PLASMA INSTABILITY IN A STRONG HIGH-FREQUENCY FIELD

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It is shown that a collisionless plasma in a strong external high-frequency field is unstable against the excitation of solenoidal oscillations. This instability is possible when the frequency of the external field is either lower or higher than the electron plasma frequency; this feature distinguishes it qualitatively from instabilities associated with irrotational oscillations.[1] The instability considered here arises when the amplitude of the electron oscillations in the external field is greater than the Debye radius.

1. In the present work we consider the stability of a plasma in a strong high-frequency field with respect to the excitation of solenoidal oscillations. It has been shown earlier[1] that irrotational oscillations can be excited when the field frequency ω_0 is of the order of the electron plasma frequency $\omega_{Le} = (4\pi N_e e^2/m)^{1/2}$ or lower. It will be shown below that solenoidal oscillations can be excited when the frequency of the external field is higher than ω_{Le}. Everywhere below collisions are neglected entirely. This means that the growth rates we obtain are large compared with the collision frequency.

The present problem arises in connection with the possibility of using radiative methods for acceleration of a transparent plasma.[2] The problem treated below is similar, in certain ways, to one that has been investigated earlier by Volkov.[3] Strictly speaking, however, the regions investigated in [3] and those studied here are different.

As in [1,4] we limit our analysis to the case of perturbations with wavelengths appreciably smaller than the characteristic dimensions of the inhomogeneity of the plasma and of the external field. Under these conditions we assume everywhere below that $\omega_0 \gg \omega_{Le} v_{Te}/c$ which means that the penetration of the external field is characterized by a normal skin effect. Then we can assume that the external field and the particle distribution in the unperturbed state are independent of coordinates and neglect the high-frequency magnetic field. The dependence of the electric field on time is written in the form $\mathbf{E}_0(t) = \mathbf{E}_0 \sin \omega_0 t$. The particle distribution function in the ground state, which is assumed to be isotropic in the coordinate system fixed in the particles, is given by

$$f_{\alpha 0}\left(\left|\mathbf{p}_\alpha - e_\alpha \int_{-\infty}^{t} dt' \mathbf{E}_0(t')\right|\right) = f_{\alpha 0}\left(\left|\mathbf{p}_\alpha + \frac{e_\alpha \mathbf{E}_0}{\omega_0}\cos\omega_0 t\right|\right).$$

Solving the kinetic equation for small deviations from this equilibrium state and writing the nonequilibrium electric field as a series

$$\delta\mathbf{E}(\mathbf{r}, t) = e^{i\mathbf{k}\mathbf{r} - i\omega t} \sum_{n=-\infty}^{\infty} \delta\mathbf{E}^{(n)} e^{-in\omega_0 t}, \tag{1}$$

we obtain the following expression for the perturbed distribution function:

$$\delta f_\alpha(\mathbf{p}, \mathbf{r}, t) = \exp(i\mathbf{k}\mathbf{r} + ia_\alpha \sin\omega_0 t)\Psi_\alpha\Big(\mathbf{p} - e_\alpha \int_{-\infty}^{t} dt' \mathbf{E}_0(t'),\ t\Big),$$

$$a_\alpha = e_\alpha \mathbf{E}_0\mathbf{k} / m_\alpha \omega_0^2, \qquad \Psi_\alpha(\mathbf{p}, t) = \sum_{n=-\infty}^{\infty} \Psi_\alpha^{(n)}(\mathbf{p}) e^{-i(\omega + n\omega_0)t},$$

$$\Psi_\alpha^{(n)}(\mathbf{p}) = -\frac{ie_\alpha}{n\omega_0 + \omega - \mathbf{k}\mathbf{v}} \frac{\partial f_{\alpha 0}}{\partial p_i} \sum_{l=-\infty}^{\infty} J_{n-l}(a_\alpha)$$
$$\times \left\{\delta_{ij} + \frac{(n-l)\omega_0}{l\omega_0 + \omega}\left[\delta_{ij} - \frac{k_i E_{0j}}{\mathbf{k}\mathbf{E}_0}\right]\right\} \delta E_j^{(l)}. \tag{2}$$

Here, $J_n(a)$ is the Bessel function of order n.

