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QUANTUM ELECTRODYNAMICS WITH TWO FERMIONS

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It is shown that if two fermions are taken into account and the Lagrangian formalism of quantum electrodynamics is partially dropped it becomes possible to formulate the equations in such a way that no infinite expressions appear in them. Both fermions are physical and an equation is obtained for the ratio of their masses.

1. FORMULATION OF THE PROBLEM

CONTEMPORARY quantum electrodynamics has all the features of a completed physical theory: it is capable of uniquely calculating in advance any observable effect with a prescribed degree of accuracy if it is not necessary to take into account any interactions other than electromagnetic ones. The divergent expressions of quantum electrodynamics are partly neglected if they appear in gauge non-invariant terms, and are partly eliminated by means of mass and charge renormalization. Although the operation of renormalization is carried out in accordance with a well defined prescription there remains one unsatisfactory detail: the product of a divergent integral by the fine structure constant is regarded as a small quantity. To justify this it is said that in the "future" theory this integral will become finite. But it is difficult to expect that this future theory will differ essentially from the present one. It is not very probable that the same set of phenomena will be described by two theories quite unlike one another. It is much more probable that in the apparatus of the theory an alteration is needed only in some detail which, moreover, at the present level of experimental knowledge does not exert any noticeable influence on any observable effect.

The search for such a detail is facilitated by the following considerations. Landau, Abrikosov, and Khalatnikov[1] have found that the most essential divergence is in the self-energy part of the photon. If it is calculated exactly and not in the first order with respect to the fine structure constant, then the physical charge of the electron will tend to zero. Apparently, the fundamental difficulties of quantum electrodynamics are rooted specifically in the photon Green's function. Johnson, Baker, and Willey[2] have shown that if the photon Green's function at high energies is assumed to have the same form as in the case of a free photon in zero order approximation then the electron will have a finite mass of purely electromagnetic origin starting with zero "bare" mass. In reference [2] the authors did not set as their goal the elimination of the divergence from the photon Green's function, assuming that contributions to it must be made by all the charged particles that exist in nature. But in this way it was assumed that in principle quantum electrodynamics cannot be constructed as an internally complete and consistent theory because of the fact that the majority of charged particles are nuclear-active.

But this is to some extent hard to reconcile with the fact that quantum electrodynamics in its modern form is almost complete as a physical theory. Therefore, one can attempt to alter the apparatus of quantum electrodynamics in such a way as to be able to formulate it without introducing fields of nonelectric nature. But one must necessarily take into account the fact that there exist two purely electromagnetic particles—the electron and the μ meson. In present electrodynamics they are taken into account in a purely formal manner: for example, in the calculation of the self-energy part of the photon the divergent integrals from the electron and the μ-meson loops are added.

The alteration of the theory proposed in the present paper consists of subtracting the integrals arising from the two loops instead of adding them. As a result all the divergent expressions in any order approximation with respect to the fine structure constant will cancel and only finite operations will remain associated with the elimination of the gauge noninvariant part and charge renormalization. As has been shown in reference[2] all the expressions of quantum electrodynamics can be made finite together with the photon Green's function.

The proposed procedure of calculation apparently cannot be obtained from a Lagrangian formalism. One must assume that the vacuum loop differs essentially from an open fermion line, and that, in particular, quantization rules do not apply to it. At the same time the former procedure remains in force for open lines, so that there are no reasons to expect any violations of unitarity which occur when fictitious regularizing particles are introduced into the Lagrangian. At the same time the theory remains, as before, local and, naturally, relativistically invariant.

In the proposed scheme the basic equations must be assumed to be the Dyson equations for the Green's functions for the fermion and for the quantum. In this procedure both fermions must appear in the theory in a symmetric manner, but in such a way that all possible fermion loops on a photon line would mutually cancel their infinite parts. If one wishes, the quantum can be regarded in this case as an "alloy" of the two fermions, a totality of all possible loops with divergences which cancel in pairs. As is well known, there exists no exact equation for the vertex part. Therefore, any result is obtained only in some order of approximation with respect to the fine structure constant. We shall consider here expansions up to terms quadratic in the fine structure constant.

