PION-NUCLEON INTERACTION IN THE AXIOMATIC APPROACH

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Charge-symmetric interaction between π mesons and nucleons is considered within the framework of a new axiomatic approach.[1] A relativistically covariant formulation of the principle of minimal singularity is presented. Together with other familiar principles the latter enables us to obtain a closed system of equations for the elements of the S-matrix subject to leaving the mass shell only with respect to a single momentum variable. The arbitrariness in the theory is reduced to two undefined real constants corresponding to the π-N-vertex and to the 4-point meson diagram. A perturbation theory solution of the equations yields a renormalized series.

INTRODUCTION

In a recent paper by Faĭnberg[1] a new axiomatic approach was proposed for the construction of a causal and unitary S-matrix. The advantage of the new approach compared to the previously known ones[2,4] consists, first, of a minimal departure from the mass shell, which leaves a tremendous degree of arbitrariness in the construction of the Green's functions, and second, of the elimination from the equations for the r-functions of indefinite quasilocal terms due, in particular, to the use of the invariant properties of the r-functions.[5] Finally, within the framework of perturbation theory the solutions of the new axiomatic equations are determined up to a fixed number of constants (for comparison cf. [4,6]). In the course of this a class of renormalizable theories is singled out.

The object of the present paper is to extend the new method to the case of the interaction of π mesons with nucleons.

In future we intend to obtain approximate equations for π-N scattering in the domain of low energies, where, apparently, the neglect of other strongly interacting particles is justified. These equations will be equations of the Low type with additional terms expressing the π-π interaction. In carrying out the indicated program it is proposed to alter the system of the basic axioms of Faĭnberg[1] by making it approach the system of Bogolyubov et al.[3] and by adding to it the principle of minimal singularity of quasilocal operators. The latter enables us to retain the principal accomplishment of the new method—a constructive elimination of quasilocal terms. The advantage of the proposed alteration consists of an explicit relativistic covariance of the formulation of the principle of minimal singularity.[1)]

1. FORMULATION OF THE BASIC AXIOMS

We consider the charge-symmetric interaction of π mesons with nucleons, assuming that there are no bound states. Up to the condition of causality the formulation of the basic postulates may be found in [3] (cf. also [7]). The initial asymptotic fields are the in-fields which satisfy the corresponding free equations of motion and free commutation relations. In this connection only the accepted notation is used.

The meson field is given by

$$\varphi_\alpha(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d\mathbf{k}}{2\omega_k}(a_\alpha^+(\mathbf{k})e^{ikx}+a_\alpha^-(\mathbf{k})e^{-ikx}),$$

$$k^0=\omega_k=(\mathbf{k}^2+\mu^2)^{1/2}. \tag{1.1}$$

The nucleon field is given by

$$\psi_\tau(x)=\frac{1}{(2\pi)^{3/2}}\int d\mathbf{p}\frac{\sqrt{2m}}{2E_p}\sum_{r=1,2}\{u^{(+)r}(\mathbf{p})b_\tau^{+r}(\mathbf{p})e^{ipx}$$
$$+u^{(-)r}(\mathbf{p})b_\tau^{-r}(\mathbf{p})e^{-ipx}\};$$

$$\bar\psi_\tau(x)=\frac{1}{(2\pi)^{3/2}}\int d\mathbf{p}\frac{\sqrt{2m}}{2E_p}\sum_{r=1,2}\{\bar u^{(+)r}(\mathbf{p})\overset{*}{b}_\tau^{+r}(\mathbf{p})e^{ipx}$$
$$+\bar u^{(-)r}(\mathbf{p})b_\tau^{-r}(\mathbf{p})e^{-ipx}\};$$

$$\bar\psi_\tau(x)=(\psi_\tau(x))^+\gamma_0,\qquad p_0=E_p=(p^2+m^2)^{1/2}, \tag{1.2}$$

1) It is proposed to publish the details in a separate preprint.

