THE LORENTZ GROUP AS A DYNAMIC SYMMETRY GROUP OF THE HYDROGEN ATOM

A. M. PERELOMOV and V. S. POPOV

Institute of Theoretical and Experimental Physics, State Atomic Energy Commission

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The hidden (dynamic) symmetry of the hydrogen atom is discussed from the point of view of group theory. It is shown that the transition from the compact group O(4) to its noncompact analog—the Lorentz group—provides the possibility of including in the description both the discrete and the continuous spectrum. In this case the wave functions of the continuous spectrum with a given energy $E > 0$ generate an infinite-dimensional irreducible representation $D(0, \rho)$ of the Lorentz group ($\rho = \sqrt{2/E}$) belonging to the so-called fundamental series of unitary representations; the wave functions for the states of the discrete spectrum with a given principal quantum number n generate a finite-dimensional representation $D((n-1)/2, (n-1)/2)$ of the Lorentz group. A symmetry of the wave functions is found which is specific for the Coulomb potential (cf., formulas (20) and (20a) respectively for the states of the continuous and the discrete spectra). The special case of states with $E = 0$ (in an attractive field) when the Lorentz group degenerates into the nonrelativistic Galilean group is also considered. An expansion of the Coulomb Green's function (for $E > 0$) is obtained in terms of the irreducible representations of the Lorentz group. An explicit expression is obtained for the Green's function for the Kepler problem in n-dimensional space and the geometric meaning of the symmetry possessed by the Coulomb wave functions is determined.

1. INTRODUCTION

In 1935 Fock[1] discovered the reason for the "accidental" degeneracy of the levels of the hydrogen atom having the same principal quantum number. He showed that the Hamiltonian of the hydrogen atom in addition to the explicit symmetry with respect to the rotation group O(3) also possesses a "hidden" symmetry of a wider group O(4) (for $E < 0$). The method utilized by Fock consisted of the transformation of the Schrödinger equation for the hydrogen atom (in the p-representation) into the integral equation for the four-dimensional spherical harmonics. Later Alliluev[2] has investigated by the same method the Kepler problem in a space of n dimensions and has shown that the Hamiltonian of the n-dimensional "hydrogen atom" possesses the hidden symmetry of the group $O(n+1)$. In a number of papers[3-6] it was shown that in order to obtain the energy of the bound states there is no necessity to utilize the integral method[1] and it is sufficient to restrict oneself to a consideration of the algebra of the operators generating the hidden symmetry group. In essence such an approach is already contained in Pauli's paper[7] who first determined the spectrum of the levels of the hydrogen atom in accordance with quantum mechanics.

According to Fock all the states of the discrete spectrum of the hydrogen atom are described by the irreducible representations of the group O(4). The question arises whether it is possible in the language of group theory to describe in a unified manner both the discrete and the continuous spectrum. It is clear that this cannot be achieved within the framework of the group O(4), since all its irreducible representations are finite-dimensional (just as in the case of any compact group, i.e., a continuous group of finite volume). Therefore the "complete" dynamic symmetry group which together with the bound states also encompasses the continuous spectrum must of necessity be noncompact. It turns out that in the case of the hydrogen atom such a "complete" symmetry group coincides with the Lorentz group. In Sec. 2 it is shown which irreducible representations of the Lorentz group occur in the continuous and the discrete spectra. In Sec. 3 properties of the wave functions of the continuous spectrum are investigated and a characteristic symmetry of the wave functions is found (cf., formula (20)) which is directly related to the topology of ξ-space (i.e., the space in which the dynamic symmetry group operates; cf., (10), (14)). It is shown that an analogous symmetry also exists for the states of the discrete spectrum (cf., formula (20a)). Section 4 is devoted to a discussion of the special case E = 0 for an attractive potential; it turns out that in this case the dynamic symmetry group degenerates into the nonrelativistic group of Galilean transformations.

In recent years a number of papers has appeared[8-11] devoted to the calculation of the nonrelativistic Green's function for the Coulomb potential; in such a case the Green's function is usually sought in the x-representation[8-10]. Schwinger[11] drew attention to the fact that in virtue of the dynamic symmetry of the hydrogen atom (which manifests itself in the transition to the variables ξ_μ associated with the momentum (cf., (10)) the Coulomb Green's function takes on its simplest form in the p-representation. This representation is also important because the Green's function appears in just such a form in the Feynman diagrams for nuclear reactions. In Secs. 6 and 7 new expansions are obtained for the Coulomb Green's function in terms of an orthogonal system of functions, which replace Schwinger's expansion in the case E > 0. From the mathematical point of view the resultant formulas give an expansion of the Coulomb's Green's function in terms of the irreducible representations of the Lorentz group.

