INVARIANT PARAMETRIZATION OF PRODUCTS OF LOCAL OPERATORS

A. A. CHESHKOV

Submitted to JETP editor July 5, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **50**, 144-155 (January, 1966)

An invariant parametrization is obtained for the matrix elements of products of local operators of arbitrary spinor or tensor rank, taken between states with one or several particles of arbitrary masses and spins. The example of polarizability of a scalar particle in an electromagnetic field is discussed.

1. THE EQUIVALENCE OF SOME REPRESENTATIONS OF THE HOMOGENEOUS AND INHOMOGENEOUS LORENTZ GROUPS

An important role is played in quantum field theory by matrix elements of various local operators (fields, currents) and of products of such operators (commutators, time-ordered products) taken in the Heisenberg picture. It is interesting to find the most general form for such quantities, taking into account only their properties of relativistic covariance. This means carrying out a relativistically invariant parametrization, expressing these matrix elements in terms of a finite number of invariant functions (form factors).

For matrix elements of local operators this problem was solved in [1,2]. By means of the method developed in [1] we shall effect in this paper a parametrization of matrix elements of products of local operators of arbitrary spinor or tensor rank, taken between states containing one or several particles with arbitrary masses and spins.

One of the essential steps in the parametrization of matrix elements of local operators and their products is the operation of "transplanting" the spin of a particle from one momentum to another. This is connected with the fact that the state vector $|\mathbf{p}, \kappa, j, m\rangle$ of a particle with momentum $\mathbf{p}$, mass κ, spin j, and spin projection on the z-axis m, defined in the center-of-mass system (c.m.s.) of the particle, transforms according to a representation of class p^j_κ of the inhomogeneous Lorentz group (ILG) [3] and its transformation matrix under a Lorentz rotation depends on the particle momentum $\mathbf{p}$. Figuratively speaking, the spin of the particle "rides" on its momentum. Therefore the operation of addition of the spins of two or more particles according to the usual rules is not invariant under Lorentz transformations.

In order to be able to add the spins of particles in an invariant manner it is necessary to "transplant" the spins involved in the addition onto a single momentum. The transplantation of the spin of a particle from the momentum vector $\mathbf{p}$ onto the momentum vector $\mathbf{p}'$ is realized by means of the Lorentz rotation matrix $D^j(\mathbf{p}, \mathbf{p}')$. From the point of view of group theory the transplantation operation for spins consists in going over from the Kronecker product of representations of the ILG $p^j_{\pm\kappa} \otimes p^0_{\pm\kappa}$ to the equivalent representation $p^0_{\pm\kappa} \otimes p^j_{\pm\kappa}$ the equivalence being realized by means of the unitary operator D^j, since this operator maps a base for one representation onto a base for the other one:

$$|\mathbf{p}, \varkappa, j, m; \mathbf{p}', \varkappa\rangle = \sum_m |\mathbf{p}, \varkappa; \mathbf{p}', \varkappa, j, m'\rangle D_{m'm}(\mathbf{p}, \mathbf{p}'). \quad (1)$$

If the momenta of two particles are equal the quantization axis for their spin states is the same, therefore the spins of such particles can be added according to the usual rule of vector addition:

$$|\mathbf{p}, \varkappa, j, m; \mathbf{p}, \varkappa, j', m'\rangle = \sum_{JM} |\mathbf{p}, \varkappa; \mathbf{p}, \varkappa; j, j', JM\rangle \times \langle JM|jmj'm'\rangle, \quad (2)$$

where $\langle JM|jmj'm'\rangle$ are the Clebsch-Gordan coefficients. This operation is Lorentz invariant, since under a Lorentz transformation spins which ride on the same momentum vector undergo identical rotations.

For two particles of different momenta $\mathbf{p}$ and $\mathbf{p}'$, the rest systems do not coincide and the quantization axes z and z' will in general be different. The operator $D^j(\mathbf{p}, \mathbf{p}')$, which transplants the spin of the particle with momentum $\mathbf{p}$ onto the momentum $\mathbf{p}'$ of the other particle, will bring the new quantization axis of this particle, z'', into coincidence with the axis z'. Thus after the transformation $D^j(p, p')$ the spins of both particles can be added according to the standard procedure:

$$|\mathbf{p}, \varkappa, j, m; \mathbf{p}', \varkappa, j', m'\rangle = \sum_{m''JM} D^j_{m''m}(\mathbf{p}, \mathbf{p}')\,|\mathbf{p}, \varkappa; \mathbf{p}', \varkappa; j, j', JM\rangle \times \langle JM | jm''j'm'\rangle. \quad (3)$$

From the group-theoretic point of view the transformation (3) is an equivalence transformation among the following two representations of ILG:

$$P_\varkappa{}^j \otimes P_\varkappa{}^{j'}$$

and

$$P_\varkappa{}^0 \otimes P_\varkappa{}^{j+j'} + P_\varkappa{}^0 \otimes P_\varkappa{}^{j+j'-1} + \cdots + P_\varkappa{}^0 \otimes P_\varkappa{}^{|j-j'|}. \quad (4)$$

In reality the relativistic spin rotation matrix depends only on the velocity vector $\mathbf{u} = \mathbf{p}/\kappa$ of the particle:

$$D^j(\mathbf{p}, \mathbf{p}') = D^j(\mathbf{u}, \mathbf{u}').$$

Therefore the operator D^j realizes an equivalence of the representation $p^j_{\pm\kappa} \otimes p^0_{\pm\kappa}$ with the representation $p^0_{\pm\kappa} \otimes p^j_{\pm\kappa}$, with $\mathbf{u} = \mathbf{p}/\kappa$ and $\mathbf{u}' = \mathbf{p}'/\kappa$. For the same reason the representations

$$P^j_{\pm\varkappa} \otimes P^{j'}_{\pm\varkappa'},\ P^0_{\pm\varkappa} \otimes P^{j+j'}_{\pm\varkappa'} + P^0_{\pm\varkappa} \otimes P^{j+j'-1}_{\pm\varkappa'} + \cdots + P^0_{\pm\varkappa} \otimes P^{|j-j'|}_{\pm\varkappa'},$$

$$P^{j+j'}_{\pm\varkappa} \otimes P^0_{\pm\varkappa'} + P^{j+j'-1}_{\pm\varkappa} \otimes P^0_{\pm\varkappa'} + \cdots + P^{|j-j'|}_{\pm\varkappa} \otimes P^0_{\pm\varkappa'} \quad (5)$$

are also equivalent.