How can one justify the proposed pragmatic prescription? First of all, the whole existing apparatus of the theory remains practically unaltered. In principle one could detect a difference between the results of the proposed scheme and the usual ones only in the second order radiation corrections to the magnetic moment of the μ meson. The μ-meson loop gives a contribution to the corresponding corrections for the electron which is smaller than the electron contribution in the ratio of the square of the masses of the two particles, while in the case of the μ meson the contribution of the electron loop is larger. But here the correction due to the polarization of the vacuum is itself very small, so that the accuracy of contemporary experiments is insufficient to use as an argument for or against the proposed modification of the theory.

The following remark is due to L. A. Kruzhkova. The sign of the electron loop must in any case be regarded as known from the Lamb shift for the electron. Therefore, the sign of the meson loop must be the opposite one in order that the infinite parts would cancel.

2. THE FERMION GREEN'S FUNCTION

The equation for the fermion Green's function has the form (cf., [3], (44.4) p. 475 Russian text, p. 602 English Transl.):

$$G(p) = S(p) + \frac{e^2}{(2\pi)^4} S(p) \int \gamma_\mu G(p-k) \Gamma_\nu (p, p-k; k) \times G(p) D_{\mu\nu}(k) d^4k, \tag{1}$$

where $S(p) = -(\hat{p} - im)^{-1}$, $D_{\mu\nu}$ is the photon Green's function, Γ_ν is the "exact" vertex part. Equation (1) on division by G(p)S(p) is brought to the form

$$G^{-1}(p) = S^{-1}(p) - \frac{e^2}{(2\pi)^4} \int \gamma_\mu G(p-k) \Gamma_\nu (p, p-k; k) \times D_{\mu\nu}(k) d^4k. \tag{2}$$

But in future, following reference [2], we shall assume that mass is of purely electromagnetic origin. Then instead of S(p) one must simply write $(\hat{p})^{-1}$.

We obtain the equation for the mass in the zero order approximation by assuming

$$\Gamma_\nu = \gamma_\nu, \quad D_{\mu\nu} = \frac{1}{ik^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right).$$

Here the Landau gauge has been chosen. We seek the function G(p) in the form

$$G(p) = [-X(p^2)\hat{p} + iY(p^2)]^{-1}. \tag{3}$$

Then, separating (2) into its even and odd parts we obtain

$$X(p^2)\hat{p} = \hat{p} + \frac{e^2}{(2\pi)^4} \int \frac{X(p'^2)[\hat{p}' + 2\hat{n}(p'n)]}{X^2(p'^2)p'^2 + Y^2(p'^2)} \frac{d^4p'}{i(p-p')^2}, \tag{4}$$

$$Y(p^2) = \frac{3e^2}{(2\pi)^4} \int \frac{Y(p'^2)}{X^2(p'^2)p'^2 + Y^2(p'^2)} \frac{d^4p'}{i(p-p')^2}, \tag{5}$$

$$n \equiv \frac{p-p'}{|p-p'|} \equiv \frac{k}{|k|}.$$

In order to carry out the integration over the angles it is convenient to go over from the Galilean to the four-dimensional Euclidean space by writing for the element of volume $4\pi p'^3 dp' \sin^2\chi d\chi$. In doing this one must assume that p has only a fourth component. Then after some elementary calculations it turns out that the integral appearing in (4) vanishes, as has been noted already in reference [2]. We obtain $X(p^2) = 1$. This result can also be used in the approximation to equation (5) of the next order with respect to e^2. Integrating in the same manner over the angles in (5) we reduce it to the form

$$f(x) = \lambda \left(\int_0^x \frac{x' f(x') dx'}{x[x' + f^2(x')]} + \int_x^\infty \frac{f(x') dx'}{x' + f^2(x')} \right). \tag{6}$$

Here $Y(z) = mf(z/m^2)$, $3e^2/16\pi^2 = \lambda$. In particular, for $x = 0$ we have

$$1 = \lambda \int_0^\infty \frac{f(x)\,dx}{x + f^2(x)}. \tag{7}$$

For further calculations we shall need only the condition $f(0) = 1$. The asymptotic form of the solution $f(x) \sim x^{-\lambda}$, obtained in [2], will be used only to prove the validity of some simplifications.