2. THE HYDROGEN ATOM AND THE LORENTZ GROUP

We consider a hydrogen-like atom ($H = p^2/2m + Z_1Z_2e^2/r$). It is well known that in the case of a Coulomb field there exists a vector **A** which is an additional integral of the motion[1]:

$$A_i = -m^{1/2}\left(\alpha\frac{x_i}{r} + \frac{1}{2m}(L_{ij}p_j + p_jL_{ij})\right), \quad \alpha = Z_1Z_2e^2. \quad (1)$$

The commutation relations between the operators L_i (components of the orbital angular momentum) and A_j have the form[3,4,7]

$$[L_i, L_j] = i\varepsilon_{ijk}L_k, \qquad [L_i, A_j] = i\varepsilon_{ijk}A_k,$$
$$[A_i, A_j] = -2iH\varepsilon_{ijk}L_k, \quad (2)$$

with $[L_i, H] = [A_i, H] = 0$. Going over to the operators $N_i = (2H)^{-1/2}A_i$ we obtain[2]

$$[L_i, L_j] = -[N_i, N_j] = i\varepsilon_{ijk}L_k,$$
$$[L_i, N_j] = i\varepsilon_{ijk}N_k, \quad (3)$$

which coincides with the commutation relations between the generators of the homogeneous Lorentz group (the fact was apparently noted for the first time by Klein; cf., footnote in the paper by Hulthen[3]).

Since the appearance of the vector **A** is characteristic of the Coulomb field the Lorentz group describes the hidden (dynamic) symmetry of the Kepler problem. The Hamiltonian H can be expressed in terms of the generators $M_{\mu\nu}$ in the following manner:

$$H^{-1} = -2E_C^{-1}(F+1), \qquad F = {}^1/_2 M_{\mu\nu}M^{\mu\nu} = \mathbf{L}^2 - \mathbf{N}^2,$$
$$E_C = m\alpha^2/\hbar^2. \quad (4)$$

The homogeneous Lorentz group has irreducible representations of two types: 1) the finite-dimensional representations $D(j_1, j_2)$ which, with the exception of the scalar D(0, 0), are nonunitary[3], and 2) infinite-dimensional unitary representations discovered by Gelfand and Naĭmark[13]. The latter, in turn, can be divided into two classes (representations of the principal and the supplementary series, cf., [14]) of which to date applications in physics have been found[15,16] only for representations of the fundamental series which we shall in future denote by $D(m, \rho)$. Referring the reader for a detailed exposition of the theory of representations of the Lorentz group to the book by Naĭmark[14] (cf. also [17]) we shall enumerate only those properties of infinite-dimensional unitary representations of the fundamental series an understanding of which is necessary for the subsequent discussion.

The numbers m and ρ uniquely characterize an irreducible representation, with m being an integer, and ρ being any arbitrary real number. The representations $D(m, \rho)$ and $D(-m, -\rho)$ are equivalent, as a result of which it is sufficient to consider values of $\rho \geq 0$ (for either sign of m). The invariant operators F and G for the Lorentz group (scalar and pseudoscalar) take on the following values for the representation $D(m, \rho)$:

$$F = {}^1/_2 M_{\mu\nu}M^{\mu\nu} = -[1 + {}^1/_4(\rho^2 - m^2)],$$
$$G = {}^1/_8 i\varepsilon_{\mu\nu\sigma\rho}M^{\mu\nu}M^{\sigma\rho} = {}^1/_4 m\rho. \quad (5)$$

From the representations $D(m, \rho)$ it is possible by means of analytic continuation to obtain also the usual finite dimensional representations for which it is necessary to take into account the connection between the numbers m, ρ and j_1, j_2:

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1)The so-called Runge-Lenz vector[12].

2)For E = 0 such a transition is impossible; this special case is considered in Sec. 4.

3)We recall that the spinor with $2j_2$ dotted and $2j_1$ undotted indices transforms according to the representation $D(j_1, j_2)$.

$$m = 2(j_1 - j_2), \quad \rho = -2i(j_1 + j_2 + 1). \qquad (6)$$

From this it follows that for finite-dimensional representations $\rho = -ik$ ($k = 2, 3, 4, \ldots$). It can be shown that in the ρ-plane exact representations of the Lorentz group correspond only to the points of the semi axis $0 \le \rho \le +\infty$ and to the integral points $\rho = -ik$ lying on the imaginary axis.

From (1) and (4) we obtain the values of the invariants F and G in the case of the hydrogen atom:

$$F = -(1 + E_C/2E), \quad G = -(2H)^{-1/2}(\mathbf{LA}) = 0. \qquad (7)$$

Comparing with (5) we have:

$$m = 0,$$

$$\rho = \begin{cases} (2E_C/E)^{1/2} = 2p_C/p & \text{for the continuous spectrum } (E > 0) \\ -2in & \text{for the discrete spectrum} \end{cases} \qquad (8)$$

(n is the principal quantum number). Here $p_C = |Z_1Z_2me^2/\hbar|$ is a momentum characteristic for the Coulomb field.