The proof of the equivalence of the representations

$$P_{\varkappa_1}{}^{j_1} \otimes P_{\varkappa_2}{}^{j_2} \otimes \cdots \otimes P^{j_n}_{\varkappa_n},$$

$$\sum_J P_{\varkappa_1}{}^0 \otimes \cdots \otimes P_{\varkappa_r}{}^J \otimes \cdots \otimes P^0_{\varkappa_n}, \quad r = 1, 2, \ldots, n \quad (6)$$

is completely analogous to the preceding one. The sum over J in (6) is over all possible values of J which are allowed by the rules of vector addition.

For $j = 1/2$ the operator $D^j(\mathbf{u}, \mathbf{u}')$ has been constructed in [4]

$$D^{1/2}(\mathbf{u}, \mathbf{u}') = \frac{(u_0+1)(u_0'+1) - \mathbf{u}\mathbf{u}' + i\boldsymbol{\sigma}[\mathbf{u}\mathbf{u}']}{[2(u_0+1)(u_0'+1)(1 - u_\lambda u_\lambda')]^{1/2}}, \quad (7)$$

where $\boldsymbol{\sigma}$ are the Pauli matrices, and $u^2 = -1$. For an arbitrary value of j the operator D^j was found explicitly in [5] in terms of finite rotations with Euler angles α, β, γ, depending on $\mathbf{p}$ and $\mathbf{p}'$. In the Appendix we give a simple method for finding this operator for arbitrary values of j, in a form which is analogous to (7).

Let us finally consider the quantities which transform according to representations of the homogeneous Lorentz group (HLG): the symmetric dotted spinors $\dot{\tau}_{s\mu}$ and the undotted spinors $\tau_{s\mu}$. We show that a representation of the ILG of the class p^j_κ is equivalent to the product of a representation of the HLG of the class $T^{j,0}$ or $T^{0,j}$ with a representation of the ILG of class $p^0_{\pm\kappa}$, which coincides with the translation group of the pseudo-euclidean space.

The vector $|p, \kappa, j, m\rangle$ which transforms according to p^j is an eigenvector of the operators $\mathbf{p}$, $-p_\lambda^2 = \kappa^2$, $s^2 = j(j+1)$ and s_3. Here $\mathbf{s}$ is the angular momentum operator of the particle in the c.m.s.:

$$s_i = {}^1/_2\, \varepsilon_{ijk} M_{jk}^{(\mathrm{cm})} = {}^1/_2\, \varepsilon_{ijk}\, \alpha_{j\mu}(\mathbf{v})\, \alpha_{k\nu}(\mathbf{v}) M_{\mu\nu}^{(\mathrm{lab})}. \quad (8)$$

Here $M^{(\mathrm{lab})}_{\mu\nu}$ is the angular momentum tensor in the laboratory system (lab) and $M^{(\mathrm{cm})}_{jk}$ is the spatial part of the angular momentum tensor in the c.m.s., $\alpha_{\lambda\sigma}$ is the Lorentz transformation matrix from the LAB system to the c.m.s., defined by the velocity $\mathbf{v} = \mathbf{p}/\kappa$. The vector $|p\kappa\rangle|j\mu\rangle$ which transforms according to the representation $T^{j,0} \otimes p^0_\kappa$ is an eigenvector of the operators $\mathbf{p}$, $-p_\lambda^2 = \kappa^2$, $\mathbf{s}'^2 = j(j+1)$, s_3' where the operator s' is the lab angular momentum operator

$$s_i' = {}^1/_2\, \varepsilon_{ijk} M_{jk}^{(\mathrm{lab})}. \quad (9)$$

Therefore

$$|\mathbf{p}, \varkappa, j, m\rangle = \sum_\mu \Lambda_{m\mu}{}^j(\mathbf{v})\,|\mathbf{p}, \varkappa\rangle\,|j, \mu\rangle, \quad (10)$$

where $\Lambda^j(\mathbf{v})$ is the Lorentz transformation of symmetric undotted spinors from the lab system to the c.m.s.

$$|\mathbf{p}, \varkappa, j, m\rangle = \sum_{\dot\mu} \widetilde{\Lambda}^j_{m\dot\mu}(\mathbf{v})\,|\mathbf{p}, \varkappa\rangle\,|j, \dot\mu\rangle, \quad (11)$$

$$\widetilde{\Lambda}^j = (\Lambda^j)^{-1},$$

$$\Lambda^{1/2}(\mathbf{v}) = (v_0 + 1 + \boldsymbol{\sigma}\mathbf{v}) \,/\, [2(v_0+1)]^{1/2}. \quad (12)$$

The method for finding the Λ^j for arbitrary j is given in the Appendix.

It follows from (10) and (11) that the operator Λ^j realizes an equivalence mapping from the representation $p^0_{\pm\kappa} \otimes T^{j,0}$ to the representation $p^j_{\pm\kappa}$ and the operator $\widetilde{\Lambda}^j$ realizes the equivalence between the representations $p^0_{\pm\kappa} \otimes T^{0,j}$ and $p^j_{\pm\kappa}$. It can be said that the operator (v) "transplants" onto the momentum $\mathbf{p} = \kappa\mathbf{v}$ a spin which was not riding on any momentum (i.e., the spin indices of symmetric spinors are covariant by themselves).