Using (2) to find the current density and substituting it in Maxwell's equations we obtain the following equation for the field in the plasma:

$$\{c^2(k_i k_j - k^2\delta_{ij}) + (\omega + n\omega_0)^2\delta_{ij}\}\delta E_j^{(n)}$$
$$+ \sum_{m, r=-\infty}^{\infty} \sum_\alpha \frac{n\omega_0 + \omega}{m\omega_0 + \omega} \delta E_j^{(m)}$$
$$\times J_{r-n}(a_\alpha) J_{r-m}(a_\alpha) \Big\{(\omega + r\omega_0)^2 \delta\varepsilon_\alpha^{tr}(\omega + r\omega_0, k)$$
$$\times \left[\delta_{ij} - \frac{k_i k_j}{k^2}\right] + \delta\varepsilon_\alpha^{l}(\omega + r\omega_0, k)\Big[\frac{\mathbf{k}}{k}(n\omega_0 + \omega)$$
$$+ \frac{(r-n)\omega_0}{k(\mathbf{k}\mathbf{E}_0)}[\mathbf{k}[\mathbf{k}\mathbf{E}_0]]\Big]_i \Big[\frac{\mathbf{k}}{k}(m\omega_0 + \omega)$$

$$+\frac{(r-m)\omega_0}{k(\mathbf{kE}_0)}[\mathbf{k}[\mathbf{kE}_0]]\Big]_j\Big\}=0, \qquad (3)^*$$

$$\delta\varepsilon_\alpha{}^l(\omega,k)=\frac{4\pi e_\alpha^2}{k^2}\int d\mathbf{p}\frac{1}{\omega+i0-\mathbf{kv}}\mathbf{k}\frac{\partial f_{\alpha0}}{\partial\mathbf{p}}, \qquad (4)$$

$$\delta\varepsilon_\alpha{}^{tr}(\omega,k)=\frac{2\pi e_\alpha^2}{\omega k^2}\int d\mathbf{p}\frac{[\mathbf{k}[\mathbf{vk}]]}{\omega+i0-\mathbf{kv}}\frac{\partial f_{\alpha0}}{\partial\mathbf{p}}. \qquad (5)$$

2. From the field equations (3) for the irrotational oscillations there follows immediately the result obtained by one of the authors [1] in which it has been shown, in particular, that the external electric field is important for irrotational oscillations only when it is not perpendicular to the wave vector. Therefore, we will be primarily interested in the case of transverse propagation in which the wave vector $\mathbf{k}$ is perpendicular to $\mathbf{E}_0$.

Under the assumption of transverse propagation the field equations (3) assume the form

$$\Big\{c^2(k_ik_j-k^2\delta_{ij})+(n\omega_0+\omega)^2\Big[\Big(\delta_{ij}-\frac{k_ik_j}{k^2}\Big)\varepsilon^{tr}(n\omega_0+\omega,k)$$
$$+\frac{k_ik_j}{k^2}\varepsilon^l(n\omega_0+\omega,k)\Big]\Big\}\frac{\delta E_j{}^{(n)}}{n\omega_0+\omega}$$
$$-(n\omega_0+\omega)k_iE_{0j}\Big[\frac{\delta E_j{}^{(n-1)}}{(n-1)\omega_0+\omega}+\frac{\delta E_j{}^{(n+1)}}{(n+1)\omega_0+\omega}\Big]$$
$$\times\sum_\alpha\frac{e_\alpha}{2m_\alpha\omega_0}\delta\varepsilon_\alpha{}^l(n\omega_0+\omega,k)$$
$$-k_jE_{0i}\sum_\alpha\frac{e_\alpha}{2m_\alpha\omega_0}\{\delta\varepsilon_\alpha{}^l([n+1]\omega_0+\omega,k)\,\delta E_j{}^{(n+1)}$$
$$+\delta\varepsilon_\alpha{}^l([n-1]\omega_0+\omega,k)\,\delta E_j{}^{(n-1)}\}$$
$$+E_{0i}E_{0j}\sum_\alpha\frac{e_\alpha{}^2k^2}{4m_\alpha{}^2\omega_0{}^2}\Big\{\delta\varepsilon_\alpha{}^l([n+1]\omega_0+\omega,k)\Big[\frac{\delta E_j{}^{(n)}}{n\omega_0+\omega}$$
$$+\frac{\delta E_j{}^{(n+2)}}{(n+2)\omega_0+\omega}\Big]+\delta\varepsilon_\alpha{}^l([n-1]\omega_0+\omega,k)$$
$$\times\Big[\frac{\delta E_j{}^{(n-2)}}{(n-2)\omega_0+\omega}+\frac{\delta E_j{}^{(n)}}{n\omega_0+\omega}\Big]\Big\}=0,$$

$$\varepsilon^l=1+\sum_\alpha\delta\varepsilon_\alpha{}^l, \qquad \varepsilon^{tr}=1+\sum_\alpha\delta\varepsilon_\alpha{}^{tr}. \qquad (6)$$