r is the sign of the component of the spin along the momentum, $\alpha = 1, 2, 3$; $\tau = 1, 2$ are the isotopic spin indices, for the sake of brevity the symbol in and the spinor indices are omitted;

$$[a_\alpha^-(\mathbf{k}),\ a_{\alpha'}^+(\mathbf{k}')]_- = \delta_{\alpha\alpha'} 2\omega_k \delta(\mathbf{k}-\mathbf{k}'),$$
$$\{b_\tau^{-r}(\mathbf{p}),\ \overset{*}{b}_{\tau'}^{+r'}(\mathbf{p}')\}_+ = \{\overset{*}{b}_\tau^{-r}(\mathbf{p}),\ b_{\tau'}^{+r'}(\mathbf{p}')\}_+ = \delta_{rr'}\delta_{\tau\tau'}2E_p\delta(\mathbf{p}-\mathbf{p}'). \quad (1.3)$$

The spinor amplitudes satisfy the conditions of normalization and orthogonality

$$\bar{u}^{(\pm)r}(\mathbf{p})\, u^{(\mp)r'}(\mathbf{p}) = \pm\delta_{rr'}, \qquad \bar{u}^{(\pm)r}(\mathbf{p})\, u^{(\pm)r'}(\mathbf{p}) = 0. \quad (1.4)$$

The existence of a unitary S-matrix is postulated. The theory is required to be separately C- and P-invariant. CPT-invariance is a consequence of locality and of Jost's theorem (cf., below). Passage outside the mass shell in the S-matrix is performed subject to the conditions of unitarity and of the necessary symmetries. It is achieved, for example, by the addition to the in-fields of appropriate classical terms with respect to which functional differentiation is carried out. The symbol of the variational derivative with respect to the in-field means that after the corresponding functional differentiation the classical additional terms are made to approach zero.

Current operators are introduced

$$j_\alpha(x) = iS^+\frac{\delta S}{\delta\varphi_\alpha(x)},$$
$$\eta_\tau(x) = iS^+\frac{\delta S}{\delta\bar{\psi}_\tau(x)} \qquad \bar{\eta}_\tau(x) = iS^+\frac{\delta S}{\delta\psi_\tau(x)} \quad (1.5)$$

(the derivative with respect to $\bar{\psi}_\tau(x)$ is a left derivative, and with respect to $\psi_\tau(x)$ is a right derivative). According to the conditions of passage outside the mass shell the currents satisfy the conditions of hermitian conjugation

$$j_\alpha^+(x) = j_\alpha(x), \quad \eta_\tau^+(x)\gamma_0 = \eta_\tau(x) \quad (1.6)$$

and have definite transformation properties. The causality conditions

$$\frac{\delta j_\alpha(x)}{\delta\varphi_\beta(y)} = \frac{\delta j_\alpha(x)}{\delta\bar{\psi}_\tau(y)} = \frac{\delta j_\alpha(x)}{\delta\psi_\tau(y)} = 0, \qquad x \lesssim y, \quad (1.7)$$

are imposed on the current operators and the same conditions are imposed on the currents $\eta_\tau(x)$, $\bar{\eta}_\tau(x)$. In contrast to reference [3] the causality conditions are required to be satisfied in going outside the mass shell only with respect to two variables (cf., however, Sec. 4).

The solution of the conditions of causality and unitarity is given by the equations

$$\frac{\delta j_\alpha(x)}{\delta\varphi_\beta(y)} = -i\theta(x^0-y^0)[j_\alpha(x), j_\beta(y)]_- + \Lambda_{\alpha\beta}(x,y);$$
$$\frac{\delta j_\alpha(x)}{\delta\bar{\psi}_\tau(y)} = -i\theta(x^0-y^0)[j_\alpha(x), \eta_\tau(y)]_- + \Lambda_{\alpha,\tau}(x,y);$$
$$\frac{\delta j_\alpha(x)}{\delta\psi_\tau(y)} = -i\theta(x^0-y^0)[j_\alpha(x), \bar{\eta}_\tau(y)]_- + \Lambda_\alpha^\tau(x,y) \quad (1.8)$$

plus the analogous equations for the currents $\eta_\tau(x)$, $\bar{\eta}_\tau(x)$; $\Lambda_{\alpha\beta}(x,y)$, $\Lambda_{\alpha,\tau}(x,y)$, $\Lambda_\alpha^{\bar\tau}(x,y)$ etc. are the so-called quasilocal operators which satisfy definite symmetry conditions:

$$\Lambda_{\alpha\beta}(x,y) = \Lambda_{\beta\alpha}(y,x), \qquad \Lambda_{\alpha,\tau}(x,y) = \Lambda_{\tau,\alpha}(y,x) \quad (1.9)$$

etc. From (1.8) and (1.9) follows the local commutativity of the Heisenberg field operators (defined in the usual manner, cf. (1.12) and (4.1)). Consequently, in accordance with Jost's theorem, the theory is CPT-invariant.