Tensors transform according to reducible representations of the HLG. Therefore, in order to "transplant" tensor indices onto a momentum $\mathbf{p}$ we have to decompose the tensor first into symmetric spinors, and then carry through the transformations (9) or (10). It is simpler, however, to transform directly a given tensor $A_{\mu\nu\ldots\lambda}$ from the lab system to the c.m.s. (i.e., the system where $\mathbf{p} = 0$, $p_0 = \kappa$):

$$\widetilde{A}_{\mu'\nu'\ldots\lambda'}(\mathbf{p}) = \alpha_{\mu'\mu}(\mathbf{v})\,\alpha_{\nu'\nu}(\mathbf{v})\ldots\alpha_{\lambda'\lambda}(\mathbf{v})\,A_{\mu\nu\ldots\lambda}, \tag{13}$$

and then to decompose $\widetilde{A}_{\mu\nu\ldots\lambda}(\mathbf{p})$ according to irreducible representations of class p_κ^j. This decomposition will coincide with the decomposition of $\widetilde{A}_{\mu\nu\ldots\lambda}$ with respect to irreducible representations of the three-dimensional rotation group. The matrices $\alpha_{\mu\mu'}(\mathbf{v})$ are ordinary Lorentz transformation matrices.

2. THE PARAMETRIZATION OF THE PRODUCT OF TWO SCALAR OPERATORS

We consider the matrix element of the product of two scalar operators, taken between one-particle states. As a consequence of translation invariance

$$\langle \mathbf{p}, \varkappa, j, m\,|\,\varphi(x)\,\psi(y)\,|\,\mathbf{p}', \varkappa, j, m'\rangle$$

$$= e^{-i(p-p')R}\langle \mathbf{p}, \varkappa, j, m\,|\,\varphi(r)\,\psi(-r)\,|\,\mathbf{p}', \varkappa, j, m'\rangle, \tag{14}$$

where $R = (x+y)/2$, $r = (x-y)/2$. Therefore it suffices to parametrize the operator $\varphi(r)\,\psi(-r) \equiv \chi(r)$ or its Fourier transform

$$\chi(q) = \frac{1}{(2\pi)^2}\int e^{iqr}\chi(r)\,d^4r.$$

If the spin j of the particle vanishes, the one-particle matrix element can be expressed in terms of one form factor $f(t, q^2, qK, qK')$, depending on the four invariants:

$$\langle \mathbf{p}, \varkappa\,|\,\chi(q)\,|\,\mathbf{p}', \varkappa\rangle = [(2\pi)^3(4p_0p_0')^{1/2}]^{-1}f(t, q^2, qK, qK');$$

$$t = -K^2, \quad K_\mu = p_\mu - p_\mu', \quad K_\mu' = p_\mu + p_\mu'. \tag{15}$$

The appearance of the factor $(4p_0p_0')^{1/2}$ in the denominator of the right hand side of (15) is due to the choice of normalization of the state vectors.

For the parametrization of the scalar operator $\chi(q)$ for nonvanishing spin j, one can find a complete set of scalar matrices of rank $2j+1$, in terms of which $\chi(q)$ is decomposed. From the vectors at our disposal p_μ, p_μ', q_μ and $\Gamma_\mu(p')$, where

$$\mathbf{\Gamma}(\mathbf{p}) = \varkappa\mathbf{s} + \mathbf{p}\frac{\mathbf{ps}}{p_0+\varkappa}, \quad \Gamma_0(\mathbf{p}) = \mathbf{ps},$$

$$[s_i, s_j] = i\varepsilon_{ijk}s_k, \quad \mathbf{s}^2 = j(j+1), \tag{16}$$

one can construct all $(2j+1)^2$ linearly independent scalar matrices N_{nml}:

$$N_{nml} = D^j(\mathbf{p}, \mathbf{p}')\,Q_1^nQ_2^mQ_3^l, \tag{17}$$

where $2j \geq n+m+l \geq 2j-1$, $n, m, l \geq 0$ and

$$Q_1 = p_\lambda\Gamma_\lambda(\mathbf{p}'), \quad Q_2 = q_\lambda\Gamma_\lambda(\mathbf{p}'),$$

$$Q_3 = \varepsilon_{\mu\nu\lambda\sigma}q_\mu p_\nu p_\lambda'\Gamma_\sigma(\mathbf{p}'). \tag{18}$$

The N_{nml} are scalars (pseudoscalars) for $n+m+l$ even (odd). The operator D^j which enters as a multiplier in N_{nml} realizes the above-mentioned transplantation of the spin j from the momentum vector $\mathbf{p}$ onto the momentum vector $\mathbf{p}'$, and the matrices Q_i realize an invariant addition of spins.

We thus obtain the final expression:

$$\langle \mathbf{p}, \varkappa, j, m\,|\,\chi(q)\,|\,\mathbf{p}', \varkappa, j, m'\rangle$$

$$= (2\pi)^{-3}(4p_0p_0')^{-1/2}\sum_{m''}D^j_{mm''}(\mathbf{p}, \mathbf{p}')$$

$$\times\sum_{n, m, l}f_{nml}(t, q^2, qK, qK')\langle m''\,|\,N_{nml}\,|\,m'\rangle. \tag{19}$$

3. PARAMETRIZATION OF THE PRODUCT OF n SCALAR OPERATORS

It is easy to generalize the parametrization obtained above from the case of a product of two scalar local operators to the case of an arbitrary number n of such operators $\varphi_1(x_1)$, $\varphi_2(x_2), \ldots, \varphi_n(x_n)$. Making use of the translation invariance and a Fourier transformation, we find that

$$\langle \mathbf{p}, \varkappa, j, m\,|\,\varphi_1(x_1)\ldots\varphi_n(x_n)\,|\,\mathbf{p}', \varkappa, j, m'\rangle$$

$$= \exp[-i(p-p')x_n]\int d^4q_1\ldots d^4q_{n-1}$$

$$\times\exp\{-i[q_1(x_1-x_n)\ldots+q_{n-1}(x_{n-1}-x_n)]\}$$

$$\times\langle \mathbf{p}, \varkappa, j, m\,|\,\varphi_1(q_1)\ldots\varphi_{n-1}(q_{n-1})\,\varphi_n(0)\,|\,\mathbf{p}', \varkappa, j, m'\rangle$$

$$= \exp\{-i(p-p')x_n\}\int d^4q_1\ldots d^4q_{n-1}$$

$$\times\exp\{-i[q_1(x_1-x_n)\ldots+q_{n-1}(x_{n-1}-x_n)]\}$$

$$\times\langle \mathbf{p}, \varkappa, j, m\,|\,\Phi(q_1\ldots q_{n-1})\,|\,\mathbf{p}', \varkappa, j, m'\rangle. \tag{20}$$