It follows from Eq. (6) that oscillations for which the vector $\delta\mathbf{E}$ is perpendicular to the plane formed by $\mathbf{k}$ and $\mathbf{E}_0$ are stable. Hence we consider the case in which the electric vector for the wave $\delta\mathbf{E}$ lies in the plane of $\mathbf{k}$ and $\mathbf{E}_0$. Using Eq. (6) and neglecting quantities of the order of the electron-ion mass ratio we find

$$\{(n\omega_0+\omega)^2\varepsilon^{tr}(n\omega_0+\omega,k)-c^2k^2\}\varphi^{(n)}$$
$$+{}^1/_4\beta^2c^2k^2\Big\{[\varphi^{(n)}+\varphi^{(n+2)}]$$
$$\times\delta\varepsilon_e{}^l([n+1]\omega_0+\omega,k)\Big[1-\frac{\delta\varepsilon_e{}^l([n+1]\omega_0+\omega,k)}{\varepsilon^l([n+1]\omega_0+\omega,k)}\Big]$$
$$+[\varphi^{(n)}+\varphi^{(n-2)}]\,\delta\varepsilon_e{}^l([n-1]\omega_0$$
$$+\omega,k)\Big[1-\frac{\delta\varepsilon_e{}^l([n-1]\omega_0+\omega,k)}{\varepsilon^l([n-1]\omega_0+\omega,k)}\Big]\Big\}=0,$$

$$\beta=\frac{v_E}{c},\qquad \mathbf{v}_E=\frac{e\mathbf{E}_0}{m\omega_0},\qquad \varphi^{(n)}=\frac{\mathbf{E}_0\delta\mathbf{E}^{(n)}}{m\omega_0+\omega}. \qquad (7)$$

The infinite determinant corresponding to this system of equations determines the dispersion relation for the characteristic frequencies ω. The condition that the wavelength of the perturbations be small compared with the characteristic dimensions of the inhomogeneity of the external field implies the inequality $c^2k^2 \gg |\omega_0^2-\omega_{Le}^2|$. Furthermore we keep in mind the fact that the velocity associated with the electron oscillations in the external field v_E is small compared with the velocity of light (i.e., $\beta \ll 1$) so that the dispersion relation corresponding to the field equation (7) can be written in the form

$$1+2\beta^2B_n-\beta^4B_nB_{n+2}\Big[1+2\beta^2B_{n+2}$$
$$-\beta^4B_{n+2}B_{n+4}\frac{1}{1+2\beta^2B_{n+4}+\ldots}\Big]^{-1}-\beta^4B_nB_{n-2}\Big[1$$
$$+2\beta^2B_{n-2}-\beta^4B_{n-2}B_{n-4}\frac{1}{1+2\beta^2B_{n-4}+\ldots}\Big]^{-1}=0, \quad (8)$$

where n is an integer

$$B_n\equiv-\frac{1}{4}\frac{\delta\varepsilon_e{}^l(\omega+n\omega_0,k)}{\varepsilon^l(\omega+n\omega_0,k)}[1+\delta\varepsilon_i{}^l(\omega+n\omega_0,k)].$$

If we are interested in external field frequencies large compared with the electron plasma frequency, to obtain possible spectra of oscillations for transverse propagation we need only consider the first two terms in Eq. (8), writing the dispersion relation in the form

$$1=\frac{1}{2}\beta^2\frac{\delta\varepsilon_e{}^l(i\gamma,k)}{\varepsilon^l(i\gamma,k)}[1+\delta\varepsilon_i{}^l(i\gamma,k)], \qquad (9)$$

where $\omega\equiv-n\omega_0+i\gamma$.

Let $|\gamma|\gg kv_{Te}\gg kv_{Ti}$ where v_{Te} and v_{Ti} are the electron and ion thermal velocities. Using the asymptotic expression $\delta\epsilon_\alpha^l$ which is obtained from Eq. (4) under these conditions, we find that one solution of Eq. (9) has the form

$$\gamma=\pm\frac{1}{\sqrt{2}}\beta\omega_{Li}\equiv\pm\frac{v_E}{c\sqrt{2}}\omega_{Li}\equiv\pm\frac{eE_0}{mc\sqrt{2}}\omega_{Li}\frac{1}{\omega_0}, \quad (10)$$

where ω_{Li} is the ion plasma frequency.