The system of wave functions of the continuous spectrum with a given value of E forms the irreducible representation $D(0, \rho)$ of the Lorentz group; this representation is infinite-dimensional and unitary. Physically the infinite dimensionality of the representation consists of the fact that the values of the orbital angular momentum l are not restricted in the continuous spectrum; the unitarity follows from the fact that the generators L_i and N_i are Hermitian for $H > 0$. The states of the discrete spectrum belonging to the level with the principal quantum numbers n form a finite-dimensional representation $D(j_1, j_2)$ with $j_1 = j_2 = (n-1)/2$. Since the generators N_i for $H < 0$ become anti-hermitian this representation is nonunitary. Its multiplicity is equal to $(2j_1+1)(2j_2+1) = n^2$. As n varies from unity to infinity j assumes all the integral and half integral values: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

Thus, the wave functions of the hydrogen atom generate all the representations of the Lorentz group with $m = 0$. The last restriction is natural since non-vanishing values of the quantum number m appear only when the particle has spin[15,16]. It is well known that between the finite-dimensional representations of the group O(4) and the Lorentz group there exists a one-to-one correspondence: if the representation $D(j_1, j_2)$ is continued analytically into the domain of purely imaginary angles of rotation in the planes (x_i, x_0) the corresponding representation of the Lorentz group is generated. Therefore, if we restrict ourselves to a discussion of only bound levels, then the group O(4) can be regarded as the group of hidden symmetry. However, in such a case the continuous spectrum is left out of consideration, and in order to describe it, it is necessary to go over to a non-compact group. Transition to the Lorentz group provides a possibility of a unified description both of the discrete and the continuous spectra. Therefore, it is natural to regard the Lorentz group in particular as the true dynamic symmetry group for the hydrogen atom.

3. WAVE FUNCTIONS OF THE CONTINUOUS SPECTRUM

The Schrödinger equation for a hydrogen-like atom has the form

$$\left(\frac{p^2}{2m} - E\right)\psi(\mathbf{p}) + \frac{Z_1Z_2e^2}{2\pi^2\hbar}\int \frac{\psi(\mathbf{p}')\,d\mathbf{p}'}{|\mathbf{p}-\mathbf{p}'|^2} = 0. \qquad (9)$$

Following Fock[1] it is convenient to regard momentum space as a stereographic projection of a four-dimensional hypersphere (for $E < 0$) or of a hyperboloid of two sheets (for $E > 0$). We introduce the coordinates ξ_μ:

$$\xi_i = \begin{cases} (2p_0p_i)/(p^2 + p_0^2) & \text{for } E < 0 \\ 2p_0p_i/(p^2 - p_0^2) & \text{for } E > 0 \end{cases},$$

$$\xi_0 = \begin{cases} (p_0^2 - p^2)/(p_0^2 + p^2) & \text{for } E < 0 \\ (p^2 + p_0^2)/(p^2 - p_0^2) & \text{for } E > 0 \end{cases} \qquad (10)$$

satisfying the conditions

$$\xi_0^2 + \xi^2 = 1 \quad (E < 0), \qquad \xi_0^2 - \xi^2 = 1 \quad (E > 0). \qquad (11)$$

Here $p_0 = (2m|E|)^{1/2}$.

Going over from the wave function $\psi(\mathbf{p})$ to a new function $\psi(\xi)$ defined on the hypersphere or on the hyperboloid:

$$\psi(\xi) = \text{const}\cdot(p^2 \pm p_0^2)^2\psi(\mathbf{p}), \qquad (12)$$

and taking into account the relations

$$(\xi - \xi')^2 = \pm\frac{4p_0^2\,|\mathbf{p}-\mathbf{p}'|^2}{(p^2 \pm p_0^2)(p'^2 \pm p_0^2)}, \quad d\mathbf{p} = \left|\frac{p^2 \pm p_0^2}{2p_0}\right|\frac{d^3\xi}{\xi_0}, \qquad (13)$$

we bring Eq. (9) to the following form:

$$\psi(\xi) \pm \frac{\eta}{2\pi^2}\int \frac{d^3\xi'}{\xi_0'}\frac{\psi(\xi')}{|\xi - \xi'|^2} = 0. \qquad (14)$$

In Eqs. (12)—(14) the upper signs refer to the case $E < 0$, the lower signs refer to the case $E > 0$. η is used to denote the Coulomb parameter: $\eta = Z_1Z_2e^2m/\hbar p_0$. It should be emphasized that although in external appearance equation (14) is the same for positive and for negative energies actually an essential difference exists between these cases (which has already been noted by Fock[1]): integration in (14) is carried out for $E < 0$ over a singly connected region (the surface of a unit hypersphere), while for $E > 0$ it is carried out over the surface of a unit hyperboloid consisting of two sheets (the upper sheet given by $1 \leq \xi_0 < \infty$, the lower sheet by $-1 \geq \xi_0 > -\infty$).