The parametrization of the matrix element in the integrand of the right hand side of the last equality (20) differs from (19) only through the fact that the form factors depend on $2(2n-1)$ invariant parameters and not only on four, as in the case $n = 2$, or on eight, as for $n = 3$:

$$\langle \mathbf{p}, \varkappa, j, m\,|\,\Phi(q_1\ldots q_{n-1})\,|\,\mathbf{p}', \varkappa, j, m'\rangle = (2\pi)^{-3}(4p_0p_0')^{-1/2}$$

$$\times\sum_{m''}D^j_{mm''}(\mathbf{p}, \mathbf{p}')\sum_{2j\geqslant\alpha+\beta+\gamma\geqslant 2j-1}\langle m''\,|\,N_{\alpha\beta\gamma}\,|\,m'\rangle$$

$$\times\Phi_{\alpha\beta\gamma}(t, q_1^2, q_2^2, Kq_i\ldots, K'q_j, \ldots, q_1q_r, \ldots, q_2q_s, \ldots). \tag{21}$$

Here q_1 and q_2 are two vectors among the $q_1, \ldots, q_{n-1}$ such that K, K', q_1, q_2 are linearly independent. The choice of invariant parameters is of course, not unique.

4. PARAMETRIZATION OF THE PRODUCT OF TWO VECTOR OPERATORS

We now consider the parametrization of the one-particle matrix element of the product of two local vector operators

$$\langle \mathbf{p}, \varkappa, j, m | A_\mu(x) B_\nu(y) | \mathbf{p}', \varkappa, j, m' \rangle.$$

As in Sec. 2, we consider the quantity

$$\langle \mathbf{p}, \varkappa, j, m | \tau_{\mu\nu}(q) | \mathbf{p}', \varkappa, j, m' \rangle,$$

where

$$\tau_{\mu\nu}(q) = \int e^{iqr} A_\mu(r) B_\nu(-r)\, d^4r. \tag{22}$$

Obviously such a matrix element is a tensor of rank two. Therefore, in order to parametrize it we must construct a complete set of tensor operators from the quantities we have at our disposal, namely, the vectors:

$$K, \quad K', \quad q, \quad R_\mu^{(1)} = \varepsilon_{\mu\nu\lambda\sigma} K_\nu K_\lambda' \Gamma_\sigma,$$
$$R_\mu^{(2)} = \varepsilon_{\mu\nu\lambda\sigma} K_\nu q_\lambda \Gamma_\sigma, \qquad R_\mu^{(3)} = \varepsilon_{\mu\nu\lambda\sigma} K_\nu' q_\lambda \Gamma_\sigma;$$

pseudovectors

$$\Gamma(\mathbf{p}), \qquad R_\mu = \varepsilon_{\mu\nu\lambda\sigma} K_\nu K_\lambda' q_\sigma$$

and the scalar

$$D^j(\mathbf{p}, \mathbf{p}').$$

For this it is sufficient to select four linearly independent vectors or pseudovectors, for example q, K, K′, R, and form out of these the 16 linearly independent tensors $S_{\mu\nu}^{(i)}$, $i = 1, 2, \ldots, 16$. Multiplying each of these tensors from the left by $D(\mathbf{p}, \mathbf{p}')$ and from the right with all possible scalar operators $N_{\alpha\beta\gamma}$, we obtain a complete set of tensor operators for the parametrization of $\tau_{\mu\nu}(q)$:

$$\langle \mathbf{p}, \varkappa, j, m | \tau_{\mu\nu}(q) | \mathbf{p}', \varkappa, j, m' \rangle = (2\pi)^{-3} (4p_0 p_0')^{-1/2}$$
$$\times \sum_{m''} D^j_{mm''}(\mathbf{p}, \mathbf{p}') \sum_{i=1}^{16} S_{\mu\nu}^{(i)} \langle m'' | T_i | m' \rangle, \tag{23}$$

$$T_i = \sum_{2j+1 \geqslant \alpha+\beta+\gamma \geqslant 2j} g_{i,\alpha\beta\gamma}(t, q^2, Kq, K'q) N_{\alpha\beta\gamma}. \tag{24}$$

The parametrization of one particle matrix elements of the product of n local tensor operators of arbitrary rank r_k will differ from (23) in two respects. First we will have to construct all linearly independent tensors of rank

$$r = \sum_{k=1}^{n} r_k$$

from any four linearly independent vectors at our disposal. Secondly, the form factors $g_{\alpha\beta\gamma}$ defined here in analogy with (24) will depend on $2(2n-1)$ invariant parameters for $n \geq 4$, and on eight parameters for $n = 3$, in the same manner as the formfactors $\Phi_{\alpha\beta\gamma}$ in Eq. (21).

5. THE CASE OF ZERO MASS PARTICLES

If the mass of the particle vanishes, the state vector is determined by the momentum and the helicity $\lambda = \pm j$ which is an invariant under proper Lorentz transformations (cf. e.g., [5,6]). For $\kappa = 0$ the vector $\Gamma_\mu(\mathbf{p})$ is not independent but equals λp_μ. Consequently all spin matrices $N_{\alpha\beta\gamma}$ are multiples of the unit matrix, but the formfactors will depend on the additional invariant variables λ and λ'.

