpha = -\frac{4}{\pi}\int \frac{dp}{p}\delta'(p),$$

which assures that $f(0) = g$.

We rewrite the last equation in the form

$$\frac{\delta''(k)}{\delta'(k)} + 2ik\delta'(k) = \frac{4}{\pi}k^2\int_0^\infty \frac{dp}{p(p^2 - k^2 - i\varepsilon)}\delta'(p).$$

The left member here is equal to $f''(k)/f'(k)$, and when we use the relation $f|_{g\to 0} = g$ we find

$$\ln f'(k) = \frac{4}{\pi}k^2\int_0^\infty \frac{dp}{p(p^2 - k^2 - i\varepsilon)}\delta(p).$$

The right member is a function which has no singularities when continued on the first sheet. Therefore the scattering amplitude also must have no singularities, and its derivative with respect to charge also can have no zeroes. It is important that the condition that the scattering amplitude itself has no zeroes does not arise in the axiomatic method. This fact is closely connected with the occurrence of an additional solution, which we shall proceed to elucidate.[8)]

The dynamical solution of (18) is distinguished by the fact that the quantity

$$k / \delta'(k) = (1 + 2ikf(k)) / f'(k)$$

is a quadratic form in g. Direct calculation of the quantity $[k/\delta'(k)]''$ shows that any solution of (22) has this same property. Setting

$$k / \delta'(k) = \alpha(k) + gk^2\beta(k) + g^2k^2\gamma(k), \tag{23}$$

we rewrite (22) in the form

$$\frac{\beta + 2g\gamma}{\alpha + gk^2\beta + g^2k^2\gamma} = \frac{4}{\pi}\int_0^\infty \frac{dp}{k^2 - p^2}\frac{1}{\alpha + gp^2\beta + g^2p^2\gamma}. \tag{24}$$

From the boundary condition for the phase at $g = 0$ we find that in the general case $\alpha(k) = |\nu(k)|^{-2}$. Setting $g = 0$ in (24), we have

[7)]It is assumed that there are no bound states, i.e., $g \int dp\, |\nu(p)|^2 > -1$ (see also [12]). For the point interaction we get the condition $g > 0$, which is assumed to be the case.

[8)]In the case of a "smeared out" interaction (21) acquires the factor $\nu(k)/|\nu(k)|$, (22) is of the same form as here, and the analytic properties stated belong to the quantity $f(k)/|\nu(k)|^2$.

$$\beta(k)=\frac{4}{\pi|\nu(k)|^2}\int_0^\infty\frac{dp}{k^2-p^2}|\nu(p)|^2.$$

As for γ, the determination of this quantity depends essentially on whether we consider the point interaction directly or as the limit of a "smeared out" interaction.

In the latter case, using the good convergence of the integral and comparing the terms linear in g in (24), we find

$$\gamma(k)=\frac{k^2}{2}|\nu(k)|^2\beta^2(k)+\frac{2}{\pi|\nu(k)|^2}\int_0\frac{dp\,p^2}{p^2-k^2}|\nu(p)|^4\beta(p).$$

Substitution of the quantities we have found in (24) shows that (23) is a solution for all g. Thus in the case of the "smeared out" interaction there is a unique expression for the amplitude. Naturally it agrees with the dynamical solution (18).

The situation is different if we at once set $|\nu(k)|^2=1$. We then have $\beta=0$, and Eq. (24) can be written in the form

$$\frac{\gamma(k)}{1+g^2k^2\gamma(k)}=\frac{2}{\pi}\int_0\frac{dx}{k^2g^2-x^2}\left[\frac{1}{1+x^2\gamma(x/g)}-1\right].$$

Setting $g=0$, we can convince ourselves that $\gamma(k)=\text{const}$, and finally get $\gamma=\gamma^{1/2}$. Therefore in this case there are two and only two solutions. One corresponds to $\gamma=1$ and is the same as the dynamical solution; the other, for which $\gamma=0$, leads to the following expression for the scattering amplitude:

$$f(k)=k^{-1}e^{ikg}\sin kg. \tag{25}$$

It is important to emphasize that this solution exists only if the interaction is precisely a point interaction. Any regularization leads to the loss of this solution. In this connection the question of the stability of the solution (25) is important, but it cannot be solved without a treatment of many-particle scattering.