Solutions of (14) for $E < 0$ are the four-dimensional spherical harmonics $Y_{nlm}(\xi)$ which have been studied in detail by Fock[1]. We give their explicit form:

$$Y_{nlm}(\alpha, \theta, \varphi) = \Pi_{nl}(\alpha) Y_{lm}(\theta, \varphi),$$
$$n = l+1, \quad l+2, \ldots, \quad -l \leqslant m \leqslant l;$$
$$\Pi_{nl}(\alpha) = \left[\frac{\pi}{2} n^2(n^2-1)\ldots(n^2-l^2)\right]^{-1/2} \times (\sin\alpha)^l \left(\frac{d}{d\cos\alpha}\right)^{l+1} \cos n\alpha. \tag{15}$$

The angles α, θ, φ are introduced on the hypersphere in the usual manner:

$$\xi_0 = \cos\alpha, \quad \xi_1 = \sin\alpha\sin\theta\cos\varphi,$$
$$\xi_2 = \sin\alpha\sin\theta\sin\varphi, \quad \xi_3 = \sin\alpha\cos\theta. \tag{16}$$

The functions $Y_{nlm}(\xi)$ are solutions of (14) (taking the upper sign) for $\eta = -n$ and satisfy the normalization condition

$$\int \frac{d^3\xi}{\xi_0} Y_{nlm}(\xi) Y^*_{n'l'm'}(\xi) = \delta_{nn'}\delta_{ll'}\delta_{mm'}. \tag{15a}$$

The generalized spherical harmonics in the case $E > 0$ can be obtained from (15) by the replacement

$$\alpha \to i\alpha, \quad n \to i\rho/2.$$

Here ρ is a continuous variable, $0 \leq \rho < \infty$. This has already been noted by Fock[1], and was later discussed in detail in the papers by Dolginov et al.[18,19] We give explicit formulas for these functions:

$$Y_{\rho lm}(\alpha, \theta, \varphi) = \Pi_{\rho l}(\alpha) Y_{lm}(\theta, \varphi),$$
$$\Pi_{\rho l}(\alpha) = \left[\pi\left(\frac{\rho}{2}\right)^2\left(\frac{\rho^2}{4}+1^2\right) \ldots \left(\frac{\rho^2}{4}+l^2\right)\right]^{-1/2} (\text{sh}\,\alpha)^l \left(\frac{d}{d\,\text{ch}\,\alpha}\right)^{l+1} \cos\frac{\rho\alpha}{2}. \tag{17*}$$

The normalization condition for $Y_{\rho lm}(\alpha, \theta, \varphi)$ has the following form:

$$\int_0^\infty \text{sh}^2\alpha\, d\alpha \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi\, Y_{\rho lm}(\alpha,\theta,\varphi) Y_{\rho' l'm'}(\alpha,\theta,\varphi) = \delta(\rho-\rho')\delta_{ll'}\delta_{mm'}. \tag{17a}$$

Formula (17) for $\Pi\ (\alpha)$ is convenient for small values of l. But if $l \gg 1$, then it is better to utilize the expression for $\Pi_{\rho l}(\alpha)$ in terms of the hypergeometric function:

$$\Pi_{\rho l}(\alpha) = N_{\rho l}(\text{sh}\,\alpha)^l F \times \left(l+1+\frac{i\rho}{2},\ l+1-\frac{i\rho}{2};\ l+\frac{3}{2};\ \frac{1-\text{ch}\,\alpha}{2}\right)$$
$$N_{\rho l} = (-)^{l+1}\left[\frac{\rho^2}{4}\left(\frac{\rho^2}{4}+1^2\right)\ldots\left(\frac{\rho^2}{4}+l^2\right)\right]^{1/2} \Big/ 2^{l+1}\Gamma\left(l+\frac{3}{2}\right). \tag{17b}$$

The analogous formula in the case of the discrete spectrum has the form

$$\Pi_{nl}(\alpha) = N_{nl}(\sin\alpha)^l F(n+l+1, -(n-l-1);\ l+3/2;\ (\sin\alpha/2)^2),$$
$$N_{nl} = [n^2(n^2-1^2)\ldots(n^2-l^2)]^{1/2} / 2^{l+1/2}\Gamma(l+3/2). \tag{15b}$$