The matrix of three-dimensional spin rotation $D(\mathbf{p}, \mathbf{p}')$ for $\kappa = 0$ becomes a phase factor

$$W_{\lambda\lambda'}(\mathbf{p}, \mathbf{p}') = \exp[-i\lambda\chi(\mathbf{p}, \mathbf{p}')]\, \delta_{\lambda\lambda'},$$

$$\operatorname{tg}\chi = \frac{\sin(\theta' - \theta)\sin(2\varphi - \varphi')}{\sin\theta\cos(\theta' - \theta) + \cos\theta\sin(\theta' - \theta)\cos(2\varphi - \varphi')} \tag{25*}$$

where θ, φ and θ', φ' are the spherical coordinates of the vectors $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$ and $\mathbf{n}' = \mathbf{p}'/|\mathbf{p}'|$.

6. OFF-DIAGONAL MATRIX ELEMENTS OF A PRODUCT OF SCALAR OPERATORS

We now consider the parametrization of the matrix element of a product of two scalar operators taken between states of arbitrary physical systems. It follows from considerations of relativistic invariance that any physical system can be described by a state vector $|P_\mu, s, m, \alpha\rangle$ where P_μ is the four-momentum, s is the total angular momentum of the system, m is its projection on the z axis and α are the other variables, invariant with respect to proper Lorentz transformations. The state vector of a system of several particles, defined as the direct product of the state vectors of one-particle state vectors, can be represented as a sum of state vectors of a selected type, by making use of one or several Clebsch-Gordan expansions for the representations if the ILG[7,8]. The normalization of the state vectors is defined by

$$\langle P_\mu, s, m, \alpha | P_\mu', s', m', \alpha' \rangle = \delta^4(P_\mu - P_\mu')\, \delta_{ss'} \delta_{mm'} \delta_{\alpha\alpha'}.$$

We shall omit the index α in the following.

In the same manner as in Sec. 1, we consider the matrix element

$$\langle P_\mu, s, m | \chi(q) | P_\mu', s', m' \rangle.$$

*tg ≡ tan.

In this matrix element the angular momenta s and s′ are in general distinct, which excludes the possibility of constructing an operator Γ_μ which is diagonal in spin. Therefore, as was done in [1] (Sec. 5) in order to achieve an invariant parametrization of the matrix element under consideration, we carry out a Lorentz transformation to the Breit system (BS), where $K'_\mu = P_\mu + P'_\mu = (0, 0, 0, \sqrt{-K'^2})$. We have then

$$\langle P_\mu, s, m | \chi(q) | P_\mu', s', m' \rangle = \sum_{\widetilde{m}, \widetilde{m}'} D^s_{m\widetilde{m}}(\mathbf{P}, \mathbf{w})$$
$$\times \langle \widetilde{P}_\mu, s, \widetilde{m} | \chi(q) | \widetilde{P}'_\mu, s', \widetilde{m}' \rangle D^{*s'}_{m'\widetilde{m}'}(\mathbf{P}', \mathbf{w}). \tag{26}$$

Here $w_\mu = K'_\mu / \sqrt{-K'^2}$ is the velocity corresponding to the Lorentz transformation: $D^s(\mathbf{P}, \mathbf{w})$ and $D^{s'}(\mathbf{P}', \mathbf{w})$ are the matrices of relativistic spin rotation (cf. [5], Eqs. (33) and (34)):

$$\widetilde{P}_\mu = \{\boldsymbol{\pi}, \widetilde{P}_0\}, \quad \widetilde{P}_\mu' = \{-\boldsymbol{\pi}, \widetilde{P}_0'\}, \quad \widetilde{P}^2 = -\widetilde{s},$$
$$\widetilde{P}'^2 = -\widetilde{s}', \quad \boldsymbol{\pi} = {}^1/_2[\mathbf{K} + \mathbf{w}(\mathbf{wK}) / (w_0 + 1) - \mathbf{w}K_0], \tag{27}$$
$$|\boldsymbol{\pi}|^2 = [\widetilde{s}^2 + \widetilde{s}'^2 + t^2 - 2(\widetilde{s}\widetilde{s}' + \widetilde{s}t + \widetilde{s}'t)] / 4(2\widetilde{s} + 2\widetilde{s}' - t);$$
$$\widetilde{\mathbf{q}} = \mathbf{q} + \mathbf{w}(\mathbf{wq}) / (w_0 + 1) - q_0\mathbf{w},$$
$$q_0 = (\mathbf{qw}) + q_0 w_0. \tag{28}$$

It is obvious that

$$\langle \widetilde{P}, s, \widetilde{m} | \chi(\widetilde{q}) | \widetilde{P}', s', \widetilde{m}' \rangle$$
$$= \langle s, \widetilde{m} | F(\widetilde{s}, \widetilde{s}', t, \mathbf{n}, \mathbf{n}', q^2, \widetilde{q}_0^2) | s', \widetilde{m}' \rangle,$$
$$\mathbf{n} = \boldsymbol{\pi} / |\boldsymbol{\pi}|, \quad \mathbf{n}' = \widetilde{\mathbf{q}} / |\widetilde{\mathbf{q}}|. \tag{29}$$

Expanding the matrix element in the right hand side of (29) with respect to a complete set of linearly independent vectors

$$\Psi^{LM}_{ll'}(\mathbf{n}, \mathbf{n}') = \sum_{\mu\mu'} \langle l\mu l'\mu' | LM \rangle Y^*_{l\mu}(\mathbf{n}) Y^*_{l'\mu'}(\mathbf{n}'), \tag{30}$$

where $L + 1 \geq l + l' \geq L$, we obtain

$$\langle s, \widetilde{m} | F(\widetilde{s}, \widetilde{s}', t, q^2, \widetilde{q}_0^2, \mathbf{n}, \mathbf{n}') | s', \widetilde{m} \rangle$$
$$= \sum_{L=0}^{\infty} \sum_{L+1 \geqslant l+l' \geqslant L} \sum_M \widetilde{\Psi}^{LM}_{ll'}(\mathbf{n}, \mathbf{n}')$$
$$\times \langle s, \widetilde{m} | F^{LM}_{ll'}(\widetilde{s}, \widetilde{s}', t, q^2, \widetilde{q}_0^2) | s', \widetilde{m} \rangle. \tag{31}$$

The expansion (30) is Lorentz-invariant, since the spherical harmonics $Y_{l\mu}(\mathbf{n})$, $Y_{l'\mu'}(\mathbf{n}')$ are defined in the BS.

