DESCRIPTION OF TWO PARTICLE SCATTERING BY MEANS OF A FUNCTIONAL OF THE WIGHTMAN TYPE

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In the framework of the Wightman axiomatic field theory, each physical theory implements a representation of the algebra A, the elements of which are finite sequences of functions $(g_0; g_1(x); \ldots; g_n(x_1 \ldots x_n); 0; 0; \ldots)$. In this paper we consider the subalgebra $A^{(+)}$ of the algebra A, consisting of elements of the form $(g_0; 0; g_2(x_1x_2); 0; g_4(x_1x_2x_3x_4); 0; \ldots)$. One of the representations of this subalgebra is constructed explicitly. An interesting peculiarity of this representation is the fact that the underlying Hilbert space contains only the vacuum and the two-particle asymptotic states, and does not contain asymptotic states with a larger number of particles.

1. INTRODUCTION

RECENT years have marked a successful development of the axiomatic direction in quantum field theory. Within this framework the physical theory is constructed as a representation theory for an abstract algebra by means of an algebra of operators on a Hilbert space. Each such representation is defined by a multiplicative positive functional on the original algebra.

Let us consider this scheme in more detail. Let A be a *-algebra which is at the same time a topological space. We denote by A′ the space of linear continuous functionals over A. A functional $\varphi \in A'$ is called multiplicatively positive (or briefly a state) if $\varphi(a^+a) \geq 0$ for all $a \in A$ (here a^+ is the adjoint of the element a under the involution).

Each state generates a representation of the algebra A as an algebra of operators $R_\varphi(A)$ on a Hilbert space H_φ. In order to construct the space H_φ and the algebra $R_\varphi(A)$ from the state φ one defines the subspace $\Omega_\varphi \subset A$:

$$\Omega_\varphi \equiv \{a \in A: \varphi(a^+a) = 0\}.$$

Then one defines the quotient space $L_\varphi \equiv A/\Omega_\varphi$. The elements of L_φ are equivalence classes $[a]_\varphi$ of elements of A, two elements $a_1, a_2 \in A$ being equivalent $[a_1]_\varphi = [a_2]_\varphi$ if and only if $a_1 - a_2 \in \Omega_\varphi$. Then the state φ defines a nondegenerate inner product on L_φ:

$$\langle [a]_\varphi, [b]_\varphi \rangle \equiv \varphi(a^+b).$$

The Hilbert space H_φ is defined by completing the pre-Hilbert space L_φ with respect to the metric defined by this inner product. Defining the operators $R_\varphi(a)$ by $R_\varphi(a)[b]_\varphi \equiv [ab]_\varphi$, it is easy to see that the family $R_\varphi(A)$ of all operators $R_\varphi(a)$ yields the required representation of the algebra A.

Let us now assume that the state (functional) φ is invariant under some group $\mathcal{T}$ of automorphisms of the algebra A, i.e., $\varphi(a_\tau) = \varphi(a)$ (where τ is an arbitrary element of the group $\mathcal{T}$ and a_τ is the result of the action of this element on $a \in A$). Then one can define a group of unitary operators on H_φ: $U_\tau[a]_\varphi \equiv [a_\tau]_\varphi$, which obviously implements a unitary representation of the group of automorphisms $\mathcal{T}$.

This construction is sufficiently general. The difference between the two fundamental approaches of Haag and Wightman, respectively, consists in the selection of the fundamental algebra.

In Haag's formalism[1] the algebra A is a C*-algebra, i.e., a *-algebra in which the topology is defined by a norm satisfying the conditions

$$\|a^+\| = \|a\|; \quad \|ab\| \leqslant \|a\| \cdot \|b\|; \quad \|a^+a\| = \|a\|^2,$$

and the algebra is complete in this norm.

In the Wightman formalism[2] the fundamental algebra has a much more complicated structure. The elements of the algebra are finite sequences of functions

$$g \equiv (g_0; g_1(x_1); \ldots; g_n(x_1, \ldots x_n); 0; 0; \ldots),$$

where $x_l \equiv (x_{l0}; \mathbf{x}_l)$, g_0 is a complex number, $g_k(x_1, \ldots x_k)$ are functions belonging to the space S_{4k} of infinitely differentiable functions of 4k variables, decreasing as $x_i \to \infty$ faster than any inverse power. The multiplication

$$(gh)_k(x_1 \dots x_k) \equiv \sum_{l=0}^{k} g_l(x_1 \dots x_l) h_{k-l}(x_{l+1} \dots x_k)$$

and involution $(g^+)_k(x_1 \dots x_k) \equiv \bar{g}(x_k \dots x_1)$ are so chosen that any physical theory which admits the introduction of a field $\varphi(x)$ generates the algebra of (distribution valued) field operators

$$a(g) \equiv \sum_{k=0}^{k} \int \dots \int dx_1 \dots dx_k g_k(x_1 \dots x_k) \varphi(x_1) \dots \varphi(x_k),$$

realizing a representation of the algebra A.