6. INVESTIGATION OF THE ADDITIONAL SOLUTION

The additional solution (25) which arises in the axiomatic method differs from the dynamical solution in that the scattering amplitude has zeroes and increases without limit in the second sheet. It is interesting that both solutions are analytic in the coupling constant and can be expanded in two different perturbation-theory series; only the first two terms of these series coincide. Iteration of Eq. (22) leads to the solution (25) for the point interaction and to (18″) when one introduces an arbitrary regularization, for example when the integral is simply "cut off."

According to the results of Sec. 4 the absence of the solution (25) in the dynamical method means that one can not find a matrix $S(t)$ which corresponds to this solution and satisfies all of the axioms. In fact, the explicit solution of Eq. (16) can be put in the form (17) with

$$f(k,k')=\left[\frac{\delta'(k')}{\delta'(k)}\right]^{1/2}e^{i\delta(k')}\frac{\sin\delta(k)}{k}.$$

In particular, for the additional solution

$$f(k,k')=k^{-1}e^{ik'g}\sin kg.$$

A direct check shows that the matrix $S(t)$ is not unitary: the matrix elements of the operators S^+-1 and $S^{-1}-1$ are respectively proportional to $\sin k'g$ and to $\tan kg\cos k'g$.[9] Of course, in the limit $t\to\infty$, when a $\delta(k^2-k'^2)$ appears, we get a unitary matrix. The fact that $S(t)$ is not unitary makes it impossible to go over to the equation (15), and consequently to the Schrödinger equation.

At first glance this assertion is in contradiction with (16), since the matrix elements of $\mathscr{L}_0$ for the system of in states are Hermitian for both solutions of Eq. (22). In fact, calculating the derivative dSS^+/dg by means of (16), we arrive at an operator whose matrix element is proportional to the quantity

$$\left[\int_0^\infty dp\int_0^\infty dp'-\int_0^\infty dp'\int_0^\infty dp\right]$$

$$\times\frac{p\delta'(p)p'\delta'(p')}{(k^2-p^2-i\varepsilon)(p^2-p'^2-i\varepsilon)(p'^2-k'^2-i\varepsilon)}$$

and only with some effort can one believe that this quantity is not zero.

Actually the order of the integrations can be changed only if the integrals converge uniformly. In our case a necessary condition for this is that the quantity $\delta'(k)/k$ decrease with increasing k. This condition is satisfied only for the dynamical solution; for it $\delta'(k)/k=(1+k^2g^2)^{-1}$, while for the additional solution this quantity is equal to unity.

In concluding this section we emphasize that we are not dealing here with the question as to whether the additional solution can compete physically with the dynamical solution in the framework of the nonrelativistic scattering prob-

[9] Here the tangent is to be taken in the sense of the principal value at points where it becomes infinite.

lem. This problem plays only the role of a mathematical model which can give some indications of the possibilities which can be expected in the relativistic case.

7. THE SCATTERING PROBLEM IN THE DISPERSION METHOD

In the dispersion method one considers only the physical scattering matrix with $g(x) = \text{const}$, which obeys the axioms of Sec. 2. Furthermore the causality axiom is replaced by the requirement that the scattering amplitude be analytic on the physical sheet. Combination of this requirement with the unitarity condition (Sec. 4) leads to an equation of the Low type for the scattering amplitude [in the case of a "smeared out" interaction one must introduce a factor $|\nu(k)|^{-2}$ in the left member of the equation and in the integrand]:

$$f(k) = g + \frac{2}{\pi} k^2 \int_0^\infty \frac{dp\,|f(p)|^2}{p^2 - k^2 - i\varepsilon}.$$

The general solution of this equation is of the form[14]

$$f(k) = g[1 - ikg + R(k)]^{-1} \tag{26}$$

and contains the well known ambiguity involved in the function $R(k)$. This function is of the following form:

$$R(k) = \sum_n \frac{r_n k^2}{k^2 - a_n},$$

where r_n and a_n are arbitrary positive quantities.