3. THE FIRST RADIATION CORRECTION TO THE PHOTON GREEN'S FUNCTION

We now utilize the proposed subtraction procedure to obtain the first radiation correction to the photon Green's function. For it we have the expression (cf., [3])

$$D_{\mu\nu} = D_{\mu\nu}{}^0 + D_{\mu\sigma}{}^0 \Pi_{\sigma\tau}{}^{(2)} D_{\tau\nu}{}^0. \tag{8}$$

Here we have

$$D_{\mu\nu}{}^0 = \frac{1}{ik^2}(\delta_{\mu\nu} - n_\mu n_\nu),$$

while $\Pi_{\sigma\tau}^{(2)}$ is given by the well-known expression:

$$\Pi_{\sigma\tau}^{(2)} = \frac{e^2}{(2\pi)^4}\,\mathrm{Sp}\int \gamma_\sigma G(p)\gamma_\tau G(p-k)\,d^4p. \tag{9}$$

Here we shall take G to stand not for the fermion Green's function in zero order approximation, i.e., not for S(p), but for the exact function (3). However, in this function it is sufficient to set $X = 1$ and $Y = Y(0) = m$. But, although G appears to refer to a free fermion, one should not assume that $\hat{p} = im$. In future we shall set $p = 0$, so that $G = (im)^{-1} = (iY(0))^{-1}$, where the mass is entirely of electromagnetic origin.

Thus, we do not have to take into account any electron self-energy corrections in the Dyson equation, nor any μ-meson corrections. We assume that they have been taken into account exactly in G(p) which is obtained from a homogeneous integral equation. Therefore, in addition to the corrections to $D_{\mu\nu}$ we will have to take into account only the corrections to the vertex part.

It can be shown that if the Landau gauge has been chosen for $D^0_{\mu\nu}$ then $D_{\mu\nu}$ will also be purely transverse if the integral (9) is convergent. Thus, in the first approximation we obtain

$$D_{\mu\nu} = D_{\mu\nu}{}^0(1 + \delta\varphi), \tag{10}$$

where $\delta\varphi$ is a difference of expressions of the form

$$\varphi = \frac{e^2}{(2\pi)^4}\frac{4}{ik^2}\int \frac{m^2 + (p, p-k) - {}^2/_3[p^2 - (np)^2]}{(p^2+m^2)[(p-k)^2+m^2]}\,d^4p, \tag{11}$$

formed for two fermions. For the divergent integrals appearing in (11) we take the well-known expressions (reference [3] p. 503 Russian text, p. 640 English translation). It is essential to note that the coefficients of these integrals do not contain any masses so that the infinite terms cancel. After this we obtain

$$\delta\varphi = \frac{e^2}{(2\pi)^4}\frac{8\pi^2}{k^2}\int_0^1 x(1-x)\,dx\left\{k^2 \ln\frac{m_2{}^2 + k^2x(1-x)}{m_1{}^2 + k^2x(1-x)} - \frac{m_1{}^2}{2} + \frac{m_2{}^2}{2}\right\}. \tag{12}$$

Because of the convergence of the integral over p we have taken only $G(0) = (im)^{-1}$.