Each functional W over the algebra A is given by the sequence of Wightman functions:

$$W \equiv (W_0;\ W_1(x_1);\ \dots;\ W_n(x_1 \ \dots \ x_n);\ \dots),$$

$$W(g) \equiv \sum_{k=0}^{k} \int \dots \int dx_1 \dots dx_k W_k(x_1 \dots x_k) g_k(x_1 \dots x_k),$$

which in addition to the requirement of multiplicative-positiveness satisfy some additional conditions imposed on the basis of physical considerations.

The description of states (multiplicatively-positive functionals) on such an algebra is a complicated mathematical problem. At present only a few examples of states are known and all correspond to the physically trivial case of noninteracting fields. Therefore any example of a nontrivial functional, even if it does not satisfy all the requirements imposed on a Wightman functional, but sufficiently close to it in mathematical structure, will command a certain amount of interest.

2. CONSTRUCTION OF THE MODEL AND ITS PHYSICAL INTERPRETATION

We shall consider the subalgebra $A^{(+)}$ of A. By definition $A^{(+)}$ consists of those elements of A for which only the components with an even subscript are different from zero, i.e. g_0; $g_2(x_1x_2)$; $g_4(x_1 \dots x_4)$ etc.

Taking as a starting point not the algebra A, but $A^{(+)}$, the condition of multiplicative positiveness will be weakened and the class of states will be correspondingly enlarged. In this case one can construct a curious example of a physically nontrivial state. The peculiarity of this functional is the fact that the corresponding Hilbert space contains only the vacuum and two-particle asymptotic states and does not contain states with a higher number of particles.

The model is defined by two (generalized) functions: $\varphi_2(x_1x_2)$ and $\varphi_4(x_1x_2x_3x_4)$ satisfying the requirement of hermiticity (i.e. such that $\varphi_2(x_1x_2) = \bar{\varphi}_2(x_2x_1)$ and $\varphi_4(x_1x_2x_3x_4) = \bar{\varphi}_4(x_4x_3x_2x_1)$). The functions $W_0^{(+)}$, $W_2^{(+)}(x_1x_2)$, ... , $W_{2n}^{(+)}(x_1 \dots x_{2n})$ which make up the functional $W^{(+)}$ are constructed as follows:

$$W_0^{(+)} = 1, \quad W_2^{(+)}(x_1x_2) = \varphi_2(x_1x_2),$$

$$W_{2n}^{(+)}(x_1 \dots x_{2n}) = W_{2n-4}^{(+)}(x_1 \dots x_{2n-4}) \varphi_4(x_{2n-3} \dots x_{2n})$$

$$+ W_{2n-2}^{(+)}(x_1 \dots x_{2n-2}) \varphi_2(x_{2n-1}x_{2n})$$

$$= \varphi_4(x_1 \dots x_4) W_{2n-4}^{(+)}(x_5 \dots x_{2n})$$

$$+ \varphi_2(x_1x_2) W_{2n-2}^{(+)}(x_3 \dots x_{2n}).$$

Let us analyze the properties of such a functional. We introduce the notation

$$g_{2n-2}^{(2)}(x_1 \dots x_{2n-2}) \equiv \int \int g_{2n}(x_1 \dots x_{2n}) \varphi_2(x_{2n-1}x_{2n}) dx_{2n-1} dx_{2n},$$

$$g_{2n-4}^{(4)}(x_1 \dots x_{2n-4})$$

$$\equiv \int \dots \int g_{2n}(x_1 \dots x_{2n}) \varphi_4(x_{2n-3} \dots x_{2n}) dx_{2n-3} \dots dx_{2n}.$$

Making use of the recursion relation for $W^{(+)}$ it is easy to see that for any $h \in A^{(+)}$

$$W^{(+)}(h^+ g_{2n}^{(2)}) = W^{(+)}(h^+ g_{2n-2}^{(2)}) + W^{(+)}(h^+ g_{2n-4}^{(4)}),$$

$$W^{(+)}(g_{2n}^+ h) = W^{(+)}(g_{2n-2}^{(2)+} h) + W^{(+)}(g_{2n-4}^{(4)+} h).$$

Here and in the following we have adopted the simplified notation, denoting by the same symbol g_{2n} both the function $g_{2n}(x_1 \dots x_{2n})$ and the element of the algebra $A^{(+)}$ with the only nonvanishing element g_{2n} (i.e. (0; 0; ...; 0; $g_{2n}(x_1 \dots x_{2n})$; 0; 0; ...)).