The functions $Y_{\rho lm}(\alpha, \theta, \varphi)$ form a complete system on each of the two sheets of the hyperboloid (on the upper sheet $\xi_0 = \cosh\alpha$, while on the lower sheet $\xi_0 = -\cosh\alpha$; the formulas for ξ_i are obtained from (16) by replacing $\sin\alpha \to \sinh\alpha$). In the paper by Dolginov and Toptygin[18] it is shown that the set of functions $Y_{\rho lm}(\alpha, \theta, \varphi)$ with fixed ρ forms a canonical basis for the infinite-dimensional representation $D(0,\rho)$ of the Lorentz group. Under Lorentz transformations of the variable ξ_μ the hyperboloid $\xi_\mu\xi^\mu = 1$ goes over into itself, and the integral equation (14) remains invariant. Therefore, the functions $\psi(\xi)$ transform in accordance with a representation of the Lorentz group, and, as follows from Sec. 2, for the states with a given energy E this representation coincides with the irreducible representation $D(0,\rho)$. Therefore, the following equation must be satisfied:

$$\psi_{Elm}(\xi) = (p^2-p_0^2)^2\psi_{Elm}(\mathbf{p}) = \begin{cases} C_1 Y_{\rho lm}(\xi) & \text{for } p > p_0 \\ C_2 Y_{\rho lm}(\xi) & \text{for } p < p_0 \end{cases}. \tag{18}$$

The angle α varies from 0 to $+\infty$; on the upper sheet $p = p_0\coth(\alpha/2)$, $\xi_0 = \cosh\alpha$, while on the lower sheet $p = p_0\tanh(\alpha/2)$, $\xi_0 = -\cosh\alpha$. The expressions for $Y_{\rho lm}(\xi)$ in terms of the angles α, θ, φ have the same form (17) for both sheets. The constants C_1 and C_2 in (18) do not depend on l and m; in order to obtain them we substitute (18) into the integral equation (14). Taking into account the expansion (A.8) (cf., Appendix A) for the kernel $[2\pi^2(\xi-\xi')^2]^{-1}$ and the normalization condition (17a) we obtain for C_1 and C_2 the system of equations (the functions F_1 and F_2 are defined in Appendix A):

$$(1-\eta F_1)C_1 + \eta F_2 C_2 = 0, \quad -\eta F_2 C_1 + (1+\eta F_1)C_2 = 0,$$

from which it follows that

$$\eta = \pm\rho/2, \quad C_2/C_1 = -e^{\pi\eta}.$$

*ch ≡ cosh, sh ≡ sinh.

If C_1, C_2 are normalized by the condition $|C_1|^2 + |C_2|^2 = 1$, then we have

$$C_1 = (1 + e^{2\pi\eta})^{-1/2}, \quad C_2 = -(1 + e^{-2\pi\eta})^{-1/2}. \tag{19}$$

We now note an interesting symmetry possessed by the wave functions of the hydrogen atom in the p-representation (for $E > 0$). We take two vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ corresponding to the same values of the angles α, θ, φ ($\mathbf{p}_1 = p_1\mathbf{n}$, $\mathbf{p}_2 = p_2\mathbf{n}$, $p_1p_2 = p_0^2$). The points $\mathbf{p}_1$ and $\mathbf{p}_2$ go over into each other under an inversion with respect to the sphere $|\mathbf{p}| = p_0$ in momentum space. From (18), (19), it follows that for any arbitrary state belonging to the given energy E the identity

$$\frac{\psi(\mathbf{p}_1)}{\psi(\mathbf{p}_2)} = -e^{-\pi\eta}\left|\frac{p_1^2 - p_0^2}{p_2^2 - p_0^2}\right|^{-2} \quad (p_1 > p_2) \tag{20}$$

is satisfied. Therefore, for a complete definition of the wave function $\psi(\mathbf{p})$ for all values of $\mathbf{p}$ it is sufficient to specify it only over one of the regions $|\mathbf{p}| > p_0$ or $|\mathbf{p}| < p_0$.

The existence of the symmetry (20) for the Coulomb functions is associated with the topology of ξ-space: the point is that the surface $\xi^\mu\xi_\mu = 1$ for $E > 0$ is doubly connected. The symmetry (20) is also preserved in the case of analytic continuation into the region $E < 0$, i.e., in going over to the states of the discrete spectrum. Since in this case $p_0 \to ip_0$ (if $p_0 = (2mE)^{1/2}$), then we must consider the values of the wave function at the points $\mathbf{p}_1$ and $\mathbf{p}_2$ which go over into one another after reflection in the origin of coordinates and inversion with respect to the sphere $|\mathbf{p}| = p_0$ (i.e., $\mathbf{p}_1 = p_1\mathbf{n}$, $\mathbf{p}_2 = -p_2\mathbf{n}$, $p_1p_2 = p_0^2 = 2m|E|$; in terms of the variables ξ_μ this transformation is simply an inversion: $\xi_\mu \to -\xi_\mu$). From (15) it can be seen that