If $\pi = 0$, $\mathbf{n}$ is undetermined, but then $l = 0$ [1]. Similarly $l' = 0$ for $\widetilde{q} = 0$.

Applying the Wigner-Eckart theorem to (31) we obtain

$$\sum_{L=0}^{\infty} \sum_M \sum_{L+1 \geqslant l+l' \geqslant L} \langle s', \widetilde{m}', LM | s\widetilde{m} \rangle \Psi^{LM}_{ll'} F^{Lss'}_{ll'}(\widetilde{s}, \widetilde{s}', t, q^2, \widetilde{q}_0^2) \tag{32}$$

This completes the parametrization of the matrix element under consideration. The $F^{Lss'}_{ll'}$ are the form factors we were looking for.

The parametrization of n scalar operators, as in Sec. 3, differs from (32) only by the number of invariant variables on which the form factor depends.

7. OFF-DIAGONAL MATRIX ELEMENTS OF PRODUCTS OF SPINOR AND TENSOR OPERATORS

In the same manner as the parametrization of matrix elements of operators having spinor or tensor character, the parametrization of the matrix element of a product of such operators is carried out by means of "transplanting" of all spinor (tensor) indices onto one single momentum, e.g., for the BS onto $\mathbf{K}'$:

$$\langle P, s, m | \tau_{\lambda_1\lambda_2} \ldots (q_1, q_2, \ldots) | P', s', m' \rangle$$
$$= \sum_{\lambda_1'\lambda_2'\ldots} \Lambda_{\lambda_1\lambda_1'}(\mathbf{w}) \Lambda_{\lambda_2\lambda_2'}(\mathbf{w}) \cdots \sum D^s_{m\widetilde{m}}(\mathbf{P}, \mathbf{w})$$
$$\times D^{*s}_{m'\widetilde{m}'}(\mathbf{P}', \mathbf{w}) \widetilde{\Psi}^{LM}_{ll'}(\mathbf{n}, \mathbf{n}') \langle {}^1/_2\lambda_1' {}^1/_2\lambda_2' | j_1\mu_1 \rangle$$
$$\times \langle {}^1/_2\lambda_3' j_1\mu_1 | j_2\mu_2 \rangle \cdots \langle j_{n-1}\mu_{n-1} LM | j\mu \rangle$$
$$\times \langle s'\widetilde{m}' j\mu | s\widetilde{m} \rangle G^{Lj}_{ss'll'}(\widetilde{s}, \widetilde{s}', t, q_1^2, \ldots). \tag{33}$$

For a product of tensor operators the parametrization procedure is similar to (33). The difference will consist only in the fact that after going over to the BS one has to expand the tensors with respect to irreducible representations of the three-dimensional rotation group and only afterwards add up the angular momenta as in (33).

In general, from the group-theoretic point of view any parametrization is a separation of a representation of the ILG of class $P^{\dot{0}}_{\pm M}$ in the Kronecker product of representations of the classes $P^j_{\pm\kappa}$, P^λ_0, $T^{0,j}$ and $T^{j,0}$.

8. THE POLARIZABILITY TENSOR. MAGNETIC MULTIPOLES OF SECOND ORDER

As an example of application of the method developed here we consider the parametrization of the matrix element of the polarizability tensor for a scalar particle placed in an external electromagnetic field with potential $A_\nu(y)$. The polarizability tensor

$$\Pi_{\mu\nu}(x, y) = \frac{\delta j_\mu\{x, A\}}{\delta A_\nu(y)}\bigg|_{A=0} \tag{34}$$

is defined as the first order term in the functional expansion of the current vector of the particle in terms of $A_\nu(y)$:

$$j_\mu\{x, A_\nu(y)\} = j_\mu^{(0)}(x) + j_\mu^{(1)}(x) + \ldots,$$

$$j_\mu^{(1)}(x) = \int d^4y A_\nu(y) \left.\frac{\delta j_\mu\{x, A\}}{\delta A_\nu(y)}\right|_{A=0}. \tag{35}$$

According to Secs. 2 and 4:

$$\langle \mathbf{p}|\Pi_{\mu\nu}(x, y)|\mathbf{p}'\rangle = \frac{e^{-iKx}}{(2\pi)^5(4p_0p_0')^{1/2}} \int d^4q e^{iq(x-y)} \langle \mathbf{p}|\Pi_{\mu\nu}(q)|\mathbf{p}'\rangle. \tag{36}$$

Gauge invariance imposes on $\Pi_{\mu\nu}(q)$ the conditions:

$$\Pi_{\mu\nu}(0) = 0, \quad \Pi_{\mu\nu}(q)q_\nu = 0, \quad \Pi_{\mu\nu}(q)(K_\mu + q_\mu) = 0. \tag{37}$$