In order to satisfy the inequality $|\gamma|\gg kv_{Te}$ i.e., if the time in which a thermal electron travels a distance of the order of a wavelength is to be

*$(\mathbf{k}\,\mathbf{E}_0)\equiv\mathbf{k}\cdot\mathbf{E}_0$, $[\mathbf{k}\,\mathbf{E}_0]\equiv\mathbf{k}\times\mathbf{E}_0$.

large compared with the growth time γ^{-1}, the following relation must be satisfied:

$$kr_{De} \ll v_E\omega_{Li} / c\omega_{Le}, \tag{11}$$

where $r_{De} = v_{Te}/\omega_{Le}$ is the electron Debye radius. Thus, a plasma in a strong high-frequency electric field can be unstable. We note that a similar instability for a plasma in a static electric field has been studied by Lovetskiĭ and Rukhadze [5]

We now examine Eq. (9) under the assumption that $kv_{Te} \gg |\gamma| \gg kv_{Ti}$. In this case

$$\gamma^2 = \omega_{Li}^2 \frac{\beta^2/2 - (kr_{De})^2}{1+(kr_{De})^2}[1+3(kr_{Di})^2], \tag{12}$$

where $r_{Di} = v_{Ti}/\omega_{Li}$ is the ion Debye radius. In the limit $\beta^2 \ll (kr_{De})^2$ this expression corresponds to the spectrum for ion acoustic oscillations; as is well known, these are excited only when the ion temperature in the plasma is much smaller than the electron temperature. In the opposite limit we obtain Eq. (10) from Eq. (12). In this case the following relation must be satisfied in place of (11):

$$\frac{\omega_{Le}}{\omega_{Li}} kr_{De} \gg \frac{v_E}{c} \gg kr_{De}, kr_{Di}. \tag{13}$$

Actually, Eq. (10) also holds when the quantities on the left and right sides of (11) are comparable. However, the right inequality in (13) must be satisfied in any case.

In view of the fact that $k \gg \omega_0/c$ when $\omega_0 \gg \omega_{Le}$ we obtain the following condition for the existence of growing instabilities with a growth rate given by (10):

$$v_E / \omega_0 \gg r_{De}, r_{Di}. \tag{14}$$

In other words, the amplitude of the electron oscillation in the external field must be large compared with the Debye radius. This condition is similar to one that has been obtained earlier [1] for excitation of irrotational oscillations. At external field frequencies smaller than the plasma frequency the analysis holds for wave numbers that satisfy the condition $k \gg \omega_{Le}/c$. Hence, in place of (14) we obtain

$$v_E \gg v_{Te}, \quad v_{Ti}(m_i/m)^{1/2}, \tag{14'}$$

where m_i is the ion mass.

In the limit $|\gamma| \ll kv_{Ti}$ we find from (9)

$$\gamma = \left(\frac{2}{\pi}\right)^{1/2} kv_{Ti}\left\{\frac{\beta^2}{2} - k^2(r_{De}^2 + r_{Di}^2)\right\}\frac{1}{k^2 r_{Di}^2}. \tag{15}$$

This solution corresponds to the possibility of excitation of oscillations with wave numbers given approximately by

$$k = \frac{1}{\sqrt{2}}\frac{\beta}{(r_{De}^2 + r_{Di}^2)^{1/2}}$$

In view of the fact that the wave vector must be large compared with ω_0/c when $\omega_0 \gg \omega_{Le}$ and comparable with ω_{Le}/c when $\omega_0 \ll \omega_{Le}$ the necessity of satisfying (14) or (14′) is evident. Since the possibility of excitation of oscillations with growth rates given in (15) can only occur within a narrow range of wave vectors, we can write (15) in the form

$$\gamma = \beta\omega_{Li}\left(\frac{2}{\pi}\right)^{1/2}\left(1+\frac{r_{De}^2}{r_{Di}^2}\right)^{1/2}\left[1 - 2\frac{k^2(r_{De}^2 + r_{Di}^2)}{\beta^2}\right].$$

The quantity in rectangular brackets is small so that this growth rate is small compared with that given in (10).

Taking account of the inhomogeneity in the external field leads to a correction in Eq. (10) which has a relative order equal to the ratio of the oscillation wavelength to the characteristic dimension of the inhomogeneity of the external field.[1)]

The expressions for the growth rates obtained here (10), (12), and (15) are roughly similar to the results of Makhan'kov and Rukhadze [6] and Sholokhov [7] concerning an instability in which the electrons move uniformly with respect to the ions. This result is to be expected because in our case the instability also derived from the motion of the plasma electrons with respect to the ions. However this motion is oscillatory rather than uniform. We therefore emphasize that the growth rates obtained above are always small compared with the frequency of oscillations of the electrons in the external field.