These relations show that the element g_{2n}, considered as the argument of the functional $W^{(+)}$, is completely equivalent to the sum $g_{2n-2}^{(2)} + g_{2n-4}^{(4)}$. Going over to the functions

$$g_{2n-4}^{(2,2)}(x_1 \dots x_{2n-4})$$

$$\equiv \int \int g_{2n-2}^{(2)}(x_1 \dots x_{2n-2}) \varphi_2(x_{2n-3}x_{2n-2}) dx_{2n-3} dx_{2n-2},$$

$$g_{2n-6}^{(2.4)}(x_1 \dots x_{2n-6}) \equiv \int \dots \int g_{2n-2}^{(2)}(x_1 \dots x_{2n-2})$$

$$\times \varphi_4(x_{2n-5} \dots x_{2n-2})\, dx_{2n-5} \dots dx_{2n-2}; \dots$$

and repeating the same procedure a sufficient number of times, one arrives at an element $\tilde{g} \in A^{(+)}$ which has only the first two functions g_0 and $g_2(x_1x_2)$ different from zero and which is completely equivalent to the initial element g_{2n} if regarded as an argument of the functional $W^{(+)}$. Consider, for example, the element g_6. It is equivalent to $g_4^{(2)} + g_2^{(4)}$. The element $g_4^{(2)}$ is in turn equivalent to $g_2^{(2,2)} + g_0^{(2,4)}$. Consequently, the element g_6 is equivalent to the element $\tilde{g}$ with the components:

$$\tilde{g}_0 = g_0^{(2,4)}, \quad \tilde{g}_2(x_1x_2) = g_2^{(4)}(x_1x_2) + g_2^{(2,2)}(x_1x_2).$$

Therefore a necessary and sufficient condition for the multiplicative positiveness of $W^{(+)}$ considered as a functional over the algebra $A^{(+)}$ is the requirement $W^{(+)}(g^+g) \geq 0$ for all elements g of the form $g = (g_0, g_2(x_1x_2), 0, 0, \ldots)$. Since

$$W^{(+)}(g^+g) = \left| g_0 + \int\int g_2(x_1x_2)\varphi_2(x_1x_2)\,dx_1dx_2 \right|^2 + \int \ldots \int \varphi_4(x_1x_2x_3x_4)\bar{g}_2(x_2x_1)g_2(x_3x_4)\,dx_1\ldots dx_4,$$

a necessary and sufficient condition for the multiplicative positiveness of $W^{(+)}$ will be the positiveness of the function φ_4, i.e. non-negativeness of the last term in the preceding equation. We assume that this condition is satisfied below.

One can then construct in the usual manner the Hilbert space and the representation of the algebra $A^{(+)}$. The set $L \subset H$ is hence exhausted by elements of the form $(g_0, g_2(x_1x_2)\ 0, 0, \ldots)$ since all other elements depend linearly on elements of this form.

Until now no restrictions have been imposed on the functions φ_2 and φ_4, except hermiticity and the positiveness of φ_4. Imposing relativistic invariance on φ_2 and φ_4 one can construct in the usual manner unitary representations of the Poincaré group on H.

In order to give physical meaning to such a functional, one can consider an asymptotic construction. In order to construct the asymptotic states and the S-matrix, one subjects the functions $W_2^{(+)}$ and $W_4^{(+)}$ to the same restrictions that are imposed on the Wightman functions.[3, 4] The only difference consists in the fact that the set of asymptotic states is exhausted here by the two-particle states and the vacuum. Therefore such a functional describes only two-particle scattering processes.

If there exists a field theory described by the functional

$$W \equiv (W_0W_1(x_1)\ldots W_n(x_1\ldots x_n)\ldots),$$

one can set

$$\varphi_2(x_1x_2) = W_2(x_1x_2);$$
$$\varphi_4(x_1x_2x_3x_4) = W_4(x_1x_2x_3x_4) - W_2(x_1x_2)W_2(x_3x_4)$$

and reconstruct the functional $W^{(+)}$, which obviously describes two particle scattering in the same manner as the underlying field theory. In this case the S-matrix will not be a unitary operator (even if the S-matrix of the underlying field theory is unitary), since the probability of transition for a two-particle state into a two-particle state is less than one (transitions to states with three or more particles are possible).

In the case of relativistic quantum mechanics (i.e. a relativistic theory without creation and annihilation of particles) the S-matrix must be unitary. A sufficient condition for the unitarity of the S-matrix is the completeness of the set of asymptotic states.[4] One can use such a scheme also for the formal description of nonrelativistic two particle scattering. In this case, one must of course reformulate the requirements on φ_2 and φ_4 and the asymptotic conditions.

In conclusion we note that in the same manner as any multiplicative positive functional (state) W over A generates a family of field operators $\varphi(x)$ such that

$$W_n(x_1\ldots x_n) = \langle 0|\varphi(x_1)\ldots\varphi(x_n)|0\rangle,$$

any state $W^{(+)}$ on $A^{(+)}$ leads to a family of operators $\Phi(x_1x_2)$ depending on two variables, such that

$$W_{2n}^{(+)}(x_1\ldots x_{2n}) = \langle 0|\Phi(x_1x_2)\ldots\Phi(x_{2n-1}x_{2n})|0\rangle.$$

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