

RELAXATION PHENOMENA IN SUPERFLUID HELIUM

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It is shown that the relatively slowest relaxation process in superfluid helium is the establishment of equilibrium between the phonon and roton gases. A set of equations is obtained which describe the propagation of sound waves by taking relaxation processes into account; the set of equations is valid for both low and high frequencies.

THE propagation of sound waves in superfluid helium, as is well known, is distinguished by many features, the chief of which is the possibility of the propagation of sound waves of two types: waves of compression and rarefaction of the density (first sound) and undamped temperature waves (second sound). Because of the interaction of the elementary excitations (phonons and rotons) a number of mechanisms of energy dissipation are initiated and lead to the absorption of sound and to the appearance of a dependence of the sound velocity on its frequency (sound dispersion). A study of sound dispersion in a superfluid thus allows us to analyze the entire picture of the kinetic phenomena in a gas of elementary excitations. A number of theoretical^[1-4] and experimental^[5-11] researches have been devoted to this question.

At low frequencies ($\omega\tau \ll 1$, ω is the sound frequency, and τ some characteristic time) the study of the problem of sound absorption is carried out by means of the hydrodynamic equations of the superfluid with dissipation terms.^[1] The superfluid liquid is characterized by kinetic coefficients: the first viscosity coefficient η , the three second viscosity coefficients ζ_1 , ζ_2 , and ζ_3 , and the thermal conductivity χ . The absorption coefficient of first sound α_1 is equal to

$$\alpha_1 = \frac{\omega^2}{2\rho u_1^3} \left[\frac{4}{3}\eta + \zeta_2 + \frac{\rho\chi}{C} \left(\frac{c_p}{c_v} - 1 \right) \right] \quad (1)$$

(u_1 is the velocity of first sound and C the heat capacity per unit volume of the liquid). The absorption coefficient of second sound α_2 is equal to

$$\alpha_2 = \frac{\omega^2}{2\rho u_2^3} \frac{\rho_s}{\rho_n} \left[\frac{4}{3}\eta + (\zeta_2 + \rho^2\zeta_3 - 2\rho\zeta_1) + \frac{\rho_n}{\rho_s} \frac{\rho\chi}{C} \right] \quad (2)$$

(u_2 is the velocity of second sound), and the principal term in the square brackets in (2) is the term containing the thermal conductivity χ .

Thanks to the small difference between c_p and

c_v for liquid helium, the third term in (1) is negligibly small. The first experiments on the absorption of first sound in liquid helium^[9] showed that the coefficient of second viscosity¹⁾ ζ_2 , in contrast with normal liquids, where it is equal to η in order of magnitude, exceeds η by approximately one-and-a-half orders. This fact is direct proof that slow processes for establishing equilibrium occur in a gas of excitations. The coefficient of first viscosity η was computed in^[13], and the coefficient of thermal conductivity χ in^[1]. The principal mechanism of scattering of the excitations, which determines the viscosity and the thermal conductivity, is the scattering of phonons by the rotons. Two slow approaches to equilibrium were considered in^[2], where the coefficients of second viscosity were computed: the process of the conversion of two phonons into three (five-phonon process²⁾) and the process of conversion of an energetic phonon into a roton. The first of these processes guarantees the establishment of equilibrium in the number of phonons, the second has such a small probability that it does not play a significant role in the phenomena considered here.

The problem of the propagation of sound in a superfluid at high frequencies ($\omega\tau \sim 1$) obviously cannot be considered in the hydrodynamic approximation and can be solved only on the basis of the kinetic equation. The present paper is devoted to this problem. In this case, as a consequence of the very favorable situation with the establishment of energy equilibrium in the excitation gas, it is possible essentially to solve the problem exactly. The

¹⁾The coefficient of first viscosity η was measured previously by direct experiments.^[12]

²⁾The decay of one phonon into two is forbidden by conservation laws for a phonon spectrum in superfluid helium.

fact is that the effective scattering cross section of rotons by rotons is sufficiently large and therefore in a roton gas one always has a local equilibrium in practice; that is, the roton gas can be described by a quasi-equilibrium distribution function with temperature values T_r and relative velocity $v_{nr} - v_s$ which vary from point to point (v_s is the velocity of the superfluid part of the liquid).

The scattering of a phonon by a phonon is not accompanied by a change in direction of the momenta of the colliding phonons and is anomalously large for collisions of phonons at small angles. This process is more rapid than all the other processes taking place with phonons and leads to the result that there is always an energy equilibrium for phonons moving in a given direction. Moreover, as we shall see, the five-phonon process, for phonons colliding at small angles, also takes place more rapidly than the process of scattering of phonons by rotons, and therefore the phonon-number equilibrium is established. Thus, the relatively slowest process is the scattering of phonons by rotons. The slowness of this process leads to the result that the establishment of the energy equilibrium between the photon and roton gases is made more difficult, whereas in each of them it occurs separately. This circumstance leads to the appearance in the hydrodynamic approximation of a certain second viscosity, while in the general case, it leads to absorption and dispersion of first and second sound. In this case, the theory contains only one characteristic time τ_{pr} , which is associated with the scattering of a phonon by a roton, which can be computed exactly. The results obtained will naturally be valid for all frequencies ($\omega \tau_{pr} \lesssim 1$), provided they do not exceed the reciprocal of the characteristic time for establishing equilibrium in the excitation gas.

1. SCATTERING OF ELEMENTARY EXCITATIONS

Let us consider in more detail the scattering of elementary excitations in helium II. The change in the number of elementary excitations in an element of phase space can take place by the following paths: scattering of phonons by phonons, of rotons by rotons, of phonons by rotons, of rotons by phonons, and absorption and emission of phonons and rotons in inelastic collisions with one another and between themselves. The fundamental laws which characterize these scattering processes were investigated in [2,14].

The scattering of phonons by phonons takes place mainly in the collision of phonons with nearly equal directions of momenta. In phonon-

phonon scattering, the largest contribution to the matrix element of the transition of interest to us, namely H'_{AF} from the initial state A to the final state F in the second order perturbation theory

$$H_{AF'} = \sum_{i=1}^{VI} \frac{(V_3)_{Ai}(V_3)_{iF}}{E_A - E_i} + (V_4)_{AF},$$

$$V_3 = \frac{v\rho'v}{2} + \frac{1}{3!} \frac{\partial}{\partial \rho} \left(\frac{c^2}{\rho} \right) \rho'^3, \quad V_4 = \frac{1}{4!} \frac{\partial^2}{\partial \rho^2} \left(\frac{c^2}{\rho^2} \right) \rho'^4 \quad (1.1)$$

(v is the velocity of the liquid, ρ' is the deviation of the density from its value for the motionless liquid) is made by terms whose denominators vanish if the dispersion of the energy of the phonons is neglected when the angle between the momenta of the colliding phonons is equal to zero.^[14] Thus, for example, for a specific intermediate state in which there is a phonon with momentum $\mathbf{p} + \mathbf{p}_1$, the energy difference $E_A - E_i$ is equal to

$$\varepsilon(p) + \varepsilon(p_1) - \varepsilon(|\mathbf{p} + \mathbf{p}_1|),$$

and it vanishes for $|\mathbf{p} + \mathbf{p}_1| = p + p_1$, that is, when the angle between the momenta of the colliding phonons \mathbf{p} and \mathbf{p}_1 is equal to zero. Therefore, in the expression for the energy of the phonons, it is necessary to consider also the terms that are cubic in the momenta

$$\varepsilon(p) = cp(1 - \gamma p^2). \quad (1.2)$$

Inasmuch as $\gamma p^2 \ll 1$, Eq. (1.2) reaches the maximum value at small angles between the momenta of the colliding phonons. Thus, in our case of phonon scattering, the principal role is played only by colliding phonons with nearly equal directions of momenta. According to the laws of conservation of momentum and energy, these collisions are not accompanied by an appreciable change in the directions of the momenta of the colliding phonons and consequently they can only lead to the establishment of an energy equilibrium between the phonons moving in a given direction. The characteristic time t_{pp} of such a scattering process, according to [12], is determined in the case of the scattering of a phonon with high energy (and in the following we shall be interested only in this case) by the relation

$$\frac{1}{t_{pp}} = \frac{(u+1)^4 (kT/c)^3 (\pi^3/3) p^4}{(12\hbar^2 \rho)^2 c \gamma (2\pi\hbar)^3} \quad (1.3)$$