We now formulate the principle of minimal singularity of the quasilocal operators (PMS) which replaces the principle of minimal singularity of simultaneous commutators of Heisenberg fields and currents in the case of Faĭnberg.[1] In particular, we consider a class of theories for which the singularity of the quasilocal operators is not higher than a δ-function:

$$\Lambda_{\alpha\beta}(x,y) = \Lambda_{\alpha\beta}(x)\,\delta(x-y);$$
$$\Lambda_{\alpha,\tau}(x,y) = \Lambda_{\alpha,\tau}(x)\,\delta(x-y) \quad (1.10)$$

etc., where, in accordance with (1.9), we have

$$\Lambda_{\alpha\beta}(x) = \Lambda_{\beta\alpha}(x); \qquad \Lambda_{\alpha,\tau}(x) = \Lambda_{\tau,\alpha}(x) \quad (1.11)$$

etc. In the case of a self-acting scalar field for a Heisenberg operator defined in the usual manner

$$\varphi^\Gamma(x) = \varphi(x) - \int \Delta^{ret}(x-x', m)\, j(x')\, dx' \quad (1.12)$$

it follows from the PMS that

$$[\varphi^\Gamma(0,\mathbf{x}), j(0)]_- = 0$$

which is the fundamental postulate of Faĭnberg.[1]

The principles formulated above enable us to construct equations for the matrix elements of the currents, i.e., for the S-matrix with passage outside the mass shell with respect to only one variable, by effectively eliminating the unknown quasilocal operators, and to analyze the degree of arbitrariness in the theory which is related to the number of so-called invariant charges.

2. ELIMINATION OF QUASILOCAL OPERATORS. EQUATIONS FOR THE MATRIX ELEMENTS OF THE CURRENTS

Equations are written for the matrix elements of the currents with respect to the in-states of the form

$$\langle \mathbf{k},\ldots;\ \mathbf{p}_-r_-,\ldots;\ \mathbf{p}_+r_+,\ldots|\, j(0)\,|\mathbf{p}_+'r_+',\ldots;\ \mathbf{p}_-'r_-',\ldots;\ \mathbf{k}',\ldots\rangle, \quad (2.1)$$

when the states on the right hand and the left hand sides do not contain identical particles with the same momenta—the so-called r-functions corresponding to connected diagrams. The subscripts plus (minus) correspond to nucleons (antinucleons). The isotopic spin index is temporarily omitted.

The matrix elements (2.1) can always be represented as matrix elements of a commutator (anticommutator) of the given current with an arbitrary creation or annihilation operator from the state vectors (we shall say that the corresponding momentum has been commuted). Utilizing the representation of such a commutator (anticommutator) in terms of the variational derivative with respect to the in-field, Eq. (1.8), and the PMS and inserting into the retarded commutator the complete system of the in-states, we obtain equations for the matrix elements of the currents containing undefined matrix elements of the quasilocal operators, the so-called quasilocal terms. For the matrix elements of the current j(0) we have, for example,