The choices of the function R corresponding to the two solutions of Eq. (22) are: for the dynamical solution $R = 0$, and for the additional solution

$$R(k) = kg \cot kg - 1 = 2k^2g^2 \sum_{n=1}^{\infty} \frac{1}{k^2g^2 - \pi^2 n^2}.$$

An R function of this type, which has a point of condensation of poles at infinity and increase of the amplitude on one sheet of the complex plane, gives a scattering amplitude which cannot be reduced to virtual or Breit-Wigner levels (in this connection see [15]).

For all other solutions of the Low equation there exists no matrix $S[g(x)]$ with $g(x) \neq \text{const}$ which can satisfy the axioms of Sec. 2 and go over into the given expression for $g(x) = g$.[10)] Using (19) and (26), we can obtain

$$\mathcal{L}_0(k, k) = \frac{1 + R(k) - g\,\partial R(k)/\partial g}{(1 + R(k))^2 + k^2g^2}. \tag{27}$$

It can be seen from this that failure to satisfy the condition $R = 0$ for $g = 0$ leads to a contradiction with the condition (1). Furthermore it can be shown that if the derivative d^nR/dg^n for $g = 0$ is not of polynomial form, then there will be a contradiction with the causality condition for $x_0 = y_0$ (see end of Sec. 2), although there is no loss of the analytic properties of the amplitude.

8. CONCLUSION

The results of the study of the degree of uniqueness of the three methods in scattering theory which we have considered, and of the character of the analytic properties of the amplitude and the causes of the appearance of "extra" solutions, are shown in the table. Passage from the dynamical to the axiomatic and then to the dispersion method is actually accompanied by successive relaxations of the requirements imposed on the scattering matrix (fourth column). There is a corresponding decrease in the severity of the restrictions on the analytic properties of the scattering amplitude on the physical sheet (third column). At the same time there is an increase of the number of solutions of the corresponding equations (second column).

Method	Number of solutions	Analytic properties	Axioms are satisfied by matrix S[g(x)]*
Dynamical	1	f regular, $df/dg \neq 0$, $f \neq 0$	with arbitrary $g(x)$
Axiomatic	2**	f regular, $df/dg \neq 0$	with $g(x) = g + \delta g(x)$, $\delta g(x) \ll g$
Dispersion	∞	f regular	with $g(x) = g = \text{const}$

*In the dispersion method the causality axiom is replaced by the requirement of analyticity.
**The second solution disappears on regularization.

In conclusion we make some remarks regarding the scattering of elementary particles, on the assumption that the results we have obtained here are to some extent of general validity and are not restricted to the nonrelativistic problem.

1. It can be expected that the usual dispersion relations do not exhaust the set of relations that are consequences of the axioms of quantum field theory. We have in mind nonlinear relations and relations with derivatives with respect to charge, of the type of Eq. (22), which lead to restrictions on the zeroes of the scattering amplitude and of its derivatives with respect to charge.

10)If we dispense with the requirement that the in states be a complete system, then a dynamical model can be found to correspond to every solution of the Low equation.[16].

2. Regularization, as usually performed in field theory, may turn out to be an inadmissible procedure which leads to loss of additional solutions of the type (25). On the other hand, if such solutions turn out to be unstable, regularization may be a way of assuring a needed uniqueness of the equations of the axiomatic method.

3. Unlike the dynamical solution, the additional solution is not of the pole type. This offers a hope for the existence of an axiomatic expression for the Green's function which does not contain a false pole. Because regularization is impossible this solution can appear only in the framework of an unrenormalized theory, and for it there is no problem of nullification of the charge.

4. Since the dynamical and additional solutions differ from each other already at rather low energies, if it is the additional solution that is confirmed by experiment this would be a proof of the completely local character of field theory up to arbitrarily high energies.

These conclusions are of course justified only to the extent to which the results obtained here are confirmed by relativistic calculations. Such calculations are now being made.

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