In the calculation of $1/t_{pp}$ we used the following parameters:

$$\gamma = 3 \cdot 10^{37} \text{ sec}^2 \text{ g}^{-2} \text{ cm}^{-2}, \quad u = \frac{\rho}{c} \frac{\partial c}{\partial \rho} = 2.7. \quad (1.4)$$

The value of γ was computed by interpolation of the entire energy curve including rotons.^[14] It is obviously difficult at present to obtain any information on this value from existing neutronographic scattering data because of its smallness. However, the data of Henshaw and Woods^[15] point to a very insignificant dispersion of the energy of the phonons. These data, and also a comparison of the values of the absorption coefficient of first sound near absolute zero computed by Andreev and one of the authors^[3,4], together with the values of Chase and Herlin,^[6] show that γ must be appreciably smaller than $10^{37} \text{ sec}^2 - \text{g}^{-2} - \text{cm}^{-2}$. We shall see below that this assumption confirms the data of our research on the absorption of first sound at low temperatures. Consequently, the real value of the time t_{pp} is evidently much less than that which is determined from Eq. (1.3) with the value of γ from (1.4).

The value of the derivative of c with respect to the density was computed from the data of Atkins and Stasior^[16] on the pressure dependence of the velocity of first sound in helium II, and also from the data of Keesom and Miss Keesom^[17] on the pressure dependence of the density of helium II; it is in excellent agreement with the recent neutronographic data of Henshaw and Woods^[15].

In what follows we shall need the value of the derivatives of the density with respect to the parameters P_0 and Δ ³⁾ (P_0 , Δ , and also μ are parameters of the energy of the rotons $\mathcal{E} = (P - P_0)^2/2\mu$, \mathbf{P} is the momentum of the roton). From current data^[14,15,18,19] the most probable values of the derivatives are

$$\frac{\rho}{P_0} \frac{\partial P_0}{\partial \rho} = 0.4, \quad \frac{\rho}{\Delta} \frac{\partial \Delta}{\partial \rho} = -0.57, \quad \frac{\rho^2}{\Delta} \frac{\partial^2 \Delta}{\partial \rho^2} = -5.2. \quad (1.5)$$

The values of the derivatives of P_0 were taken by us from the neutronographic data of Henshaw and Woods. These same authors obtained $(\rho/\Delta)\partial\Delta/\partial\rho = -1$ for the first derivative of the parameter Δ . However, in the calculation of this parameter, Henshaw and Woods did not consider that Δ has a large second derivative with respect to the density. Therefore, we shall use for the first derivative of Δ the value $(\rho/\Delta)\partial\Delta/\partial\rho = -0.57$, computed by Atkins and Edwards^[18] from data on the temperature dependence of the coefficient of thermal expansion of helium II, while for the second derivative, we shall use the value $(\rho^2/\Delta)\partial^2\Delta/\partial\rho^2 = -5.2$ com-

puted by us from the neutronographic data of Henshaw and Woods.

As has already been pointed out above, the scattering of phonons by phonons takes place principally by means of collisions of particles with nearly equal directions of momenta. In comparison with this specific scattering, the collisions of phonons with change of direction of momenta of the colliding particles has a low probability. Nevertheless, in the solution of our problem, one must take into account the scattering of phonons by phonons with a change of direction of momenta of the colliding particles, inasmuch as in the temperature region below 0.9°K, as we shall see, it becomes significant along with the scattering of phonons by rotons.

We shall need below, the differential effective cross section for the process under consideration. It is represented by the relation

$$d\sigma_{pp}(\mathbf{p}, \mathbf{p}_1, \mathbf{p}^*, \mathbf{p}_1^*) = (2\pi/\hbar c) |H_{AF'}|^2 \delta(\epsilon + \epsilon_1 - \epsilon^* - \epsilon_1^*) d\tau_{p^*} \quad (1.6)$$

with the matrix element of transition $H_{AF'}$ from the initial state A to the final state F in second order perturbation theory determined from Eq. (1.1), $d\tau_{p^*}$ is the element of volume in p^* space, $d\tau_{p^*} = p^{*2} dp_{p^*} / (2\pi\hbar)^3$ (do^* is the element of solid angle). The quantities without and with asterisks in (1.6) refer to incident, and scattered phonons, respectively.

The scattering of a phonon by a roton in some sense is analogous to the scattering of a light particle by a heavy one. This analogy is associated with the fact that the phonon possesses a momentum that is much smaller than the momentum of the roton. With account of the law of conservation of momentum, the law of conservation of energy for the given process is written in the form

$$cp + (P - P_0)^2/2\mu = cp^* + (|\mathbf{P} + \mathbf{p} - \mathbf{p}^*| - P_0)^2/2\mu$$

(\mathbf{p} and \mathbf{P} are the momenta of the phonon and the roton before collision, \mathbf{p}^* is the momentum of the phonon after collision), hence, after several transformations, taking into account the smallness of the phonon momenta ($p, p^* \ll P_0$), we get

$$c(p - p^*) = (\mathbf{P}, \mathbf{p} - \mathbf{p}^*)^2 / 2\mu P_0^2.$$

We now make use of the fact that the energy of the phonons is $\epsilon = cp < 3\mu c^2$ (just such an inequality, as we shall see below, is essential in the interval of collision). This allows us to describe the energy difference of interest to us in the following way:

$$\epsilon - \epsilon^* = \frac{p^2}{2\mu} (\mathbf{m}, \mathbf{n} - \mathbf{n}^*)^2, \quad (1.7)$$

³⁾The numerical values of the parameters P_0 , Δ , c , and also μ are equal to^[15] $P = 2.01 \times 10^{-19} \text{ erg-sec-cm}^{-1}$, $\Delta = 8.65^\circ\text{K}$, $c = 238 \text{ m/sec}$, and $\mu = 1.06 \times 10^{-24} \text{ g}$.

where \mathbf{m} , \mathbf{n} and \mathbf{n}^* are unit vectors directed along the vectors \mathbf{P} , \mathbf{p} , and \mathbf{p}^* . Taking into account what has been said regarding the smallness of the quantity $\epsilon - \epsilon^*$ ($\epsilon - \epsilon^* < \epsilon$ and ϵ^*), the matrix element of the transition under discussion H_{AF}' from the initial state A to the final state F in second order perturbation theory, in agreement with [14], can be written in the form

$$H_{AF}' = \frac{P_0 p}{2\rho} \left\{ (\mathbf{n} + \mathbf{n}^*, \mathbf{m})(\mathbf{n}\mathbf{n}^*) + \frac{P_0}{\mu c} (\mathbf{n}\mathbf{m})^2 (\mathbf{n}^*\mathbf{m})^2 + A \right\},$$

$$A = \frac{\rho^2}{P_0 c} \left[\frac{\partial^2 \Delta}{\partial \rho^2} + \frac{1}{\mu} \left(\frac{\partial P_0}{\partial \rho} \right)^2 \right]. \quad (1.8)$$

The value of the parameter A computed by means of the values of the derivatives with respect to the density of P_0 and Δ , given in (1.5), turns out to be very small and approximately equal to -0.1 .