As usual, the expression for $\delta\varphi$ should be set equal to zero for $k = 0$. In order to do this we must neglect the terms $(m_2^2 - m_1^2)/2$ and $k^2 \ln(m_2^2/m_1^2)$. The latter corresponds to a finite gauge. However, we note that the final form of the equation for the ratio of the masses does not depend on this procedure since the terms appearing in the charge renormalization cancel in the equation for the ratio of the masses. But we shall write $\delta\varphi$ already in gauge invariant form:

$$\delta\varphi' = \frac{e^2}{(2\pi)^4}\int_0 x(1-x)\,dx \ln\frac{1 + x(1-x)k^2/m_2{}^2}{1 + x(1-x)k^2/m_1{}^2}. \tag{13}$$

4. RADIATION CORRECTION TO THE VERTEX PART

If one adds to the vertex part in the Dyson equation (2) the first radiation correction, the following expression is obtained

$$\frac{e^4}{(2\pi)^8}\int \gamma_\mu G(p-k) D_{\alpha\beta}(q)\gamma_\alpha G(p-q-k)\gamma_\nu G(p-k) \times \gamma_\beta D_{\mu\nu}(k)\,d^4q\,d^4k.$$

Here specifically from the vertex part we have the integral

$$\Gamma_\nu{}^{(2)}(p, p-k; k) = \frac{e^2}{(2\pi)^4}\int D_{\alpha\beta}(q)\gamma_\alpha G(p-q-k) \times \gamma_\nu G(p-q)\gamma_\beta\,d^4q. \tag{14}$$

It does not vanish for $k = 0$, and, therefore, in order to obtain the correct gauge we must substitute into the equation

$$\Gamma_\nu{}^{(2)}(p, p-k; k) - \Gamma_\nu{}^{(2)}(p, p, 0).$$

Moreover, we shall be interested only in the value of the correction for $p = 0$. Terms of longitudinal nature, i.e., k_ν and q_ν which after integration again yield k_ν will be omitted immediately. Then the integrand in (14) can be reduced to the following form:

$$-(2\hat{q}\gamma_\nu\hat{q} + q^2\gamma_\nu) - \hat{q}\hat{k}\gamma_\nu + \frac{im}{q^2}\hat{q}\hat{k}\gamma_\nu\hat{q} + 2m^2\gamma_\nu + \frac{m^2}{q^2}\hat{q}\gamma_\nu\hat{q}.$$

In the final expression we shall need only those terms which contain a logarithmic divergence. This divergence is liquidated if one substitutes the correct asymptotic expression for $Y(p^2)$. But we shall derive an equation for the ratio of the fermion masses in which we shall need only $Y(0)$ just as in the formula for the radiation correction to the photon Green's function. Those terms in the vertex part which do not contain a logarithmic divergence are cancelled out of the expression for the vertex part when we go over from the fermion masses themselves to the ratio of the masses. It should be noted that the masses themselves cannot be obtained from the equations because the "bare" mass was assumed to be equal to zero, and it is not possible to construct from electrodynamic quantities a combination having the dimension of mass.

Substituting in place of $G(p')$ the simple expression $-(\hat{p}' - im)^{-1}$ we bring the integral involving the corrected vertex part in the second order with respect to e^2 to the form

$$\frac{-\pi^2e^4}{(2\pi)^8}\int\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\gamma_\mu\frac{im-\hat{k}}{k^2(k^2+m^2)}\gamma_\nu \times\left\{\int_0^1 dx_1\int_0^{x_1}dx_2\left[\frac{x_1(1-x_1)k^2}{k^2x_1(1-x_1)+m^2(1-x_2)} - \frac{im\hat{k}x_2(1-x_1)^2k^2}{[k^2x_1(1-x_1)+m^2(1-x_2)]^2}\right]+\frac{3}{2}\right\}d^4k. \quad (15)$$