$$\langle \mathbf{k},\ldots;\mathbf{p}_-r_-,\ldots;\mathbf{p}_+,r_+,\ldots|j(0)|\mathbf{p}_+'r_+',\ldots;$$
$$\mathbf{p}_-'r_-',\ldots;\mathbf{k}',\ldots\rangle$$
$$\equiv \langle \mathbf{k},l|j(0)|m\rangle = \langle l|[a^-(\mathbf{k})j(0)]_-|m\rangle$$
$$= -(2\pi)^{3/2}\sum_n \langle l|j(0)|n\rangle\langle n|j(0)|m\rangle$$
$$\times\left\{\frac{\delta(\mathbf{k}-\mathbf{p}_m+\mathbf{p}_n)}{\omega_k-P_m{}^0+P_n{}^0-i\varepsilon}-\frac{\delta(\mathbf{k}+\mathbf{p}_l-\mathbf{p}_n)}{\omega_k+P_l{}^0-P_n{}^0-i\varepsilon}\right\}$$
$$+\frac{1}{(2\pi)^{3/2}}\langle l|\Lambda_{jj}|m\rangle,$$
$$\langle \mathbf{k},\ldots;\mathbf{p}_-r_-,\ldots;\mathbf{p}_+,r_+,\ldots|j(0)|\mathbf{p}_+'r_+',\ldots;$$
$$\mathbf{p}_-'r_-',\ldots;\mathbf{k}',\ldots\rangle \equiv \langle l'|j(0)|\mathbf{p}_+'r_+',m'\rangle$$
$$= \langle l'|[j(0)\overset{*}{b}{}^{+r'}(\mathbf{p}')]|m'\rangle$$
$$= -(2\pi)^{3/2}\sqrt{2m}\sum_n\Big\{\langle l'|j(0)|n\rangle\langle n|\bar\eta(0)|m'\rangle$$
$$\times\frac{\delta(\mathbf{p}_n-\mathbf{p}'-\mathbf{p}_m)}{P_n{}^0-E_{p'}-P_m{}^0-i\varepsilon}-\langle l'|\bar\eta(0)|n\rangle\langle n|j(0)|m'\rangle$$
$$\times\frac{\delta(\mathbf{p}_l-\mathbf{p}'-\mathbf{p}_n)}{P_l{}^0-E_{p'}-P_n{}^0-i\varepsilon}\Big\}u^{(-)r'}(\mathbf{p}')$$
$$+\frac{\sqrt{2m}}{(2\pi)^{3/2}}\langle l'|\Lambda_{j\bar\eta}|m'\rangle u^{(-)r'}(\mathbf{p}') \qquad (2.2)$$

etc. plus analogous equations for the spinor curents.

An essential feature of (2.2) is the fact that the quasilocal term in virtue of the PMS does not depend on the commuted meson momentum, and depends on the commuted nucleon momentum only through the corresponding spinor terms of the state vector. Because of this it is possible to eliminate the quasilocal term from (2.2) by utilizing in addition the well-known covariant properties of the matrix elements of the currents. In order to achieve this one uses the method of expanding the matrix elements in terms of invariant amplitudes.

The matrix element of any current which is a matrix with respect to the polarization and spinor indices can be expanded in terms of well-known linearly independent matrices which have the necessary transformation properties—the so-called basis tensors constructed from the corresponding spinor amplitudes, γ-matrices, and the four-momenta of the problem (cf. [8]). The coefficients of such an expansion are the invariant amplitudes which depend only on the scalar products of the four-momenta of the problem.[5] Examples of such an expansion are given in Sec. 3. Because of the fact that the term with the retarded commutator (R-term) and the quasilocal term have the same transformation properties as the initial matrix element they give rise to analogous expansions.

We note that additional symmetries reduce the total number of nonvanishing invariant amplitudes. In virtue of the linear independence of the basis tensors one can in accordance with (2.2) equate the corresponding invariant amplitudes

$$r_\alpha(s_1,s_2,\ldots)=R_\alpha(s_1,s_2,\ldots)+K_\alpha(s_1,s_2,\ldots), \qquad (2.3)$$

where the subscript α indicates the basis tensor corresponding to the given invariant amplitude, s_1, s_2, ... are the invariant scalar products of the four-momenta of the problem. Here it is evident that the invariant amplitudes of the quasilocal term K_α do not depend on the commuted momentum.

For each given invariant amplitude one can write as many equations of the type (2.3) as there are particles contained in the left hand side and the right hand side state vectors appearing in the corresponding matrix element. This system of equations for each given invariant amplitude does not in principle differ in any way from the equations of the scalar case.[1] The quasilocal terms are eliminated by the already known procedure of differentiation with respect to the invariants or by the subtraction procedure equivalent to it (cf. [1, 5]).