The differential effective scattering cross section of a phonon by a roton is equal to

$$d\sigma_{pr} = (2\pi/\hbar c) |H_{AF}'|^2 \delta(\mathcal{E} + cp - \mathcal{E}^* - cp^*) d\tau_{\mathbf{p}^*} \quad (1.9)$$

(\mathcal{E} and \mathcal{E}^* are the energies of the rotons before and after collision). We substitute (1.8) in (1.9) and integrate over the momentum of the scattered phonon; as a result, we have

$$d\sigma_{pr} = \left(\frac{P_0 p^2}{4\pi \hbar^2 \rho c} \right)^2 \left[(\mathbf{n} + \mathbf{n}^*, \mathbf{m})(\mathbf{n}\mathbf{n}^*) + \frac{P_0}{\mu c} (\mathbf{n}\mathbf{m})^2 (\mathbf{n}^*\mathbf{m})^2 + A \right] d\omega^*.$$

In the following, we need the cross section $d\sigma_{pr}$ averaged over all the angles of the colliding particles. It is equal to

$$d\sigma_{pr} = \left(\frac{P_0 p^2}{4\pi \hbar^2 \rho c} \right)^2 \left[\frac{2}{9} + \frac{1}{25} \left(\frac{P_0}{\mu c} \right)^2 + \frac{2A}{9} \frac{P_0}{\mu c} + A^2 \right] d\omega^*. \quad (1.10)$$

In contrast with the processes described above, the scattering of a phonon by a phonon and of a phonon by a roton, for which one can calculate the effective cross section exactly, the process of roton-roton scattering is considered with the help of some interaction model. The total effective cross section σ_{rr} of roton-roton scattering was computed under the assumption that the interaction of the rotons has a δ -type character. According to [14],

$$\sigma_{rr} = \frac{2P_0 \mu |V_0|^2}{|\partial \mathcal{E} / \partial \mathbf{P} - \partial \mathcal{E}_1 / \partial \mathbf{P}_1| \hbar^4 \cos(\psi/2)}. \quad (1.11)$$

Here $|\partial \mathcal{E} / \partial \mathbf{P} - \partial \mathcal{E}_1 / \partial \mathbf{P}_1|$ is the relative velocity of the colliding rotons, ψ is the angle between \mathbf{P} and \mathbf{P}_1 (\mathcal{E} , \mathbf{P} and \mathcal{E}_1 , \mathbf{P}_1 are the energy and momentum of the colliding rotons), V_0 is the amplitude which

determines the interaction of the rotons ($V = V_0 \delta(\mathbf{r}_1 - \mathbf{r})$, V is the energy of interaction of the rotons, \mathbf{r}_1 and \mathbf{r} are their radius vectors). The quantity V_0 can be computed from the experimental values of the coefficient of viscosity of He II [13, 20-23]. It is seen to be of the order of 1.7×10^{-38} erg-cm³.

The reciprocal of the mean time between two collisions of a roton t_{rr} is obtained from (1.11) by multiplication by the total roton current with subsequent averaging over all angles formed by the momenta of the colliding rotons:

$$\frac{1}{t_{rr}} = 4P_0 \mu |V_0|^2 \frac{N_p}{\hbar^4}, \quad N_r = \frac{2P_0^2 (\mu k T)^{1/2} e^{-\Delta/T}}{(2\pi)^{3/2} \hbar^3} \quad (1.12)$$

(N_r is the number of rotons per unit volume).

From among processes of inelastic scattering of phonons and rotons, as was explained previously, [2] the process of the transformation of three phonons into two (five-phonon process) has the greatest probability. The five-phonon process, as also the process of phonon-phonon scattering considered above, is not accompanied by any change in the direction of the momenta of colliding particles. Actually, the matrix element of interest to us of a transition from the initial state A to the final state F in second order perturbation theory contains the sum

$$\sum_{I, II} \frac{(V_3)_{AI} (V_3)_{I, II} (V_3)_{II F}}{(E_A - E_I)(E_A - E_{II})}$$

some terms of which contain in their denominators the product of two equations which vanish when dispersion is neglected, when the angles between the momenta of the colliding phonons are equal to zero. Thus the probability of the process (3 \rightarrow 2) has a maximum for collisions of phonons at small angles. Such collisions, as pointed out above, do not lead to an appreciable change in directions of the colliding particles; consequently, they can lead to the establishment of equilibrium between phonons moving in a given direction.

Calculation of the probability of a five-phonon process is very cumbersome. However, inasmuch as the expression for this probability contains the quantity γ^2 in the denominator, while the value of γ is known with low accuracy, one can limit oneself to obtaining a symbolic formula without a numerical coefficient. [2] The presence of such an expression permits us to obtain without difficulty the temperature dependence of the coefficient Γ_p , which determines the rate of change in the number of phonons N_p ,

$$\dot{N}_p = -\alpha \Gamma_p$$

(α is some chemical potential; see below). According to [2], the value of Γ_p is

$$\Gamma_p = \Lambda T^4, \tag{1.13}$$

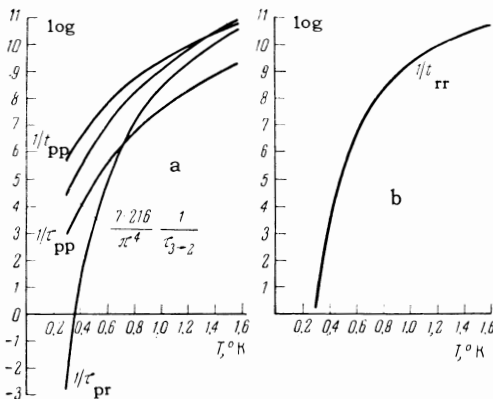
where Λ is a temperature-independent coefficient.

In addition to the five-phonon process, other inelastic processes are also possible which lead to a change in the number of phonons and rotons. Let us say a few words about one of them, which we have considered previously^[2], but which, as it turns out, is not essential for the present problem. When an energetic phonon (with energy of the order of Δ) collides with a roton it may become transformed into a roton. The probability of this process, however, is small and certainly much smaller than the probability of scattering of a phonon by a roton. Therefore, we can neglect the exchange of energy and momentum between the phonon and roton gases.

Having established the basic laws that characterize the interaction of the elementary particles, we can describe the form of the non-equilibrium distribution function. We begin with the phonons. Drawing (a) shows the temperature dependence of the times characterizing the phonon processes. In a comparison of the times τ_{pr} and $\tau_{3 \rightarrow 2}$, which describe the processes of scattering of a phonon by a roton and the transformation of three phonons into two, it must be kept in mind that these processes of scattering enter into the kinetic equation, as we shall see below, with different weights, inasmuch as the energetic phonons play more of a role in the scattering of phonons by rotons than in the five-phonon process. In this connection a graph of the quantity

$$\frac{7 \cdot 216}{\pi^4} \frac{1}{\tau_{3 \rightarrow 2}}$$

has been plotted in drawing (a). This curve should be compared with the quantity $1/\tau_{pr}$.



Temperature dependence of the times: a - t_{pp} , τ_{pr} , τ_{pp} , $\tau_{3 \rightarrow 2}$; b - t_{rr} .