When $|\gamma| \gg kv_{Te} \gg kv_{Ti}$, in addition to the solution obtained above Eq. (9) yields a stable solution with frequency $i\gamma \approx \omega_{Le}$. However, if one investi-

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[1)] In order to obtain this result in a consistent way we must make a more accurate analysis of the ground state. For example, when nonlinear effects are considered the field in the ground state is given by

$$\mathbf{E} = \mathbf{E}_0 \sin(\omega_0 t - \mathbf{k}_0\mathbf{r}) - \frac{e}{6m} E_0^2 \frac{\mathbf{k}_0}{\omega_0^2}\sin 2(\omega_0 t - \mathbf{k}_0\mathbf{r}),$$

$$\mathbf{B} = \frac{c}{\omega_0}[\mathbf{E}_0\mathbf{k}_0]\sin(\omega_0 t - \mathbf{k}_0\mathbf{r}),$$

where $\mathbf{k}_0$ is a wave vector which is related to the frequency ω_0 by the dispersion formula $k_0^2c^2 = \omega_0^2 - \omega_{Le}^2$. If the plasma particles are described in the hydrodynamic approximation the velocities and number densities for the electrons and ions are given by the following relations:

$$\mathbf{v}_e(t,\mathbf{r}) = -\mathbf{v}_E\cos(\omega_0 t - \mathbf{k}_0\mathbf{r}) + \frac{1}{3}v_E^2\frac{\mathbf{k}_0}{\omega_0}\cos 2(\omega_0 t - \mathbf{k}_0\mathbf{r}),$$

$$\mathbf{v}_i(t,\mathbf{r}) = \frac{e_i}{e}\frac{m}{m_i}\left[-\mathbf{v}_E\cos(\omega_0 t - \mathbf{k}_0\mathbf{r}) + \frac{1}{12}v_E^2\frac{\mathbf{k}_0}{\omega_0}\cos 2(\omega_0 t - \mathbf{k}_0\mathbf{r})\right],$$

$$n_e(t,\mathbf{r}) = n_{e0}\left[1 + \frac{1}{6}\frac{k_0^2}{\omega_0^2}v_E^2\cos 2(\omega_0 t - \mathbf{k}_0\mathbf{r})\right],$$

$$n_i(t,\mathbf{r}) = n_{i0}\left[1 + \frac{1}{36}\frac{e_i}{e}\frac{m}{m_i}v_E^2\frac{k_0^2}{\omega_0^2}\cos 2(\omega_0 t - \mathbf{k}_0\mathbf{r})\right].$$

gates this solution at field frequencies which are of most direct interest ($\omega_0 \sim \omega_{Le}$) it is not sufficient to consider the first two terms in the dispersion relation (8), as we have done in obtaining Eq. (9). It is also necessary to take account of the B_{n+2} and B_{n-2} terms. Carrying out this procedure and introducing the notation $\omega = -(n-1)\omega_0 + i\gamma$ we obtain the following dispersion equation:

$$\begin{aligned}\varepsilon^l(i\gamma + \omega_0, k)\,\varepsilon^l(i\gamma - \omega_0, k) - {}^1/_2\beta^2\{\varepsilon^l(i\gamma - \omega_0, k)\\ \times \delta\varepsilon_e^l(i\gamma + \omega_0, k)\,[1 + \delta\varepsilon_i^l(i\gamma + \omega_0, k)]\\ + \varepsilon^l(i\gamma + \omega_0, k)\,\delta\varepsilon_e^l(i\gamma - \omega_0, k)\,[1 + \delta\varepsilon_i^l(i\gamma - \omega_0, k)]\}\\ + {}^3/_{16}\beta^4\delta\varepsilon_e^l(i\gamma - \omega_0, k)\,[1 + \delta\varepsilon_i^l(i\gamma - \omega_0, k)]\,\delta\varepsilon_e^l(i\gamma\\ + \omega_0, k)\,[1 + \delta\varepsilon_i^l(i\gamma + \omega_0, k)] = 0.\end{aligned}$$

It is evident that a solution of this equation can be obtained when $|\gamma| \ll \omega_0$ if the frequency of the external field is approximately equal to the electron plasma frequency

$$|\omega_0^2 - \omega_{Le}^2 - \omega_{Li}^2| \ll \omega_0^2.$$

In this case we can regard the period of the external field as being smaller than the time in which a thermal electron traverses a distance equal to the wavelength of the perturbation being considered ($\omega_0 \gg kv_{Te}$). Then, using the notation

$$\omega_0^2 - \omega_{Le}^2 - \omega_{Li}^2 \equiv {}^1/_4\beta^2\omega_{Le}^2 x,$$

we find

$$\gamma^2 = -{}^1/_{64}\beta^4\omega_{Le}^2(x^2 + 4x + 3). \tag{16}$$

It then follows that an instability is possible if the following inequality is satisfied:

$$\frac{1}{4} < \frac{\omega_{Le}^2 + \omega_{Li}^2 - \omega_0^2}{\beta^2\omega_{Le}^2} < \frac{3}{4}.$$