$$\Pi_{nl}(\pi - \alpha) = (-1)^{n-l-1}\Pi_{nl}(\alpha),$$

from which it follows that

$$Y_{nlm}(\pi - \alpha, -\mathbf{n}) = (-1)^{n-1}Y_{nlm}(\alpha, \mathbf{n}).$$

For a wave function belonging to a level with the principal quantum number n we obtain the relation

$$\frac{\psi(\mathbf{p}_1)}{\psi(\mathbf{p}_2)} = (-1)^{n-1}\left|\frac{p_1^2 + p_0^2}{p_2^2 + p_0^2}\right|^{-2}. \tag{20a}$$

Formally it follows from (20), if we take into account that $\eta = in$ for a level with the principal quantum number n. Formulas (20) and (20a) show that all the states in the Coulomb field possess the characteristic symmetry[4)].

[4)]Comparison of (20) with (20a) shows that a transition from $\mathbf{p}_1$ to $\mathbf{p}_2$ in the case $E < 0$ includes the reflection ($\mathbf{p} \to -\mathbf{p}$), while for $E > 0$ it does not include it. However, if we go over to the variables ξ_μ, then we find that in both cases the points ξ_1 and ξ_2 are symmetric with respect to the origin of coordinates: $\xi_{1\mu} = -\xi_{2\mu}$. Therefore, the symmetry (20) or (20a) is a consequence of the invariance of (14) with respect to inversion in ξ-space. Investigating the stereographic projection of the unit hypersphere (or of the hyperboloid of two sheets) one can easily see the reason that the "corresponding" points $\mathbf{p}_1$ and $\mathbf{p}_2$ for $E > 0$ lie in p-space on the same side of the origin of coordinates, while for $E < 0$ they lie on opposite sides.

The behavior of the Coulomb functions $\psi(\mathbf{p})$ near the mass surface $p^2 = p_0^2$ is of a very characteristic nature. We let

$$\psi_{Elm}(\mathbf{p}) = R_{El}(p)\,Y_{lm}(\theta, \varphi);$$

and from (18) obtain

$$R_{El}(p) = \frac{\text{const}}{p^2 - p_0^2}\,e^{\mp\pi\eta/2}\sin\left\{\eta\ln\left|\frac{p - p_0}{2p_0}\right| + \sigma_l - \sigma_0\right\}, \tag{21}$$

where the upper (lower) sign corresponds to $p > p_0$ ($p < p_0$), while σ_l is the Coulomb scattering phase: $\sigma_l = \arg\Gamma(l + 1 + i\eta)$. The point $p = p_0$ is not a simple pole of $\psi_{Elm}(\mathbf{p})$, as usual, but a branch point. Figure 1 shows qualitatively that the behavior of $R_{El}(p)$ for $p \to p_0$. Going over from

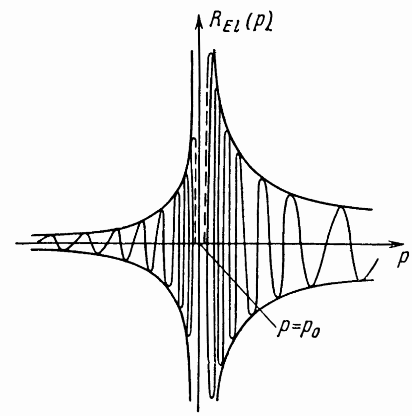

FIG. 1. Behavior of the function $R_{El}(p)$ near the mass surface $p = p_0$.

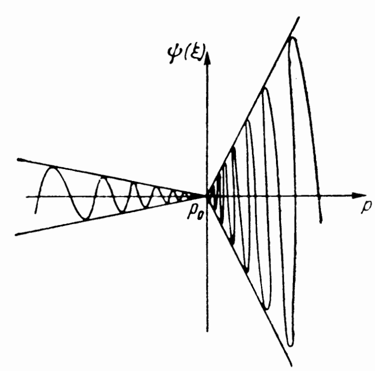

FIG. 2. Behavior of the radial part of the function $\psi_{Elm}(\xi)$ near the mass surface $p = p_0$.

$\psi(\mathbf{p})$ to the function $\psi(\xi)$ in accordance with (12) we see that at the point $p = p_0$ the function $\psi(\xi) = 0$ and is continuous (cf., Fig. 2); moreover, $p = p_0$ corresponds in ξ-space to infinite values of α. Thus, the transition from the momentum p to the variables ξ^μ and from $\psi(\mathbf{p})$ to $\psi(\xi)$ leads to the singularity in the wave function on the mass surface $p = p_0$ being smoothed out and removed to infinity.