It is seen from the drawing (a) that in the temperature region below 1.2°K, in which we are most interested, the four-phonon and five-phonon processes guarantee a rapid establishment of equilibrium both in energies and in the number of phonons, for phonons moving in a given direction. Therefore, the phonons moving in a given direction can be characterized by a certain temperature T_p (a function of the direction of momentum of the phonon). For temperatures above 1.2°K, the characteristic time for the five-phonon process becomes comparable with the time τ_{pr} and the establishment of equilibrium in the number of phonons will not be so rapid. In this case, the distribution function for phonons moving in a given direction also depends on some chemical potential α . Thus the distribution function of the phonons is some quasi-equilibrium function

$$n = [\exp(\epsilon / kT_p + \alpha) - 1]^{-1}, \tag{1.14}$$

in which the temperature T_p and the chemical potential α depend on the direction of the phonon's momentum. For temperatures below 1.2°K, in accord with what has been said, α can be equated to zero⁴⁾, and the departure of the distribution function from a constant equilibrium value n_0 (for small departures) is represented in the form

$$n - n_0 = \frac{\partial n_0}{\partial \epsilon} \left(\frac{\partial \epsilon}{\partial p} \rho' + \epsilon \nu \right). \tag{1.15}$$

The first term reflects the dependence of the energy spectrum on the density, the second is obtained by expanding T_p in a series in the difference $T_p - T_0$ (T_0 is the constant-equilibrium temperature), ν is some function of the direction of the momentum of the phonon, which is found by solving the kinetic equation. Generally speaking, the function n depends not on ϵ but on the difference $\epsilon - (\mathbf{p} \cdot \mathbf{v}_{np} - v_s)$, where $\mathbf{v}_{np} - v_s$ is some relative velocity of the phonon gas, which is also a function of the direction of the phonon momentum. One can see, however, that inasmuch as ϵ is a linear function of p , the function ν in Eq. (1.15) automatically takes into account the presence of such a term.

We now proceed to the rotons. Equation (1.12) determines the temperature dependence of the characteristic time t_{rr} of roton-roton scattering (drawing b). A comparison of t_{rr} with the time for scattering of rotons by phonons shows that at temperatures above 0.6°K the roton-roton scattering takes place more rapidly, which assures the estab-

⁴⁾Account of α will be taken in a subsequent work of the authors.^[14]

lishment of a certain local equilibrium in the roton gas. Thus the rotons can be described by some quasi-equilibrium distribution function with a temperature T_p and a relative velocity $\mathbf{v}_{nr} - \mathbf{v}_s$, which is a point function:

$$N = \exp \left[- \frac{\mathcal{E} - (\mathbf{P}, \mathbf{v}_{nr} - \mathbf{v}_s)}{kT_r} \right]. \quad (1.16)$$

Expanding this function in a series in the deviations of all the quantities from their constant equilibrium values, we get (for small deviations)

$$N - N_0 = \frac{\partial N}{\partial \mathcal{E}} \left[\frac{\partial \mathcal{E}}{\partial \rho} \rho' - \mathcal{E} \frac{T_r'}{T_0} - (\mathbf{P}, \mathbf{v}_{nr} - \mathbf{v}_s) \right] \quad (1.17)$$

Here T_r' is the deviation of the temperature of the roton gas from a constant equilibrium value T_0 , N_0 is the constant equilibrium distribution function.

We note that, for generality, one should have written a term in the square bracket in the expression for the contribution to the roton distribution function that corresponds to some chemical potential α_r . However, inasmuch as the roton spectrum is such that its energy is almost constant and equal to Δ , such a term is impossible to distinguish by its dependence on momentum and energy from the term $\mathcal{E} T_r'/T_0 \approx \Delta T_r'/T_0$. This circumstance permits us to combine both terms from the very beginning. The physical meaning of this circumstance is also simple. Any local change in the number of rotons can be described by the local change in temperature.

Let us consider the problem of the limits of applicability of the results obtained with respect to the frequencies of the sound. It is evident that two circumstances are significant for all the considerations that have been made. First, the presence of an energy equilibrium for phonons moving in a given direction; second, the presence of a local equilibrium in the roton gas. Therefore, in order that these two conditions not be violated, it is obviously necessary that

$$\omega t_{rr} \ll 1 \quad \text{and} \quad \omega t_{pp} \ll 1, \quad (1.18)$$

where the time t_{rr} is determined by Eq. (1.12) and the time t_{pp} by Eq. (1.3). To estimate t_{pp} , it is necessary to substitute the energy of the phonons, which plays an important role in the range of collisions and is significant for the phenomenon of sound dispersion (as we shall see, $\epsilon \approx (7-8) \cdot kT_0$). Here, it must be kept in mind that the real value of the time t_{pp} , as shown above, is much smaller than that determined by Eq. (1.3) with the value of γ from (1.4).

An appreciable dispersion takes place for first sound when $\omega \tau_{pr} \sim 1$, and for second sound, when $\omega \tau_{pr} \sim u_2/u_1$. Inasmuch as $u_2/u_1 \lesssim 1$, the disper-

sion of second sound sets in at much lower frequencies than for first sound. It is easy to see that in this and in the other case, dispersion begins when the wavelength of the corresponding sound is comparable with the mean free path of the phonon. It is seen from the drawings (a) and (b) that there is a large range of frequencies for which observation of sound dispersion is possible.

2. KINETIC EQUATIONS

As was pointed out above, the free paths of the phonons and rotons, in the general case of an arbitrary value of the parameter $\omega \tau$, can be shown to be comparable with and even exceed the sound wavelength. Therefore, it is impossible to apply the hydrodynamic equations in the normal part of the fluid and one must see the kinetic equation for the phonon distribution function $n(\mathbf{p}, \mathbf{r}, t)$ and roton distribution function $N(\mathbf{P}, \mathbf{r}, t)$:

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}} \frac{\partial H_p}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}} \frac{\partial H_p}{\partial \mathbf{r}} = J(n), \quad (2.1)$$

$$\frac{\partial N}{\partial t} + \frac{\partial N}{\partial \mathbf{r}} \frac{\partial H_r}{\partial \mathbf{P}} - \frac{\partial N}{\partial \mathbf{P}} \frac{\partial H_r}{\partial \mathbf{r}} = J(N). \quad (2.2)$$

Here $J(n)$ and $J(N)$ are collision integrals, which are due to processes of elastic and inelastic scattering of phonons and rotons. The Hamiltonians H_p and H_r are equal to

$$H_p = \epsilon(p) + \mathbf{p}\mathbf{v}_s, \quad H_r = \mathcal{E}(P) + \mathbf{P}\mathbf{v}_s,$$

where $\epsilon(p)$ and $\mathcal{E}(P)$ are the energies of the phonons and rotons in a reference system moving with a velocity \mathbf{v}_s and depending on the liquid density ρ . The quantities ρ and \mathbf{v}_s play the role of external conditions for the excitation gas.

In order to get a complete set of equations, it is necessary to add to Eqs. (2.1) and (2.2) the equation of mass continuity

$$\frac{\partial \rho}{\partial t} + \text{div } \mathbf{j} = 0 \quad (2.3)$$

and also the equation of superfluid motion^[25]

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \left(\mu + \frac{\mathbf{v}_s^2}{2} \right) = 0, \quad (2.4)$$

where \mathbf{j} is the momentum density of the liquid and $\mu = \partial E / \partial \rho$, where E is the energy density in a set of coordinates in which $\mathbf{v}_s = 0$. The derivative of the energy with respect to density is taken for a constant distribution of the excitations, i.e., for constant $n(p)$ and $N(P)$.

To find \mathbf{j} , we note that the momentum in a system of coordinates where the superfluid portion is at rest is equal to

$$\int \mathbf{p} n d\tau_p + \int \mathbf{P} N d\tau_P.$$

Using the well known transformation formula of Halliday, we get

$$\mathbf{j} = \rho \mathbf{v}_s + \int \mathbf{p} n d\tau_p + \int \mathbf{P} N d\tau_p. \quad (2.5)$$

For the energy E, we have

$$E = E_0 + \int \varepsilon n d\tau_p + \int \mathcal{E} N d\tau_p,$$

where E_0 is the energy at absolute zero.