The maximum growth rate,

$$\gamma_{max} = \frac{1}{8}\beta^2\omega_{Le} \equiv \frac{1}{8}\frac{v_E^2}{c^2}\omega_{Le} \equiv \frac{1}{8}\frac{e^2E_0^2}{m^2\omega_0^2c^2}\omega_{Le}, \tag{17}$$

obtains when

$$\omega_0^2 = \omega_{Le}^2 + \omega_{Li}^2 - {}^1/_2\beta^2\omega_{Le}^2.$$

We note that for the branch being considered above, in addition to the instability at $\omega_0 \approx \omega_{Le}$ it is also possible to have an instability when the frequency of the external field approaches ω_{Le}/n where n is an integer. In studying the solutions for these values of ω_0 in the dispersion equation (8) we must retain quantities B_i of higher order than those that apply for the case $\omega_0 \sim \omega_{Le}$. However, the growth rates obtained in this analysis are found to be smaller than those in (17). Thus, when $\omega_0 \sim \omega_{Le}/2$ the order of the growth rate $\beta^2\omega_{Li}$ is not only smaller than (17), but is also smaller by a factor β^{-1} than the growth rate in (10).

In contrast with the instabilities that arise when the frequency of the external field approaches the plasma frequency or when there is a resonance between an overtone of the external field and the electron plasma frequency, the instability with the growth rate in (10) obtains for a wide frequency range ($c \gg v_E \gg \omega_0 r_{De}$). This is a qualitative difference between the instability of the plasma against solenoidal oscillations and the case of irrotational oscillations that has been treated earlier; [1] in the latter case the instability does not appear if the frequency of the external field is appreciably greater than the electron Langmuir frequency.

We note that in the limit of small wavenumbers Eq. (10) coincides with Eq. (2.12) given by Volkov, [3] who investigated the effect of a traveling wave on plasma oscillations. In this case one considers oscillations characterized by a wavevector perpendicular to the direction of the external field and the problem is similar to that being considered in the present section. It should be noted that in [3] consideration was given only to external field frequencies greater than the electron plasma frequency; moreover, a single-fluid hydrodynamic model was used and this is valid only when the wavelength of the oscillations is large compared with the mean free path. Volkov [3] has noted that his solution of the problem "cannot be regarded as completely correct because of the neglect of the thermal motion of the particles." On the other hand, our analysis does not have the limitations of the Volkov analysis [3] and is suitable for the object of interest to us, a collisionless plasma. A consistent account of the thermal motion shows that the expression that arises (12) differs from that obtained in [3]. Furthermore, the kinetic spectrum (15) obviously cannot be obtained in a hydrodynamic theory. We note further that the instability with the growth rate given by (17), which obtains in the hydrodynamic limit of a cold plasma, was not noted by Volkov. [3] Finally, all of the analysis in his work is limited to the case of plasma waves propagating along the traveling wave, that is to say, the case of purely transverse propagation. In the following section we consider plasma waves with arbitrary wave vectors.

3. We now consider oscillations propagating at an arbitrary angle with respect to the external electric field, neglecting a_i because the amplitude of the ion oscillations in the external field is small compared with that of the electron oscillations ($a_e \equiv a$). For oscillations whose electric vector lies in the plane formed by the vectors $\mathbf{k}$ and $\mathbf{E}_0$

$$\delta\mathbf{E}^{(n)} = \frac{\mathbf{k}}{k}\delta E_l^{(n)} + \frac{[\mathbf{k}[\mathbf{E}_0\mathbf{k}]]}{k|[\mathbf{E}_0\mathbf{k}]|}\delta E_{tr}^{(n)},$$

from Eq. (3) we have

$$\delta E_l^{(n)}\{1 + \delta\varepsilon_i^l(n\omega_0 + \omega, k)\} + \sum_{m=-\infty}^{\infty}\delta E_l^{(m)}\sum_{r=-\infty}^{\infty}J_{r-n}(a)J_{r-m}(a)$$
$$\times\,\delta\varepsilon_e^l(\omega + r\omega_0, k) + \frac{|[\mathbf{kE}_0]|}{\mathbf{kE}_0}\sum_{m=-\infty}^{\infty}\frac{\delta E_{tr}^{(m)}}{m\omega_0 + \omega}$$
$$\times\sum_{r=-\infty}^{\infty}J_{r-n}(a)J_{r-m}(a)\,\delta\varepsilon_e^l(\omega + r\omega_0, k)\,(m - r)\,\omega_0 = 0,$$

$$[c^2k^2 - (\omega + n\omega_0)^2]\frac{\delta E_{tr}^{(n)}}{n\omega_0 + \omega} - \sum_{m=-\infty}^{\infty}{}'\frac{\delta E_{tr}^{(m)}}{m\omega_0 + \omega}$$
$$\times\sum_{r=-\infty}^{\infty}J_{r-n}(a)J_{r-m}(a)\,(\omega + r\omega_0)^2\,\delta\varepsilon_e^{tr}(\omega + r\omega_0, k)$$
$$= \frac{a\omega_0}{2}\frac{|[\mathbf{kE}_0]|}{\mathbf{kE}_0}\,\{\delta E_l^{(n+1)}[1 + \delta\varepsilon_i^l(\omega + [n+1]\,\omega_0,\ k)]$$
$$+\delta E_l^{(n-1)}[1 + \delta\varepsilon_i^l(\omega + [n-1]\omega_0, k)]\}.$$