For the lower matrix elements (up to the five-pronged term inclusive) we obtain a system of integrodifferential equations of the form

$$\frac{\partial r_\alpha(s_1,s_2,\ldots)}{\partial s_i}=\frac{\partial R_\alpha{}^{(x)}(s_1,s_2,\ldots)}{\partial s_i},$$

$$i=1,2,\ldots;\quad x=k,p,\ldots \qquad (2.4)$$

together with subsidiary conditions of the form

$$\frac{\partial R_\alpha^{(k)}(s_1, s_2, \ldots)}{\partial s_i} = \frac{\partial R_\alpha^{(p)}(s_1, s_2, \ldots)}{\partial s_i}; \qquad (2.5)$$

the indices k,p, ... indicate a commuted momentum. In going over to higher matrix elements the form of the equations and of the subsidiary conditions becomes more complicated.

One can also write for r_α difference or integral equations which in their form do not differ in any way from the equations of Kallosh and Faĭnberg.[5]

The presence of the isotopic spin index leads to the necessity of an additional expansion in terms of isotopically-invariant amplitudes, so that (2.4) and (2.5) should now be interpreted as the equations and the subsidiary conditions respectively for Lorentz- and isotopically-invariant amplitudes.

The formulation of the boundary conditions for any given invariant amplitude also in principle does not differ in any respect from the scalar case.

It is necessary to emphasize that the elimination of the quasilocal terms from the equations for the r-functions can be successfully carried out due to 1) the principle of minimal singularity of the quasilocal operators, 2) the consideration in the case of each matrix element of a system of equations which takes into account the causal properties of the currents with respect to all the particles participating in the process, and 3) the utilization of definite transformation properties of the matrix elements of the R-functions and quasilocal terms. Apparently, this precisely is the principal difference between the approach proposed here and that of reference [3].

3. FORMULATION OF BOUNDARY CONDITIONS, THE NUMBER OF INVARIANT CHARGES IN THE THEORY.

In virtue of (2.4), each invariant amplitude r_α, if it does not vanish at infinity with respect to the invariants (cf. [1]), is defined up to an arbitrary constant—the invariant charge which is the value of the invariant amplitude at some fixed point in the domain of variation of the invariants (just as in the scalar case,[1] the matrix elements of the same currents for the same particles but defined in different physical domains of variation of the invariants are regarded as different functions giving rise to independent invariant charges).

However, it is not possible to conclude that the number of invariant charges in the theory is equal to the number of invariant amplitudes which do not vanish at infinity with respect to the invariants. Firstly, not all the invariant charges turn out to be independent. Secondly, not each invariant amplitude which does not vanish at infinity with respect to the invariants gives rise to an invariant charge. The presence of the first restriction is associated with the existence in the theory of additional symmetries and of the PMS on the one hand, and with the correspondence of different matrix elements to the same process on the mass shell on the other hand. The presence of the second restriction is associated with the fact that in a given method of expansion in terms of the invariant amplitudes the quasilocal term contains, generally speaking, fewer terms in the expansion than the initial matrix element. This occurs when the meson momentum appearing explicitly in the basis tensors commutes with the current. For invariant amplitudes corresponding to such basis tensors Eq. (2.3) does not contain indefinite quasilocal terms and degenerates into an integral equation of the form

$$r_\alpha(s_1, s_2, \ldots) = R_\alpha(s_1, s_2, \ldots). \qquad (3.1)$$

Finally, we must take care that the introduction of definite boundary conditions and the specification of corresponding invariant charges should not contradict the initial equations (2.4). Specifically, already in the scalar case there exist strong suspicions that the matrix elements of higher processes beginning with the five-point term cannot have nonvanishing values at infinity with respect to the invariants. In the opposite case the system of initial equations turns out to be contradictory—the integrals in R_α over the intermediate states diverge in an unacceptable manner. But this means that the matrix elements of higher processes cannot give rise to independent invariant charges.

These considerations can be transferred in their entirety to the given case with the subsidiary condition that at infinity with respect to the invariants the elastic π–N and N–N amplitudes must also vanish. Within the framework of perturbation theory (assuming analyticity with respect to the invariant charges) this assertion can be proved rigorously. Specifically, in solving the initial system of equations by perturbation theory (with respect to the invariant charges) finite solutions are possible only when the independent invariant charges arise only from a π–N vertex and a meson four-point term.