Carrying out differentiation of E with respect to the density ρ , and using (2.3), (2.4), and (2.5), we get the desired equation^[26]

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \left(\rho \mathbf{v}_s + \int \mathbf{p} n d\tau_p + \int \mathbf{P} N d\tau_p \right) = 0, \quad (2.6)$$

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \left(\mu_0 + \int \frac{\partial \varepsilon}{\partial \rho} n d\tau_p + \int \frac{\partial \mathcal{E}}{\partial \rho} N d\tau_p \right) = 0, \quad (2.7)$$

where μ_0 is the chemical potential at $T = 0$.

In a plane sound wave, all the thermodynamic quantities are composed of constant equilibrium components and small additions which change according to the law $\exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ (\mathbf{k} is the wave vector). The velocities \mathbf{v}_s , \mathbf{v}_{np} and \mathbf{v}_{nr} in the sound wave are also small quantities⁵⁾ which vary according to the same law. Therefore, one can set

$$n = n_0 + n', \quad N = N_0 + N', \quad \rho = \rho_0 + \rho',$$

in Eqs. (2.1), (2.2), (2.6), and (2.7). Here n_0 , N_0 , ρ_0 are the constant equilibrium values and n' , N' , ρ' and also \mathbf{v}_s are small additions proportional to $\exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$. After linearization these equations take the form

$$\left(1 - \frac{(\mathbf{k}\mathbf{v})}{\omega} \right) n' + \frac{(\mathbf{k}\mathbf{v})}{\omega} \frac{\partial n_0}{\partial \varepsilon} \left(\frac{\partial \varepsilon}{\partial \rho} \rho' + (\mathbf{P}\mathbf{v}_s) \right) = -\frac{1}{i\omega} J(n), \quad (2.8)$$

$$\left(1 - \frac{(\mathbf{k}\mathbf{V})}{\omega} \right) N' + \frac{(\mathbf{k}\mathbf{V})}{\omega} \frac{\partial N_0}{\partial \mathcal{E}} \left(\frac{\partial \mathcal{E}}{\partial \rho} \rho' + (\mathbf{P}\mathbf{v}_s) \right) = -\frac{1}{i\omega} J(N), \quad (2.9)$$

$$-\omega \rho' + k \rho v_s + \int (\mathbf{k}\mathbf{p}) n' d\tau_p + \int (\mathbf{k}\mathbf{P}) N' d\tau_p = 0, \quad (2.10)$$

$$-\omega v_s + k \left(\frac{c^2}{\rho} + \int \frac{\partial^2 \varepsilon}{\partial \rho^2} n_0 d\tau_p + \int \frac{\partial^2 \mathcal{E}}{\partial \rho^2} N_0 d\tau_p \right) \rho' + k \int \frac{\partial \varepsilon}{\partial \rho} n' d\tau_p + k \int \frac{\partial \mathcal{E}}{\partial \rho} N' d\tau_p = 0, \quad (2.11)$$

where $\mathbf{v} = \partial \varepsilon / \partial \mathbf{p}$ and $\mathbf{V} = \partial \mathcal{E} / \partial \mathbf{P}$. We drop the zero subscript on the density and temperature⁶⁾ and use the fact that $d\mu_0 = c^2 d\rho / \rho$.

⁵⁾One must keep in mind the smallness of the velocities \mathbf{v}_s , \mathbf{v}_{np} , and \mathbf{v}_{nr} in comparison with the velocities of first and second sound.

⁶⁾In what follows, we shall drop the zero subscript everywhere for equilibrium thermodynamic quantities.

We introduce a spherical set of coordinates with the polar axis along the vector \mathbf{k} . The angle between \mathbf{k} and \mathbf{p} will be denoted by θ and between \mathbf{k} and \mathbf{P} by ϑ . In this set of coordinates n' and N' , in accord with (1.15), (1.17), are equal to

$$n' = \frac{\partial n_0}{\partial \varepsilon} \left(\frac{\partial \varepsilon}{\partial \rho} \rho' + \varepsilon v(\cos \theta) \right), \quad (2.12)$$

$$N' = \frac{\partial N_0}{\partial \mathcal{E}} \left(\frac{\partial \mathcal{E}}{\partial \rho} \rho' - \mathcal{E} \frac{T_r'}{T} - P w_r \cos \theta \right) \quad (2.13)$$

($w_r = |\mathbf{v}_{nr} - \mathbf{v}_s|$). We shall seek the unknown function $\nu(\cos \theta)$ in the form

$$\nu(\cos \theta) = \sum_{i=0}^{\infty} v_i P_i(\cos \theta)$$

($P_i(\cos \theta)$ are spherical harmonics).

We substitute (2.12) and (2.13) in Eqs. (2.10) and (2.11) and integrate over all \mathbf{p} and \mathbf{P} space:

$$-\tilde{\omega} \rho' + \left(v_s + \frac{\rho_n}{\rho} w_r \right) - \frac{\rho_{np}}{\rho} (v_1 + w_r) = 0, \quad (2.14)$$

$$-\tilde{\omega} v_s + \frac{\rho}{c^2} \left(\frac{\partial \mu}{\partial \rho} \right)_T \rho' + \frac{T}{c^2} \left(\frac{\partial \mu}{\partial T} \right)_\rho T_r'$$

$$-3u \frac{\rho_{np}}{\rho} (v_0 + T_r') = 0. \quad (2.15)$$

Here ρ_{np} is the phonon part of the normal density, μ is the chemical potential and ν_0 and ν_1 are coefficients for the zero and first harmonics in the expansion of $\nu(\cos \theta)$ in spherical harmonics. In Eqs. (2.14) and (2.15), we have introduced the notation $\tilde{\omega} = \omega/kc$ and transformed to nondimensional quantities.⁷⁾

The collision integral $J(n)$ on the right side of Eq. (2.8) consists of four components: $J_{pr}(n)$, due to scattering of phonons by rotons, $J_{pp}(n)$ due to phonon-phonon scattering, $J'_{pp}(n)$ due to the specific phonon-phonon scattering at small angles, and finally $J_{3 \rightarrow 2}(n)$ which characterizes the change in the number of phonons which occurs by way of the five-phonon process.

The total energy of the phonons moving in a given direction, for a five-phonon process and for scattering of phonons by phonons at small angles is conserved. We integrate the left and right sides of Eq. (2.8) [with account of (2.12)] over all possible energies and divide by

$$\int \frac{\partial n_0}{\partial \varepsilon} \varepsilon^2 p^2 dp.$$

⁷⁾Here we shall everywhere denote by ρ' and T_r' the ratios of the deviations of the liquid density and the temperature of the roton gas, respectively, from ρ_0 and T_0 , and by \mathbf{v}_s and w_r the ratios of the velocity of the superfluid part of the liquid and the relative velocity of the roton gas to c .