Using the fact that ω_0 is small compared with kc, we use the second equation to eliminate $\delta E_{tr}^{(n)}$ from the first. As a result we find

$$\delta E_l^{(n)}[1 + \delta\varepsilon_i^l(\omega + n\omega_0, k)] + \sum_{m,r=-\infty}^{\infty}\delta E_l^{(m)}J_{r-n}(a)$$
$$\times\,J_{r-m}(a)\,\delta\varepsilon_e^l(\omega + r\omega_0, k) - \frac{1}{4}\frac{[\mathbf{kv}_E]^2}{c^2k^2}$$
$$\times\sum_{m,r=-\infty}^{\infty}\delta E_l^{(m)}[1 + \delta\varepsilon_i^l(\omega + m\omega_0, k)]\,\delta\varepsilon_e^l(\omega$$
$$+ r\omega_0, k)J_{r-n}(a)[2J_{r-m}(a) + J_{r-m+2}(a) + J_{r-m-2}(a)] = 0.$$

Multiplying the right side of this equation by $J_{s-n}(a)$ and summing over n we obtain an equation which allows us to eliminate

$$\sum_m \delta E_l^{(m)}J_{r-m}(a).$$

Then the longitudinal field components are described by the following system of equations:

$$\delta E_l^{(n)}[1 + \delta\varepsilon_i^l(\omega + n\omega_0, k)] - \sum_{m,s=-\infty}^{\infty}\delta E_l^{(m)}\delta\varepsilon_i^l(\omega + m\omega_0, k)$$
$$\times\frac{J_{s-n}(a)J_{s-m}(a)}{1 + [\delta\varepsilon_e^l(\omega + s\omega_0, k)]^{-1}} - \frac{1}{4}\frac{[\mathbf{kv}_E]^2}{c^2k^2}$$
$$\times\sum_{m,s=-\infty}^{\infty}\delta E_l^{(m)}[1 + \delta\varepsilon_i^l(\omega + m\omega_0, k)]J_{s-n}(a)$$
$$\times\frac{2J_{s-m}(a) + J_{s-m-2}(a) + J_{s-m+2}(a)}{1 + [\delta\varepsilon_e^l(\omega + s\omega_0, k)]^{-1}} = 0. \qquad (18)$$

In view of the fact that the growth rate (10) is small compared with the ion Langmuir frequency, we shall use this approximation in analyzing (18).

In this case we assume that $\omega_0 \gg \omega_{Li}$ and consequently that $|\delta\epsilon_i^l(\omega_0, k)| \ll 1$ and $|\delta\epsilon_i^l(\omega, k)| \gg 1$. Then, using Eq. (18) we obtain the following approximate relation when $n \neq 0$:

$$\delta E_l^{(n)} \cong \delta E_l^{(0)}\delta\varepsilon_i^l(\omega, k)\sum_{s=-\infty}^{\infty}J_{s-n}(a)J_s(a)\frac{\delta\varepsilon_e^l(s\omega_0 + \omega, k)}{1 + \delta\varepsilon_e^l(s\omega_0 + \omega, k)}.$$

The system of equations in (18) can be used in conjunction with this relation and m = 0 to write the following dispersion relation:

$$-\frac{1}{\delta\varepsilon_i^l(\omega, k)} + \sum_{s=-\infty}^{\infty}\frac{J_s^2(a)}{1 + \delta\varepsilon_e^l(s\omega_0 + \omega, k)}$$
$$= \frac{1}{4}\frac{[\mathbf{kv}_E]^2}{c^2k^2}\sum_{s=-\infty}^{\infty}J_s(a)\frac{\delta\varepsilon_e^l(s\omega_0 + \omega, k)}{1 + \delta\varepsilon_e^l(s\omega_0 + \omega, k)}$$
$$\times\Big\{[2J_s(a) + J_{s+2}(a) + J_{s-2}(a)]$$
$$\times\left[1 - \sum_{r=-\infty}^{\infty}J_r^2(a)\frac{\delta\varepsilon_e^l(r\omega_0 + \omega, k)}{1 + \delta\varepsilon_e^l(r\omega_0 + \omega, k)}\right] + 2J_s(a)$$
$$\times\frac{\delta\varepsilon_e^l(s\omega_0 + \omega, k)}{1 + \delta\varepsilon_e^l(s\omega_0 + \omega, k)}$$
$$+ J_{s+2}(a)\frac{\delta\varepsilon_e^l([s+2]\,\omega_0 + \omega, k)}{1 + \delta\varepsilon_e^l([s+2]\,\omega_0 + \omega, k)}$$
$$+ J_{s-2}(a)\frac{\delta\varepsilon_e^l([s-2]\,\omega_0 + \omega, k)}{1 + \delta\varepsilon_e^l([s-2]\,\omega_0 + \omega, k)}\Big\}. \qquad (19)$$