Taking into account the conservation laws implied by space and time reflections, we obtain the parametrization of the matrix element $\langle \mathbf{p}|\Pi_{\mu\nu}(q)|\mathbf{p}'\rangle$:

$$\begin{aligned}\langle \mathbf{p}|\Pi_{\mu\nu}(q)|\mathbf{p}'\rangle &= f_1\{q_\mu q_\nu + q_\mu K_\nu - \delta_{\mu\nu}(Kq + q^2)\} \\ &+ f_2\{\delta_{\mu\nu}(K'q)^2 + K_\mu' K_\nu'(Kq + q^2) \\ &- (K_\mu' q_\nu + q_\mu K_\nu')(K'q) - K_\mu' K_\nu (K'q)\} \\ &+ f_3\{K_\mu' K_\nu' q^2(K^2 + Kq) - K_\mu K_\nu' q^2 (K'q) \\ &- K_\mu' q_\nu (K'q)(K^2 + Kq) + K_\mu q_\nu (K'q)^2\} \\ &+ f_4\{K_\mu' K_\nu'(Kq)(Kq + q^2) - K_\mu' K_\nu (K'q)(Kq + q^2) \\ &- q_\mu K_\nu'(Kq)(K'q) + q_\mu K_\nu (K'q)^2\} \\ &+ f_5\{\delta_{\mu\nu} q^2 (K^2 + Kq) - K_\mu K_\nu q^2 - q_\mu q_\nu (K^2 + Kq) \\ &+ K_\mu q_\nu (Kq)\}.\end{aligned} \tag{38}$$

Here

$$f_i = f_i(t, r, a, b), \quad t = -K^2, \quad r = -q^2, \quad a = Kq, \quad b = -K'q.$$

In the parametrization of the current of a free particle one obtains form factors which depend only on the single variable t. These form factors are easily identified with multipole moments of the particle and their derivatives at $t = 0$ are the mean square, mean fourth power (fourth moment) etc., of the radii of distribution of the corresponding multipoles. In the case under consideration here we obtain a completely different picture. The five form factors in (38) together with their partial derivatives at $K = 0$ and $q = 0$ determine the multipole moments induced by the external field, for arbitrarily high multipolarity. The appearance of an infinite number of induced multipoles is not unexpected, if one takes into account the fact that the form factors which determine these moments, depend on several variables. The external field does not affect the total charge of the particle, however the mean square radius of the charge distribution is modified

$$\overline{\Delta r^2} = \frac{2\pi^2}{\varkappa}\left\{6f_5(0) - 24\varkappa^2 f_3(0) - 6\frac{\partial f_1}{\partial t}(0) - 24\varkappa^2\frac{\partial f_2}{\partial t}(0) + 3\frac{\partial f_1}{\partial a}(0) + 12\varkappa^2\frac{\partial f_2}{\partial a}(0) - 2f_4(0)\right\}\Delta\varphi. \tag{39}$$

Homogeneous electric and magnetic fields, **E** and **H**, induce the electric dipole moment

$$d_i = \frac{2\pi^2}{\varkappa}\{f_1(0) + 4\varkappa^2 f_2(0)\} E_i \tag{40}$$

and the usual magnetic dipole moment (we shall call it magnetic dipole moment of the first kind)

$$\mu_i = -4\pi^2\varkappa^{-1} f_1(0) H_i. \tag{41}$$

An external current $\mathbf{J} = \operatorname{curl} \mathbf{H}$ induces a magnetic dipole moment of the second kind, named "anapole" by Ya. B. Zel'dovich [9]

$$\lambda_i = \frac{3\pi^2}{2\varkappa}\left\{6f_5(0) + 4\frac{\partial f_1}{\partial t}(0) + 3\frac{\partial f_1}{\partial a}(0)\right\}\operatorname{curl}_i \mathbf{H}. \tag{42}$$

Coordinate-dependent electric and magnetic fields induce electric and magnetic dipoles and also electric and magnetic quadrupoles of the first kind:

$$Q_{ij} = \frac{2\pi^2}{\varkappa}\left\{3\frac{\partial f_1}{\partial a}(0) + 12\varkappa^2\frac{\partial f_2}{\partial a}(0) - 2f_4(0)\right\} \times \left(\frac{\partial^2\varphi}{\partial x_i \partial x_j} - \frac{1}{3}\delta_{ij}\Delta\varphi\right), \tag{43}$$

$$\mu_{ij} = \frac{3\pi^2}{2\varkappa}\left(\frac{\partial f_1}{\partial a}\right)_0 \left(\frac{\partial H_i}{\partial x_j} + \frac{\partial H_j}{\partial x_i}\right). \tag{44}$$

An external current with density depending linearly on the coordinates induces a magnetic quadrupole moment of the second kind, etc.

In the absence of an external field the n-pole magnetic moments of the second kind, determined by the form factors $f_{2,n-1}$ (cf. [1], Eqs. (14) and (15)) are forbidden by either invariance under space reflection (n even) or under time reversal (n odd).

There are no such interdictions in classical electrodynamics. The magnetic multipoles of the second kind for a classical system with current density j are defined by:

$$\lambda_i = \int [x_i(\mathbf{xj}) - j_i \mathbf{x}^2]\, d^3x,$$

$$\lambda_{ij} = \frac{1}{18}\int [2x_i x_j(\mathbf{xj}) - \mathbf{x}^2(x_i j_j + x_j j_i)]\, d^3x,$$

$$\begin{aligned}\lambda_{ikl} = \frac{1}{600}\int [&15 x_i x_k x_l(\mathbf{xj}) - 5\mathbf{x}^2(x_i x_k j_l + x_i x_l j_k + x_k x_l j_i) \\ &- \mathbf{x}^2(\mathbf{xj})(\delta_{ik}x_l + \delta_{il}x_k + \delta_{kl}x_i) + \mathbf{x}^4(\delta_{ik}j_l + \delta_{il}j_k \\ &+ \delta_{kl}j_i)]\, d^3x \text{ etc.}\end{aligned} \tag{45}$$

In the static limit they interact not with the external field, but with an external current, in such a manner that the energy of such a system, situated in an external magnetic field **H** produced by a current with density **J** has the expression

$$W = \mu_i H_i + \lambda_i J_i + \mu_{ik}\nabla_k H_i + \lambda_{ik}\nabla_k J_i + \mu_{ikl}\nabla_k\nabla_l H_i + \lambda_{ikl}\nabla_k\nabla_l J_i + R_i^2\Delta H_i + r_i^2\Delta J + R^2{}_{ik}\nabla_k\Delta H_i + \ldots \quad (46)$$

Here R_i^2, R_{ik}^2, r_i^2 are the mean square radii of the magnetic dipole moment of the first kind, of the magnetic quadrupole moment of the first kind and of the magnetic dipole moment of the second kind, respectively.