In accord with the above, the integrals

$$\int J_{pp'}(n) \varepsilon p^2 dp, \quad \int J_{3 \rightarrow 2}(n) \varepsilon p^2 dp$$

are equal to zero, and we have

$$\begin{aligned} & (\tilde{\omega} - \cos \theta) v(\cos \theta) + \tilde{\omega} \rho' + \cos^2 \theta v_s \\ &= -\frac{\tilde{\omega}}{i\tilde{\omega}} \left(\int J_{pr}(n) \varepsilon p^2 dp \left| \int \frac{\partial n_0}{\partial \varepsilon} \varepsilon^2 p^2 dp \right. \right. \\ & \left. \left. + \int J_{pp}(n) \varepsilon p^2 dp \left| \int \frac{\partial n_0}{\partial \varepsilon} \varepsilon^2 p^2 dp \right. \right). \end{aligned} \quad (2.16)$$

The collision integral $J(N)$ on the right side of Eq. (2.9) consists of two parts: $J_{rp}(N)$ which is due to scattering of rotons by phonons, and $J_{rr}(N)$ due to roton-roton scattering. We multiply the left and right sides of Eq. (2.9) first by \mathcal{E} and then by \mathbf{P} , and integrate in both cases with account of (2.13) over all \mathbf{P} space.

Inasmuch as

$$\begin{aligned} \int J_{rp}(N) \mathcal{E} d\tau_p &= -\int J_{pr}(n) \varepsilon d\tau_p, \\ \int J_{rp}(N) \mathbf{P} d\tau_p &= -\int J_{pr}(n) \mathbf{p} d\tau_p, \end{aligned}$$

in accordance with the laws of conservation of energy and momentum, and

$$\int J_{rr}(N) \mathcal{E} d\tau_p = 0, \quad \int J_{rr}(N) \mathbf{P} d\tau_p = 0,$$

we have⁸⁾

$$\begin{aligned} & -\tilde{\omega} T \left(\frac{\partial S_r}{\partial T} \right)_\varepsilon T_r' - \tilde{\omega} \rho \left(\frac{\partial S_r}{\partial \rho} \right)_T \rho' + S_r (w_r + v_s) \\ &= -\frac{\tilde{\omega}}{i\tilde{\omega} T} \int J_{pr}(n) \varepsilon d\tau_p, \end{aligned} \quad (2.17)$$

$$-\tilde{\omega} \rho_{nr} w_r + \frac{T S_r}{c^2} T_r' = -\frac{\tilde{\omega}}{i\omega c} \int J_{pr}(n) p \cos \theta d\tau_p, \quad (2.18)$$

where ρ_{nr} is the roton part of the normal density and S_r is the entropy density of the roton gas.

We now calculate the collision integrals $J_{pr}(n)$ and $J_{pp}(n)$ entering in Eqs. (2.16)–(2.18):

$$J_{pr}(n) = -\int c d\sigma_{pr} \{nN(n^* + 1) - n^*N^*(n + 1)\} d\tau_p. \quad (2.19)$$

Here the quantities with the asterisks refer to scattered phonons and rotons, and $d\sigma_{pr}$ is the differential effective scattering cross section of a phonon by a roton, determined by Eq. (1.9).

Substituting n , n^* , N , and N^* into (2.19) expressed in the form of sums of the constant equilibrium values and the additions (2.12) and (2.13),

and keeping only terms that are linear in ν , ρ' , T_r' , and w_r , we get

$$\begin{aligned} J_{pr}(n) &= -\int c d\sigma_{pr} N_0 \frac{\partial n_0}{\partial \varepsilon} \{ \varepsilon [v(\cos \theta) - v(\cos \theta^*)] \\ & \quad + w_r (\cos \theta - \cos \theta^*) \} + (\varepsilon - \varepsilon^*) [v(\cos \theta^*) + T_r'] d\tau_p. \end{aligned} \quad (2.20)$$

The quantity $\varepsilon - \varepsilon^*$ is small in comparison with the energy ε . However, the calculation of this difference in the collision integral (2.20) is seen to be important for establishing the energy equilibrium between the phonon and roton gases. As we shall see below, the relative slowness of this latter process leads to an appreciable dissipation and dispersion of sound and plays a fundamental role in all subsequent considerations. It is natural in this case that we shall consider the difference $\varepsilon - \varepsilon^*$ only in those terms in the collision integral (2.20) which correspond to the establishment of the energy equilibrium mentioned.

Calculation of the collision integral (2.20) with $d\sigma_{pr}$ from (1.9) leads to a very cumbersome expression which contains the first six harmonics of the function $\nu(\cos \theta^*)$. However, there is hardly any need for such a calculation, since all the harmonics involved except the first make a trivial contribution to the final result. Therefore, without great error in accuracy, one can substitute in (2.20) the roton-roton scattering cross section (1.9) averaged over the angles of the scattering particles.⁹⁾ The collision integral (2.20) after substitution of $d\sigma_{pr}$ [which is determined by (1.10)] and $\varepsilon - \varepsilon^*$ from (1.7) and integration over all \mathbf{P} space and over all the scattering angles of the phonon, is equal to

$$\begin{aligned} J_{pr}(n) &= -\frac{1}{\tau_{pr}(p)} \frac{\partial n_0}{\partial \varepsilon} \left\{ \varepsilon [v(\cos \theta) - v_0 + \cos \theta w_r] \right. \\ & \quad \left. + \frac{\varepsilon^2}{3\mu c^2} (v_0 + T_r') \right\}, \\ \frac{1}{\tau_{pr}(p)} &= 4\pi c N_r \left(\frac{P_0 p^2}{4\pi \hbar^2 \rho c} \right)^2 \left[\frac{2}{9} + \frac{1}{25} \left(\frac{P_0}{\mu c} \right)^2 \right. \\ & \quad \left. + \frac{2A}{9} \frac{P_0}{\mu c} + A^2 \right]. \end{aligned} \quad (2.21)$$

⁹⁾We also note that a more exact calculation would not have any meaning, for even in the derivation of the formula for the matrix element of the transition H_{AF}' in the second order perturbation theory [Eq. (18)] a number of terms comparable with this effect, which can make a contribution to the higher harmonics, were neglected. Thus, for example, in the derivation of (1.8) terms were omitted which contain $P - P_0$. Moreover, the previously mentioned inequality $\varepsilon - \varepsilon^* < 3\mu c^2$ is satisfied with small margin, and this also limits the accuracy of the calculations.

⁸⁾The values of the integrals encountered in (2.14), (2.15), (2.17), and (2.18) are given in the Appendix.

With account of (2.21) the integrals containing $J_{\text{pr}}(n)$ on the right sides of Eqs. (2.16)–(2.18) are easily computed and are equal to

$$\int J_{\text{pr}}(n) \varepsilon p^2 dp \left/ \int \frac{\partial n_0}{\partial \varepsilon} \varepsilon^2 p^2 dp = -\frac{1}{\tau_{\text{pr}}} [\nu(\cos \theta) - \nu_0 + \cos \theta w_r + \beta(\nu_0 + T_r')], \right. \quad (2.22)$$

$$\frac{1}{T} \int J_{\text{pr}}(n) \varepsilon d\tau_p = \frac{1}{\tau_{\text{pr}}} \beta C_p(\nu_0 + T_r'),$$

$$\frac{1}{c} \int J_{\text{pr}}(n) p \cos \theta d\tau_p = \frac{1}{\tau_{\text{pr}}} \rho_{\text{np}}(\nu_1 + w_p),$$

where $\beta = 3kT/\mu c^2$ and τ_{pr} is the characteristic time for the process of scattering of phonons by rotons:

$$\frac{1}{\tau_{\text{pr}}} = \frac{4\pi^3}{c} N_r \left[\frac{P_0(kT/c)^2}{\rho \hbar^2} \right]^2 \left[\frac{2}{9} + \frac{1}{25} \left(\frac{P_0}{\mu c} \right)^2 + \frac{2A}{9} \frac{P_0}{\mu c} + A^2 \right]. \quad (2.23)$$

Finally, we calculate $\int J_{\text{pp}}(n) \varepsilon p^2 dp$, which turns out to be convergent without account of dispersion of the phonons:

$$\int J_{\text{pp}}(n) \varepsilon p^2 dp = - \int c d\sigma_{\text{pp}} \{ n n_1 (n^* + 1) (n_1^* + 1) - n^* n_1^* (n + 1) (n_1 + 1) \} \varepsilon p^2 dp d\tau_{\text{p}_1}. \quad (2.24)$$

Here the quantities with asterisks refer to scattered phonons and $d\sigma_{\text{pp}}$ is the differential effective scattering cross section of a phonon with momentum \mathbf{p}_1 by a phonon with momentum \mathbf{p} , Eq. (1.6).