The expression appearing in the right side of Eq. (19) differs from that obtained in the equation in an earlier work[1] in which we investigated irrotational oscillations. Since this term contains a small factor it has the greatest effect on changing the spectrum of the irrotational oscillations for small values of a. In this case the dispersion equation assumes the form

$$\frac{1}{\delta\varepsilon_i^l(\omega, k)} + \frac{1}{1 + \delta\varepsilon_e^l(\omega, k)} + \frac{1}{2}\frac{a^2\omega_0^2}{\omega_0^2 - \omega_{Le}^2} - \frac{1}{2}\frac{[\mathbf{kv}_E]^2}{k^2c^2}$$
$$\times\frac{\delta\varepsilon_e^l(\omega, k)}{1 + \delta\varepsilon_e^l(\omega, k)} = 0. \qquad (20)$$

We can now obtain the solution of Eq. (20) for different values of the phase velocity:

$$\omega^2 = \omega_{Li}^2\left\{\frac{1}{2}\frac{(\mathbf{kv}_E)^2}{\omega_0^2 - \omega_{Le}^2} - \frac{1}{2}\frac{[\mathbf{kv}_E]^2}{c^2k^2}\right\},\quad |\omega| \gg kv_{Te}; \qquad (21)$$

$$\omega^2 = \omega_{Li}^2\left\{(kr_{De})^2 + \frac{1}{2}\frac{(\mathbf{kv}_E)^2}{\omega_0^2 - \omega_{Le}^2} - \frac{1}{2}\frac{[\mathbf{kv}_E]^2}{c^2k^2}\right\},$$
$$1 \gg kr_{De} \gg \frac{|\omega|}{\omega_{Le}}; \qquad (22)$$

$$i\omega = \left(\frac{2}{\pi}\right)^{1/2}kv_{Ti}\left\{-(kr_{De})^2 - (kr_{Di})^2 + \frac{1}{2}\frac{(\mathbf{kv}_E)^2}{\omega_{Le}^2 - \omega_0^2}\right.$$
$$\left.+\frac{1}{2}\frac{[\mathbf{kv}_E]^2}{c^2k^2}\right\}\frac{1}{k^2r_{Di}^2},\quad |\omega| \ll kv_{Ti}. \qquad (23)$$

It is evident that at external field frequencies appreciably greater than the ion plasma frequency the instability will arise for oscillations propagating in a small range of angles $\delta\theta \lesssim \omega_0/ck$ almost transverse to the external electric field.

Considering non-transverse propagation ($a \ll 1$) for instabilities for which the frequency of the external field is close to the electron plasma frequency we find that Eq. (17) for γ is replaced by the following equation:

$$4\frac{\gamma^4}{\omega_{Le}^4} + \frac{\gamma^2}{\omega_{Le}^2}\left[\Delta^2 - \beta^2\Delta + \frac{3}{16}\beta^4 + \frac{3}{2}a^2\frac{\omega_{Li}^2}{\omega_{Le}^2}\right] - \Delta\frac{a^2}{2}\frac{\omega_{Li}^2}{\omega_{Le}^2} = 0,$$

$$\Delta = 1 - (\omega_{Li}^2 - \omega_0^2)/\omega_{Le}^2. \tag{24}$$

The presence of the term with β^2 in Eq. (24) does not introduce a qualitative change in the results obtained in [1] in the analysis of equations similar to (24) but with $\beta = 0$. This statement obviously holds only when the propagation if distinctly not transverse.

Summarizing the results given above we see that instabilities will arise in either a transparent plasma or an opaque plasma in a strong high-frequency field. The solenoidal instabilities that have been considered for an opaque plasma exhibit a growth rate which is smaller than the growth rate for the irrotational oscillations that were investigated earlier.[1] In the region in which the plasma is transparent with respect to the external field the most rapidly growing solenoidal oscillations have a growth rate given by Eq. (10). Hence the notion of a quiescent collisionless plasma under conditions of radiative acceleration of transparent plasmoids is valid only for times smaller than $c/v_E\omega_{Li}$.

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