Below it is assumed that at infinity with respect to the invariants the only nonvanishing matrix elements are those corresponding to the π–N vertex and the meson four-pronged term (the 3-

pion vertex is forbidden by G-parity) and an analysis is made of the number of independent invariant charges to which the corresponding invariant amplitudes give rise. For the invariant charges it is convenient to take the value of the invariant amplitudes at zero values of the momenta of the particles.

A. π–N Vertex

The matrix element

$$\langle \mathbf{p}_+ r\tau | j_\alpha(0) | \mathbf{p}_+' r'\tau'\rangle = (\sigma_\alpha)_\tau{}^{\tau'} \bar{u}^{(+)r}(\mathbf{p})\gamma_5 u^{(-)r'}(\mathbf{p}') r((p-p')^2) \tag{3.2}$$

gives rise correspondingly to one invariant charge

$$\lambda = r(0). \tag{3.3}$$

The matrix element

$$\langle \mathbf{k}\alpha | \eta_\tau(0) | \mathbf{p}_+' r'\tau'\rangle = (\sigma_\alpha)_\tau{}^{\tau'} [\gamma_5 u^{(-)r'}(\mathbf{p}') r_1((p-k)^2) + \gamma_5 \hat{k} u^{(-)r'}(\mathbf{p}') r_2((p-k)^2)] \tag{3.4}$$

contains two invariant amplitudes, but only r_1 can give rise to an invariant charge, since for r_2 one can write an equation of the form (3.1) containing no quasilocal terms by commuting the momentum $\mathbf{k}$ with $\eta_\tau(0)$. Let

$$\lambda_1 = r_1((m-\mu)^2). \tag{3.5}$$

It can be easily seen, however, that in virtue of the PMS and of the symmetry of quasilocal operators ($\Lambda_{\alpha,\tau} = \Lambda_{\tau,\alpha}$) there exists between the matrix elements (3.2) and (3.4) a relation which expresses their difference in terms of definite matrix elements:

$$\langle \mathbf{p}_+ r\tau | j_\alpha(0) | \mathbf{p}_+' r'\tau'\rangle - \sqrt{2m}\, \bar{u}^{(+)r}(\mathbf{p}) \langle \mathbf{k}\alpha | \eta_\tau(0) | \mathbf{p}_+' r'\tau'\rangle$$
$$= (2\pi)^{3/2}\sqrt{2m}\, \bar{u}^{(+)r}(\mathbf{p}) \sum_n \Big\{ \langle 0 | \eta_\tau(0) | n\rangle\langle n | j_\alpha(0) | \mathbf{p}_+' r'\tau'\rangle \times \Big[\frac{\delta(\mathbf{p}-\mathbf{p}_n)}{E_p - P_n^0 - i\varepsilon} + \frac{\delta(\mathbf{k}-\mathbf{p}'+\mathbf{p}_n)}{\omega_k - E_{p'} + P_n^0 - i\varepsilon}\Big]$$
$$- \langle 0 | j_\alpha(0) | n\rangle \langle n | \eta_\tau(0) | \mathbf{p}_+' r'\tau'\rangle \Big[\frac{\delta(\mathbf{p}-\mathbf{p}'+\mathbf{p}_n)}{E_p - E_{p'} + P_n^0 - i\varepsilon} + \frac{\delta(\mathbf{k}-\mathbf{p}_n)}{\omega_k - P_n^0 - i\varepsilon}\Big]\Big\}. \tag{3.6}$$

In virtue of (3.6) λ_1 is not an independent constant, but is a function of λ and of other possible invariant charges.

In a similar manner one can show with the aid of the PMS, the symmetry of quasilocal operators, the Hermitian properties of the currents and the invariance of the theory with respect to charge conjugation, that all the matrix elements corresponding to a π–N vertex cannot give rise to invariant charges independent of λ. Consequently, a π–N vertex can introduce into the theory only one invariant charge–the constant describing the coupling of π mesons with nucleons $\lambda_3 = \lambda$ (the fact that it is real follows from the Hermitian nature of the current $j_\alpha(0)$).