The statements made above referred to the regular solution of the Schrödinger equation in a Coulomb field. As is well known, there also exists an irregular solution which in x-space is expressed in terms of Whittaker[20] functions. We describe the principal properties of the irregular solution[21]. For $p \to p_0$ it has the form

$$G_{El}(p) = \frac{\text{const}}{p^2 - p_0^2} e^{\mp\pi\eta/2} \cos\left\{\eta \ln \left| \frac{p - p_0}{2p_0} \right| + \sigma_l - \sigma_0\right\}. \tag{22}$$

The points $p = 0$ and $p = \infty$ are also singular points; the behavior of the corresponding functions $\psi(\xi)$ at these points is of the following nature:

$$\psi_{El}^{(\text{reg})}(\xi) \sim \begin{cases} (p_0/p)^l & \text{for } p \to \infty \\ (p/p_0)^l & \text{for } p \to 0 \end{cases}, \tag{23}$$

$$\psi_{El}^{(\text{sing})}(\xi) \sim \begin{cases} (p/p_0)^{l+2} & \text{for } p \to \infty \\ (p_0/p)^{l+2} & \text{for } p \to 0 \end{cases}. \tag{24}$$

Thus, the irregular solution becomes infinite at the vertices of both sheets of the hyperboloid ($\xi_0 \pm 1$, $\xi = 0$) and, therefore, cannot be expanded in terms of the system of functions $Y_{\rho l m}(\xi)$.

Finally, we note that the symmetry property (20) is preserved also for the irregular solution in a Coulomb field[21].

4. THE CASE E = 0

In an attractive Coulomb field there exist states with $E = 0$. From (2) it can be seen that in this case $[A_i, A_j] = 0$, and the operators L_i, A_j are converted into a system of generators of the nonrelativistic Galilean group[5)]. Carrying out in the Schrödinger equation (9) the following change of variables:

$$\xi = \frac{2p_c\mathbf{p}}{p^2}, \qquad |\mathbf{p} - \mathbf{p}'|^2 = p_c^2 \frac{|\xi - \xi'|^2}{\xi^2\xi'^2} \tag{10a}$$

and going over to the new function

$$\psi(\xi) = \text{const}\cdot(p / 2p_c)^4\psi(\mathbf{p}), \tag{12a}$$

[5)]Or of the group of displacements in three-dimensional space which is isomorphic to it; this manner of conversion of one group into another is referred to as the contraction of the group; cf., for example,[22].

we obtain

$$\psi(\xi) - \frac{1}{2\pi^2}\int \frac{\psi(\xi')}{|\xi - \xi'|^2} d\xi' = 0. \tag{14a}$$

This equation is translationally invariant in three-dimensional ξ-space and expresses the symmetry characteristic of the Coulomb functions of zero energy. Its solution has the form

$$\psi(\xi) = e^{i\mathbf{a}\xi}, \qquad |\mathbf{a}| = 1.$$

States with a definite value of the orbital angular momentum are obtained by means of the well-known expansion of a plane wave[23]; they have the form

$$\psi_{0lm}(\xi) = \text{const}\cdot j_l(\xi) Y_{lm}(\xi / \xi), \tag{25}$$

from where we obtain

$$\psi_{0lm}(\mathbf{p}) = \text{const}\cdot\left(\frac{2p_c}{p}\right)^4 j_l\left(\frac{2p_c}{p}\right) Y_{lm}\left(\frac{\mathbf{p}}{p}\right). \tag{26}$$

From this with the aid of a Fourier transformation we obtain the wave function in the x-representation:

$$\psi_{0lm}(\mathbf{r}) = \text{const}\cdot\left(\frac{8r}{r_0}\right)^{-1/2} J_{2l+1}\left(\frac{8r}{r_0}\right)^{1/2} Y_{lm}\left(\frac{\mathbf{r}}{r}\right), \tag{27}$$

well known in the literature[23].

We note that the equation for a repulsive Coulomb potential differs from (14a) only by the sign in front of the integral. It can be easily shown that such an equation has no bounded solutions except the one which is identically equal to zero. This corresponds to the fact that in a repulsive potential the wave function with $E = 0$ reduces to zero.