Substituting in (2.24) the distribution functions of the incident and scattered phonons, expressed in the form of sums of the constant equilibrium values and the additions of (2.12), and keeping only terms linear in ν and ρ' , we get

$$\int J_{\text{pp}}(n) \varepsilon p^2 dp = \int c d\sigma_{\text{pp}} n_0 n_{01} (n_0^* + 1) (n_{01}^* + 1) \times \{ (\varepsilon + \varepsilon_1 - \varepsilon^* - \varepsilon_1^*) u \rho' + \varepsilon \nu(\cos \theta) + \varepsilon_1 \nu(\cos \theta_1) - \varepsilon^* \nu(\cos \theta^*) - \varepsilon_1^* \nu(\cos \theta_1^*) \} \frac{\varepsilon}{kT} p^2 dp d\tau_{\text{p}_1}. \quad (2.25)$$

As a consequence of the laws of conservation of momentum and energy, the terms with ν_0 and ν_1 in the curly brackets of (2.25) are cancelled, as a result of which the spherical harmonics remain, beginning with the second. As we shall see below, the considered process of phonon-phonon scattering plays a role only at low temperatures (below 0.9°K) and in order not to complicate the calculations, we alter the integral (2.28) somewhat, writing it in the form

$$\int J_{\text{pp}}(n) \varepsilon p^2 dp = -\frac{1}{\tau_{\text{pp}}} [\nu(\cos \theta) - \nu_0 - \nu_1 \cos \theta] \int \frac{\partial n_0}{\partial \varepsilon} \varepsilon^2 p^2 dp, \quad (2.26)$$

where the time τ_{pp} is computed accurate to the second harmonic^[12]

$$\frac{1}{\tau_{\text{pp}}} = \frac{9(13)!(u+1)^4}{2^{13}(2\pi\hbar)^7 \rho^2 c} \left(\frac{kT}{c} \right)^9. \quad (2.27)$$

The inappreciable error which we have introduced here does not change the picture qualitatively.

It is seen from drawing (a) that at comparatively high temperatures (above 0.9°K) the phonon-phonon scattering does not have to be considered in comparison with the scattering of phonons by rotons, inasmuch as $\tau_{\text{pp}} \gg \tau_{\text{pr}}$. In the temperature range from 0.6 to 0.9°K, $\tau_{\text{pp}} \sim \tau_{\text{pr}}$ and both scattering processes are important. Finally, at very low temperatures (below 0.6°K) $\tau_{\text{pp}} \ll \tau_{\text{pr}}$ and in this case, only the effect of phonon-phonon scattering is important.

We shall first consider the most interesting temperature region from 0.9 to 1.2°K. In this region of temperatures the second term on the right side of Eq. (2.16) is much smaller in magnitude than the first and it can be neglected. Equations (2.14)–(2.18), after some transformations and substitution in them of the integrals (2.22), can be represented in the form

$$-\tilde{\omega} T_r' + \frac{\rho_s}{\rho} \frac{S_r}{C_r} w_r + \frac{C_p}{C_r} \beta (\tilde{\omega} - \tilde{z}_{\text{pr}}) (\nu_0 + T_r') - \frac{S_p}{C_r} \tilde{\omega} (3u + 1 + \delta) \rho' + \frac{S_r}{C_r} \frac{\rho_{\text{np}}}{\rho} (\nu_1 + w_r) = 0, \quad (2.28)$$

$$-\tilde{\omega} w_r + \frac{T S_r}{\rho_{\text{nr}} c^2} T_r' + \frac{\rho_{\text{np}}}{\rho_{\text{np}}} (\tilde{\omega} - \tilde{z}_{\text{pr}}) (\nu_1 + w_r) = 0, \quad (2.29)$$

$$-\tilde{\omega} \rho' + j_r - \frac{\rho_{\text{np}}}{\rho} (\nu_1 + w_r) = 0, \quad (2.30)$$

$$-\tilde{\omega} j_r + \frac{1}{c^2} \left(\frac{\partial \mathcal{P}}{\partial \rho} \right)_T \rho' + \frac{\rho_{\text{np}}}{\rho} \{ (\tilde{\omega} - \tilde{z}_{\text{pr}}) (\nu_1 + w_r) - 3u(\nu_0 + T_r') - (1 + \delta) T_r' \} = 0, \quad (2.31)$$

$$-(\tilde{z}_{\text{pr}} - \cos \theta) \nu(\cos \theta) = \tilde{\omega} u \rho' + \cos^2 \theta \left(j_r - \frac{\rho_n}{\rho} w_r \right) + (\tilde{\omega} - \tilde{z}_{\text{pr}}) [\nu_0 - \cos \theta w_r + \beta(\nu_0 + T_r')]. \quad (2.32)$$

We have introduced the notation¹⁰⁾ \mathcal{P} = pressure,

$$j_r = \frac{\rho_s}{\rho} v_s + \frac{\rho_{\text{nr}}}{\rho} v_{\text{nr}}, \quad C_r = T \left(\frac{\partial S_r}{\partial T} \right)_\rho,$$

$$\tilde{z}_{\text{pr}} = \tilde{\omega} \left(1 - \frac{1}{i\omega\tau_{\text{pr}}} \right),$$

$$\delta = \frac{T}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p \left/ \left(\frac{\rho_{\text{np}}}{\rho} \right) \right. = -3u - 1$$

¹⁰⁾In accord with the latest neutronographic data,^[15] $(\rho/\mu)\partial\mu/\partial\rho$, is approximately equal to -2.1.

$$-\frac{S_r}{S_p} \left\{ 1 - 2 \frac{\rho}{P_0} \frac{\partial P_0}{\partial \rho} - \frac{1}{2} \frac{\rho}{\mu} \frac{\partial \mu}{\partial \rho} + \frac{\Delta}{T} \frac{\rho}{\Delta} \frac{\partial \Delta}{\partial \rho} \frac{\Delta/T + 1/2}{\Delta/T + 3/2} \right\}.$$