B. The Meson Four-Point Term

In exactly the same way as in the case of a π–N vertex we can show that the meson four-point term can introduce into the theory only one real invariant charge λ_4. Thus, within the framework of the axiomatic approach proposed in reference [1] the theory of the interaction of π mesons with nucleons turns out to be defined up to two real constants—invariant charges corresponding to a π–N vertex and a meson four-point term.

If we assume that the matrix elements of the currents are analytic with respect to the coupling constants λ_3 and λ_4 then the solution of the initial equations by means of perturbation theory leads to a renormalized series corresponding to the Lagrangian

$$L_{\text{int}}(x) = -\frac{(2\pi)^3\lambda_3}{2m} :\bar{\psi}_{\tau'}(x)\gamma_5\psi_\tau(x)(\sigma_\alpha)_\tau{}^{\tau'}\varphi_\alpha(x): - \frac{(2\pi)^{9/2}\lambda_4}{4!} :(\varphi_\alpha^2)^2:.$$

A separate paper is devoted to the solution of the axiomatic equations by means of perturbation theory.

4. CONCLUSION

1. First of all, in connection with the paper of Faĭnberg,[1] we investigate what are the simultaneous commutation relations for the Heisenberg fields and currents to which the principle of minimal singularity of quasilocal operators leads us. The Heisenberg field operators are defined in the usual manner

$$\varphi_\alpha^\Gamma(x) = \varphi_\alpha(x) - \int \Delta^{ret}(x-x',\mu) j_\alpha(x')\,dx',$$
$$\psi_\tau^\Gamma(x) = \psi_\tau(x) - \int S^{ret}(x-x',m)\eta_\tau(x')\,dx',$$
$$\bar{\psi}_\tau^\Gamma(x) = (\psi_\tau^\Gamma(x))^+\gamma_0. \tag{4.1}$$

Then we have

$$[\varphi_\alpha^\Gamma(0,\mathbf{x}), j_\beta(0)]_- = [\varphi_\alpha^\Gamma(0,\mathbf{x}), \eta_\tau(0)]_- = [\varphi_\alpha^\Gamma(0,\mathbf{x}), \bar{\eta}_\tau(0)]_- = 0,$$
$$[\psi_\tau^\Gamma(0,\mathbf{x}), j_\alpha(0)]_- = \gamma_0\Lambda_{\alpha,\tau}\delta(\mathbf{x}),\ \{\psi_\tau^\Gamma(0,\mathbf{x}), \bar{\eta}_{\tau'}(0)\}_+ = \gamma_0\Lambda_\tau{}^{\tau'}\delta(\mathbf{x}),\ \{\psi_\tau^\Gamma(0,\mathbf{x}), \eta_{\tau'}(0)\}_+ = \gamma_0\Lambda_{\tau'\tau}\delta(\mathbf{x}), \tag{4.2}$$

with $\Lambda_{\tau\tau} = 0$. The remaining commutation relations are obtained from (4.2) by hermitian conjugation. It is obvious, that (2.2) are solutions of the above commutation relations satisfying all the necessary requirements.

2. The elimination of quasilocal terms from the equations for the matrix elements of the currents and the analysis of the boundary conditions for these equations show that when the PMS is applicable the degree of arbitrariness in quasilocal operators reduces to the arbitrariness in the finite number of constants—the invariant charges—in the initial equations of the theory. For a definite choice of these constants the quasilocal operators and correspondingly the quasilocal terms become quite definite (cf., also subsection 4). Moreover, from the point of view of the Lagrangian approach (in the present case—perturbation theory) it turns out that the representation of the matrix elements in the form of a sum of an R-term and a quasilocal term is a decomposition of a finite matrix element into two, generally speaking, divergent expressions. In the latter case the quasilocal terms perform renormalizing functions and the procedure of the solution coincides with the subtraction technique in the renormalizations of field theory.