The author expresses his profound gratitude to Yu. M. Shirokov who has shown constant interest in this work.

APPENDIX

A complete expression for the Euler angles α, β, γ has been obtained in [5]. These angles determine the spin rotation operator D^j under a Lorentz transformation. In most cases however, it is more convenient to express the matrix elements of this operator, $D^j_{mm'}(\mathbf{p}\cdot\mathbf{p}')$ not in terms of the Euler angles, but in terms of the spin rotation angle ω, the unit vector $\mathbf{k}$ along the rotation axis and the spin matrices $\mathbf{j}$. In order to find this form of the matrices $D^j(\omega, \mathbf{k})$ we expand them in terms of the complete set of scalar matrices $(\mathbf{j}\cdot\mathbf{k})^n$:

$$D^j(\omega, \mathbf{k}) = \sum_{n=0}^{2j} a_n{}^j(\omega)(\mathbf{jk})^n. \qquad (A.1)$$

We choose the z axis along the vector $\mathbf{k}$; then

$$e^{im\omega} = \sum_{n=0}^{2j} a_n{}^j(\omega) m^n, \quad j \geqslant m \geqslant -j, \qquad (A.2)$$

where

$$a_{2k}{}^j(\omega) = a_{2k}^{*j}(\omega), \quad a_{2k+1}^j(\omega) = -a_{2k+1}^{*j}(\omega).$$

One can now represent (A.1) in the form

$$D^j(\omega, \mathbf{k}) = \sum_n a_n{}^j(\omega)[i(\mathbf{jk})]^n \qquad (A.3)$$

where all $a_n^j(\omega)_j$ are real. The system (A.3) for the determination of $a_n^j(\omega)$ splits into two:

$$\cos m\omega = \sum_{k=0}^{A} a_{2k}{}^j(\omega)[-m^2]^k,$$

$$\sin m\omega = \sum_{k=0}^{B} a_{2k+1}^j(\omega)\, m[-m^2]^k. \qquad (A.4)$$

The quantity A takes on the values j for integral spin and $j - \frac{1}{2}$ for half-integral spins and the quantity B is $j-1$ for integral spin and $j - \frac{1}{2}$ for half-integral spin.

In order to find all $a_n^j(\omega)$ for a given j, it suffices to solve the first system in (A.4) for integral spins, and the second system in (A.4), for half-integral spin, since in the two cases we have respectively

$$\frac{da_{2k}{}^j}{d\omega} = a_{2k-1}^j, \qquad \frac{da_{2k+1}^j}{d\omega} = a_{2k}{}^j.$$

For integral j

$$a_0{}^j(\omega) = 1. \qquad (A.5)$$

Thus, for instance

$$D^{1/2} = \cos(\omega/2) + 2i(\mathbf{kj})\sin(\omega/2),$$

$$D^1 = I - i(\mathbf{kj})\sin\omega + (\mathbf{kj})^2\cos\omega,$$

$$D^{3/2} = \frac{1}{8}\left\{\cos\frac{3\omega}{2} - 9\cos\frac{\omega}{2}\right\} + \frac{i}{12}\left\{27\sin\frac{\omega}{2} - \sin\frac{3\omega}{2}\right\}(\mathbf{kj}) + \frac{1}{2}\left\{\cos\frac{\omega}{2} - \cos\frac{3\omega}{2}\right\}(\mathbf{kj})^2 + \frac{i}{3}\left\{\sin\frac{3\omega}{2} - 3\sin\frac{\omega}{2}\right\}(\mathbf{kj})^3. \qquad (A.6)$$

For a Lorentz rotation of the spin [1)]

$$\mathbf{k} = \frac{[\mathbf{pp}']}{|[\mathbf{pp}']|} \qquad \omega = 2\tan^{-1}\frac{|[\mathbf{pp}']|}{(p_0+\varkappa)(p_0'+\varkappa) - (\mathbf{pp}')} \qquad (A.7)^*$$

The transformation Λ^j for the spinors $\tau_{j\mu}$ is obtained by setting

$$\mathbf{k} = \frac{\mathbf{w}}{|\mathbf{w}|}, \quad \mathrm{tg}\frac{\omega}{2} = -i\left(\frac{w_0-1}{w_0+1}\right)^{1/2}. \qquad (A.8)$$

1)The expressions for ω and k here differ from the ones given in[10] because of a different definition of the Lorentz transformation.

*$[\mathbf{p}\,\mathbf{p}'] \equiv \mathbf{p}\times\mathbf{p}'$.

[1] A. A. Cheshkov and Yu. M. Shirokov, JETP **44**, 1982 (1963), Soviet Phys. JETP **17**, 1333 (1963).

[2] L. Durand, P. C. De Celles, and R. B. Marr, Phys. Rev. **126**, 1883 (1962).

[3] Yu. M. Shirokov, JETP **33**, 1196 (1957), Soviet Phys. JETP **6**, 919 (1958).

[4] Yu. M. Shirokov, DAN SSSR **99**, 737 (1954).

[5] A. A. Cheshkov and Yu. M. Shirokov, JETP **42**, 144 (1962), Soviet Phys. JETP **15**, 103 (1962).

[6] M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) **7**, 404 (1959).

[7] Yu. M. Shirokov, JETP **35**, 1005 (1958), Soviet Phys. JETP **8**, 703 (1959).

[8] J. S. Lomont, Journ. Math, Phys. **1**, 327 (1961).

[9] Ya. B. Zel'dovich, JETP **33**, 1531 (1957), Soviet Phys. JETP **6**, 1184 (1958).

[10] V. I. Ritus, JETP **40**, 352 (1961), Soviet Phys. JETP **13**, 240 (1961).

Translated by M. E. Mayer
23