We multiply the left and right sides of Eq. (2.32) first by unity and then by $(\tilde{z}_{\text{pr}} - \cos \theta)^{-1}$ and average in each case over $\cos \theta$. The resulting equations

$$3\tilde{\omega}u\rho' - 3\tilde{\omega}T_r' + j_r + \frac{\rho_s}{\rho} w_r + 3[\tilde{\omega} - \beta(\tilde{\omega} - \tilde{z}_{\text{pr}})] \times (v_0 + T_r') - (v_1 + w_r) = 0, \quad (2.33)$$

$$[2 + (1 - \beta)(\tilde{\omega} - \tilde{z}_{\text{pr}}) \ln \tilde{a}] (v_0 + T_r') + \tilde{\omega} u \ln \tilde{a} \rho' - [2 + (\tilde{\omega} - \tilde{z}_{\text{pr}}) \ln \tilde{a}] T_r' + \left[\tilde{z}_{\text{pr}} j_r - \left(\tilde{\omega} - \frac{\rho_s}{\rho} \tilde{z}_{\text{pr}} \right) \right] (-2 + \tilde{z}_{\text{pr}} \ln \tilde{a}) = 0,$$

$$\tilde{a} = \frac{\tilde{z}_{\text{pr}} + 1}{\tilde{z}_{\text{pr}} - 1}, \quad (2.34)$$

together with (2.28)–(2.31) represent the complete set of six linear homogeneous equations. Equations (2.28)–(2.31) contain the very small quantity $\rho_{\text{np}}/\rho = 1.28 \times 10^{-4} T^4$. We keep terms of small order not higher than ρ_{np}/ρ . In this approximation, Eqs. (2.30) and (2.31) describe the propagation of first sound, while Eqs. (2.28)–(2.29) describe second sound. The last pair [(2.33) and (2.34)] give us a dispersion equation (in zero approximation)

$$2 + (1 - \beta)(\omega - z_{\text{pr}}) \ln a = 0,$$

which for $T > 0.6^\circ\text{K}$ does not have undamped acoustic solutions. At temperatures below 0.6°K , Eqs. (2.28), (2.29) are invalid (the rotons are practically absent, while Eqs. (2.33) and (2.34), made up from terms containing $J_{\text{pp}}(n)$ [see Eqs. (2.16)], as we shall see below, describe undamped temperature waves in a phonon gas (second sound).

We note that in the derivation of Eqs. (2.28)–(2.31), (2.33) and (2.34), we have everywhere assumed the energy of phonon to be equal to $\epsilon = cp$. As simple calculations show, the term containing γp^2 in the expression for ϵ (1.2) can be neglected if the following condition is satisfied:

$$\frac{1}{\omega \tau_{\text{pr}}} \left(1 + \frac{\tau_{\text{pr}}}{\tau_{\text{pp}}} \right) \gg 3\gamma \left(2\pi \frac{kT}{c} \right)^2 \frac{B_3}{B_2}, \quad (2.35)$$

where B_2 and B_3 are Bernoulli numbers: $B_2 = 1/30$, $B_3 = 1/42$. For the frequency region under consideration ($\omega \tau_{\text{pr}}, \omega \tau_{\text{pp}} \ll 1$) the condition (2.35) is virtually always satisfied for temperatures above 0.6°K .

APPENDIX

The values of the integrals encountered in (2.14), (2.15), (2.17) and (2.18) are given below:

$$\int \frac{\partial N_0}{\partial \mathcal{E}} \mathcal{E}^2 d\tau_{\mathbf{p}} = -T \left(\frac{\partial}{\partial T} \int \mathcal{E} N_0 d\tau_{\mathbf{p}} \right)_{\rho} = -T^2 \left(\frac{\partial S_r}{\partial T} \right)_{\rho},$$

$$\int \frac{\partial N_0}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \rho} \mathcal{E} d\tau_{\mathbf{p}} = -T \left(\frac{\partial}{\partial T} \int N_0 \frac{\partial \mathcal{E}}{\partial \rho} d\tau_{\mathbf{p}} \right)_{\rho} = -T \left(\frac{\partial \mu_r}{\partial T} \right)_{\rho},$$

$$= T \left(\frac{\partial S_r}{\partial \rho} \right)_T,$$

$$\frac{1}{3} \int \frac{\partial N_0}{\partial \mathcal{E}} \mathcal{E} P V d\tau_{\mathbf{p}} = -T S_r, \quad \frac{1}{3} \int \frac{\partial N_0}{\partial \mathcal{E}} P^2 d\tau_{\mathbf{p}} = -\rho_{\text{np}},$$

$$\frac{1}{3} \int \frac{\partial n_0}{\partial \epsilon} p^2 d\tau_{\mathbf{p}} = -\rho_{\text{np}},$$

$$\frac{c^2}{\rho} + \int \frac{\partial^2 \epsilon}{\partial \rho^2} n_0 d\tau_{\mathbf{p}} + \int \frac{\partial n_0}{\partial \epsilon} \left(\frac{\partial \epsilon}{\partial \rho} \right)^2 d\tau_{\mathbf{p}} + \int \frac{\partial^2 \mathcal{E}}{\partial \rho^2} N_0 d\tau_{\mathbf{p}} + \int \frac{\partial N_0}{\partial \mathcal{E}} \left(\frac{\partial \mathcal{E}}{\partial \rho} \right)^2 d\tau_{\mathbf{p}} = \left[\frac{\partial}{\partial \rho} \left(\mu_0 + \int \frac{\partial \epsilon}{\partial \rho} n_0 d\tau_{\mathbf{p}} + \int \frac{\partial \mathcal{E}}{\partial \rho} N_0 d\tau_{\mathbf{p}} \right) \right]_T = \left(\frac{\partial \mu}{\partial \rho} \right)_T = \frac{1}{\rho} \left(\frac{\partial \mathcal{P}}{\partial \rho} \right)_T.$$

¹I. M. Khalatnikov, JETP 23, 21 (1952).

²I. M. Khalatnikov, JETP 20, 243 (1950).

³A. F. Andreev and I. M. Khalatnikov, JETP 44, 2058 (1963), Soviet Phys. JETP 17, 1384 (1963).

⁴I. M. Khalatnikov, JETP 44, 769 (1963), Soviet Phys. JETP 17, 519 (1963).

⁵K. Atkins and C. Chase, Proc. Roy. Soc. (London) 64A, 826 (1951).

⁶C. Chase and M. Herlin, Phys. Rev. 97, 1447 (1955).

⁷W. M. Whitney and C. E. Chase, Phys. Rev. Letts. 9, 243 (1962).

⁸H. A. Notarys and J. R. Pellam, IX International Conference on Low Temperature Physics, 1964.

⁹J. R. Pellam and C. Squire, Phys. Rev. 72, 1245 (1947).

¹⁰W. B. Hanson and J. R. Pellam, Phys. Rev. 95, 321 (1954).

¹¹K. R. Atkins and K. H. Hart, Canad. J. Phys. 32, 381 (1954).

¹²É. L. Andronikashvili, JETP 18, 429 (1948).

¹³L. D. Landau and I. M. Khalatnikov, JETP 19, 709 (1949).

¹⁴L. D. Landau and I. M. Khalatnikov, JETP 19, 637 (1949).

¹⁵D. V. Henshaw, Phys. Rev. Letts. 1, 127 (1958). D. V. Henshaw and A. D. B. Woods, Phys. Rev. 121, 1266 (1961). Proc. IV International Conference on Low Temperature Physics, Toronto, 1960.

¹⁶K. R. Atkins and R. A. Stasiar, Canad. J. Phys. 31, 1156 (1953).

¹⁷W. Keesom and Miss A. Keesom, Physica 1, 128 (1933).

¹⁸K. R. Atkins and M. H. Edwards, Phys. Rev. 97, 1429 (1955).

¹⁹J. deBoer, in the collection Phonons and

Phonon Interactions, ed. T. A. Bak, New York and Amsterdam, 1965, p. 395.

²⁰A. de Troyer, A. Van Itterbeek and A. van den Berg, *Physica* **17**, 50 (1951).

²¹A. C. Hollis-Hallet, *Proc. Roy. Soc. (London)* **210**, 404 (1952).

²²W. J. Heikkila and A. C. Hollis-Hallet, *Canad. J. Phys.* **33**, 420 (1955).

²³K. N. Zinov'eva, *JETP* **31**, 31 (1956), *Soviet Phys. JETP* **4**, 36 (1957).

²⁴I. M. Khalatnikov and D. M. Chernikova, *JETP* **50**, 411 (1966), *Soviet Phys. JETP* **23**, in press.

²⁵L. D. Landau, *JETP* **11**, 592 (1941).

²⁶I. M. Khalatnikov, *JETP* **23**, 8 (1952).

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252