3. Of direct interest are not the matrix elements of the currents (r-functions), but the matrix elements of the S-matrix (τ-functions). Starting with the definition of the currents one can write for the τ-functions for noncoincident values of the momenta (the connected diagrams for the S-matrix) linear integral equations the kernels of which are the well-known r-functions. For example

$$\langle \mathrm{k}\alpha; l|S|m\rangle = -i(2\pi)^{5/2}\sum_n \langle l|S|n\rangle\langle n|j_\alpha(0)|m\rangle \times \delta(k+p_n-p_m). \quad (4.3)$$

From the stability of a single-particle state it follows that the matrix elements for the transitions of two particles into n reduce directly to the matrix elements of the currents on the mass shell. For example,

$$\langle \mathrm{k}\alpha, \mathrm{k}\beta|S|n\rangle = -i(2\pi)^{5/2}\delta(k_\alpha+k_\beta-p_n)\langle \mathrm{k}\beta|j_\alpha(0)|n\rangle. \quad (4.4)$$

4. A few remarks on the arbitrariness and the choice of independent invariant charges. Firstly, the solution of the initial equations might be nonexistent for a certain choice of the invariant charges (essentially we have an eigenvalue problem with respect to the invariant charges). Secondly, when speaking, for example, of the fact that λ_1 (3.5) is a function of λ (3.3) and of other possible invariant charges, it is impossible not to admit that this function will turn out to be multivalued, and perhaps even infinitely multivalued. It is necessary also to place restrictions on arbitrariness of this type.

Until now we have been discussing arbitrariness in the initial equations themselves. But in virtue of the nonlinearity of these equations one can expect even for fixed boundary conditions the existence of a whole class of solutions the arbitrariness in which can also be functional. One should in general note that the problem of the consistency of the formulation of boundary conditions and of the possible arbitrariness in the theory is a very complex one and forms the subject of a separate investigation.

5. Finally, we consider the possible restriction of the condition of causality and the enlargement of the class of quasilocal operators leading to the same equations for the matrix elements of the currents—for simplicity using the example of a scalar field of mass m interacting with itself.

In future it will be convenient to say that in a certain expression a given spatial argument x lies on the mass shell if a Fourier transform is taken with respect to x, with the conjugate momentum p lying on the mass shell, i.e., satisfying the condition $p^2 = m^2$ (in other words, with respect to the given spatial argument an integration is carried out with a function satisfying the Klein-Gordon equation with the mass m).

In obtaining the initial equations the basic equation is an equation of the form (1.8)

$$\delta j(x) / \delta\varphi(y) = -i\theta(x^0-y^0)[j(x), j(y)]_- + \Lambda(x)\delta(x-y), \quad (4.5)$$

the validity of which is assumed for arbitrary x and y. But essentially it is utilized only for a fixed x = 0 and y lying on the mass shell (with respect to y in equations of the type (2.2) an integration is performed with $\exp(\pm ipy)$, where $p^2 = m^2$). For the elimination of the quasilocal terms by the previously described method it is sufficient to assume that the quasilocal operator $\Lambda(xy)$ for a fixed first argument (x = 0) should represent such an operator distribution with respect to the second argument which on the mass shell would effectively reduce to $\delta(y)$, i.e., on the mass shell with respect to the second argument we must have

$$\Lambda(0, y) = \Lambda\delta(y).$$

In virtue of the symmetry $\Lambda(x,y) = \Lambda(y,x)$ the quasilocal operator has this property also on the mass shell with respect to the first argument for a fixed value of the second argument. The PMS can, thus, be made weaker so that it becomes the principle of minimum singularity of quasilocal operators on the mass shell with respect to the second argument for fixed values of the first. Corresponding to this the condition of causality is now required to be satisfied also only on the mass shell with respect to the second argument.

Thus, the equations for the matrix elements of the currents are not changed if instead of (1.10) one admits the broader class of quasilocal operators (strictly speaking, they are no longer quasilocal):

$$\Lambda(x, y) = \Lambda(x)\delta(x-y) + \sum_n (\Box_y - m^2)^n \Lambda_n(x-y),$$

where the matrix elements $\Lambda_n(x-y)$ have no strong singularities on the mass shell. We note that the solution of the commutation relations in reference[1] admits such quasilocal operators.

Summarizing, we can say that the assumptions with respect to the minimal singularity of quasilocal operators together with other more usual principles are sufficient in order to write the integrodifferential or difference equations for the matrix elements of the S-matrix for a minimum departure from the mass shell and to reduce the arbitrariness in the theory to a finite number of undetermined constants.

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