The integrals over x_1 and over x_2 have appeared, as usual, as a result of the transformation of the denominators according to Feynman. The last term in the figure brackets of (15) arose as a result of the subtraction of $\Gamma_\nu^{(2)}(0, 0; 0)$. Finally the contribution of the radiation correction to the vertex part reduces to the expression

$$-\frac{3\pi^4 im}{(2\pi)^8}\frac{3}{2}\ln\frac{L^2}{m^2}. \quad (16)$$

5. EQUATION FOR THE RATIO OF THE MASSES

In the equations for the Green's functions of the two fermions it is convenient to go over to the dimensionless variables $|p|/m_1$ and $|p|/m_2$, and then to divide respectively by m_1 and m_2. In transforming to the coordinate system in which $p = 0$ we obtain equations in which only $f(k^2/m_{1,2}^2)$ appears (the quantity a in the argument of the logarithm comes from nondivergent terms):

$$1=\frac{3e^2}{(2\pi)^4}\int\frac{d^4k}{m_{1,2}^4}\frac{f(k^2/m_{1,2}^2)}{k^2/m_{1,2}^2+f^2(k^2/m_{1,2}^2)}\frac{1}{ik^2/m_{1,2}^2} \times(1+\delta\varphi_{1,2}')-\frac{3e^4}{(2\pi)^8}\frac{3}{2}\ln\frac{L^2}{am_{1,2}^2}. \quad (17)$$

Because the integrals in (17) are convergent it is possible to subtract one equation from the other one and in so doing to replace $f(k^2/m_{1,2}^2)$ by unity: the convergence is preserved for the difference of the integrals. The infinite terms in the vertex parts cancel; because of this we have in advance set $f = 1$ in the vertex parts.

In the integration it is again convenient to go over to Euclidian space. The singularities of the logarithmic term appearing in $\delta\varphi_{1,2}'$ do not hinder this since they remain to one side when the contour of integration is deformed in the k_4 plane. Then for the ratio of the masses we obtain the equation

$$\frac{3e^4}{(2\pi)^8}8\pi^2\int_0^\infty\frac{f(z)\,dz}{1+z}\int_0^1 x(1-x)\,dx \times\ln\frac{[1+zx(1-x)]^2}{[1+zm_1^2m_2^{-2}x(1-x)][1+zm_2^2m_1^{-2}x(1-x)]} = \frac{3e^4\pi^2}{(2\pi)^8}\frac{3}{2}\ln\frac{m_2^2}{m_1^2}, \quad (18)$$

where $f(z)$ in the numerator must be replaced by unity. We assume the m_1 mass to be the smaller one and we replace the ratio m_1^2/m_2^2 by μ. We finally obtain

$$\int_0^1 x(1-x)\,dx\int_0^\infty\frac{dz}{1+z} \times\ln\frac{[1+zx(1-x)]^2}{[1+\mu zx(1-x)][1+\mu^{-1}zx(1-x)]}=\frac{3}{16}\ln\mu. \quad (19)$$

This equation has a nontrivial solution $\mu \neq 1$. Indeed, it is satisfied for $\mu = 1$. But the derivative of the right hand side with respect to $\ln\mu$ is finite, while the derivative of the left hand side vanishes. Consequently, near $\mu = 1$ the right hand side rises more steeply. For very small values of μ the left hand side behaves like $-\ln^2\mu$, so that there must necessarily be another root of equation (19) in addition to unity. We immediately set $\mu \ll 1$, which will be justified by the subsequent calculation. Making use of the fact that the integral appearing in (19) is convergent it can be represented in the following form:

$$\int_0^\infty\frac{dz}{1+z}\ln\frac{[1+zx(1-x)]^2}{[1+\mu zx(1-x)][1+\mu^{-1}zx(1-x)]} = \int_0^\infty\left\{\frac{2[1-x(1-x)]}{e^u-1+x(1-x)}-\frac{1+\mu x(1-x)}{e^u-1+\mu x(1-x)} - \frac{1+\mu^{-1}x(1-x)}{e^u-1+\mu^{-1}x(1-x)}\right\}u\,du. \quad (20)$$

The first term on the right hand side integrated over x is simply a number independent of μ. It is equal to 0.359. In the second term we must neglect μ and it gives $\pi^2/6$. In the third term we neglect unity in comparison with $\mu^{-1}x(1 - x)$. Then the integral takes on the form which is encountered in Fermi statistics:

$$\int_0^\infty \frac{u\,du}{\exp\{u - \ln[\mu^{-1}x(1-x)]\}+1} \cong \frac{1}{2}\ln^2\frac{x(1-x)}{\mu} + \frac{\pi^2}{6}, \quad (21)$$

if μ is a sufficiently small number. Substituting all this into (20) we find $\ln\mu = -4.40$ or $m_2/m_1 = 9$. Since the equation referred not to μ but to its logarithm, the disagreement with the correct value $m_2/m_1 = 206$ is not excessively great. One can assume that it arose as a result of an insufficiently rapid convergence of the approximation with respect to e^2 in which the calculations were carried out. Another order of approximation with respect to e^2 can probably be obtained, but a significant improvement of the theory requires an entirely different approach.

Equations in a higher order approximation with respect to the fine structure constant could contain within themselves some possibility of determining this constant itself. Indeed, in the present paper we were forced simply to neglect the gauge noninvariant term $m_2^2/2 - m_1^2/2$. In a more exact calculation this will be a function of the fine structure constant and of the ratio of the masses which should be set equal to zero in order to make the theory invariant. Moreover, there will also be an equation of type (19) which will also contain the ratio of the masses and the fine structure constant. Therefore, in principle, both these quantities could be determined.

In conclusion I express my gratitude to L. A. Kruzhkova for her aid in calculations and for the discussion of results.

APPENDIX

THE KÄLLEN-LEHMANN THEOREM FOR THE REGULARIZED PHOTON GREEN'S FUNCTION

As is well known the photon Green's function must satisfy the relation

$$G(k) = Z\left[D^c + \int_0^\infty \rho(M^2)\Delta_M{}^c(k)\,dM^2\right],$$

$$D^c = -i(k^2 - i\varepsilon)^{-1}, \quad \Delta_M{}^c = -i(k^2 + M^2 - i\varepsilon)^{-1}, \quad (A.1)$$

where $\rho(M^2)$ is a positive definite function. The relation (A.1) follows from general requirements of unitarity.

We verify that (A.1) is satisfied for the photon Green's function regularized by our method. We first show this for the function obtained in the same order of approximation by means of the usual renormalization procedure. The relation (A.1) can be rewritten in this case in the form:

$$\frac{\alpha}{4\pi k^2}\int_{-1}^{1}(1-\eta^2)\ln\left[1+\frac{1}{4}\frac{k^2}{m^2}(1-\eta^2)\right]d\eta = \int_0^\infty \frac{\rho_m(M^2)\,dM^2}{k^2+M^2}. \quad (A.2)$$

From here the function $\rho_m(M^2)$ is determined by the well-known formula (reference [4], formula (17.85c))

$$\rho_m^{(2)} = \frac{e^2}{12\pi^2}\frac{1}{M^2}\left(1+\frac{2m^2}{M^2}\right)\left(1-\frac{4m^2}{M^2}\right)^{1/2}\theta(M-4m^2),$$

$$\theta(x) = \begin{cases}1, & x>0\\ 0, & x<0\end{cases}. \quad (A.3)$$

But then, as can be seen from (A.3), the difference $\rho = \rho_{m1} - \rho_{m2}$ is also essentially positive, and, as required, the value of ρ depending on the larger mass has been subtracted.

This Appendix was written at the suggestion of I. S. Shapiro to whom I am sincerely grateful.

[1] L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, DAN SSSR **95**, 497, 733, 1117 (1954).

[2] K. Johnson, M. Baker and R. Willey, Phys. Rev. **136**, B1111 (1964).

[3] A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya elektrodinamika (Quantum Electrodynamics), 2nd edition, Fizmatgiz, 1959. English Transl. Interscience, New York, 1965.

[4] S. Schweber, Introduction to Relativistic Quantum Field Theory (Russ. Transl. IIL, 1963, p. 631.)

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