5. THE COULOMB GREEN'S FUNCTION FOR E < 0

The Coulomb Green's function $G(\mathbf{p}, \mathbf{p}', E)$ satisfies the integral equation:

$$\left(\frac{p^2}{2m} - E\right) G(\mathbf{p}, \mathbf{p}') + \frac{Z_1Z_2e^2}{2\pi^2\hbar}\int \frac{G(\mathbf{p}'', \mathbf{p}')}{|\mathbf{p} - \mathbf{p}''|^2} d\mathbf{p}'' = -\delta(\mathbf{p} - \mathbf{p}'). \tag{28}$$

As has been shown by Fock[1], in order to bring the Schrödinger equation into a form obviously invariant with respect to the group O(4) it is necessary to carry out the transformations (10) and (12). From here follows the necessity of a similar transformation of the Green's function (cf., [11]):

$$G(\xi, \xi') = -\frac{1}{16mp_0^3}(p^2 \pm p_0^2)^2 G(\mathbf{p}, \mathbf{p}'; E)(p'^2 \pm p_0^2)^2. \tag{29}$$

The upper (lower) signs here (and in Eq. (30)) refer to the case $E < 0$ ($E > 0$); $E = 0$ represents the special case discussed in Sec. 8. The function $G(\xi, \xi')$ is defined over a four-dimensional hypersphere (or hyperboloid) and satisfies an equation invariant with respect to the appropriate group of transformations:

$$G(\xi, \xi') \pm \frac{\eta}{2\pi^2} \int \frac{d^3\xi''}{\xi_0''} \frac{1}{(\xi - \xi'')^2} G(\xi'', \xi') = \delta(\xi - \xi''). \quad (30)$$

Since $G(\xi, \xi')$ has simpler properties than the initial $G(\mathbf{p}, \mathbf{p}'; E)$, while the connection between them is trivial we shall in future call $G(\xi, \xi')$ the Coulomb Green's function. Schwinger [11] has obtained an expansion for it in terms of the irreducible representations of the O(4) group:

$$G(\xi, \xi') = \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \sum_{m=-l}^{l} \frac{Y_{nlm}(\xi) Y_{nlm}^*(\xi')}{1 + \eta/n}, \quad (31)$$

which is valid for $E < 0$ (the functions $Y_{nml}(\xi)$ are defined in (15)). For the remaining values of E one can obtain $G(\xi, \xi')$ by means of analytic continuation. One of the possible ways of doing this is shown in [11]; below a somewhat different method is given for transforming formula (31) which enables us to express $G(\xi, \xi')$ in terms of known analytic functions.

Since the sum

$$\sum_{l=0}^{n-1} \sum_{m=-l}^{l} Y_{nlm}(\xi) Y_{nlm}^*(\xi')$$

is an invariant of the representation $D((n-1)/2, (n-1)/2)$ of the O(4) group it can be evaluated in any system of coordinates. Making the fourth axis lie along the vector ξ and utilizing (15), we obtain

$$\sum_{l=0}^{n-1} \sum_{m=-l}^{l} Y_{nlm}(\xi) Y_{nlm}^*(\xi') = \frac{n}{2\pi^2} \frac{\sin n\chi}{\sin \chi}, \quad (32)$$

where $|\xi - \xi'| = 2\sin(\chi/2)$. Therefore

$$G(\xi, \xi') = \frac{1}{2\pi^2 \sin\chi} \sum_{n=1}^{\infty} n \left(1 + \frac{\eta}{n}\right)^{-1} \sin n\chi \quad (33)$$

$$= \delta(\xi - \xi') - \frac{\eta}{2\pi^2 (\xi - \xi')^2} + \frac{\eta^2}{4\pi^2 i \sin\chi} [\Phi(e^{i\chi}, \eta) - \Phi(e^{-i\chi}, \eta)].$$

Here $\Phi(z, \eta)$ denotes the following function:

$$\Phi(z, \eta) = \sum_{n=0}^{\infty} \frac{z^n}{n + \eta}, \quad |z| < 1. \quad (34)$$

It can be analytically continued into the whole complex z-plane with the exception of the cut $1 < z < \infty$. A summary of the principal properties of the function $\Phi(z, \eta)$ is given in Appendix B.

6. ANALYTIC CONTINUATION INTO THE REGION $E > 0$

Going over to positive energy we come across the cut $0 < E < \infty$ on the upper and lower edge of which two different Green's functions $G^{\pm}(\xi, \xi')$ are defined. In x-space they correspond to diverging (converging) waves. Therefore it is necessary to indicate in (30) the rule for going around singularities. Moreover, the four-dimensional hypersphere in ξ-space is converted into a hyperboloid of two sheets.

We introduce two sign functions σ and σ' defined in the following manner:

$$\sigma = \begin{cases} +1 & \text{on the upper edge of the cut } (E = E_0 + i\delta) \\ -1 & \text{on the lower edge of the cut } (E = E_0 - i\delta) \end{cases};$$

$$\sigma' = \begin{cases} +1, & \text{if } \xi_0 > 0, \ \xi_0' > 0 \\ -1, & \text{if } \xi_0 < 0, \ \xi_0' < 0 \end{cases}.$$

From the explicit expression for $\eta(E)$ it follows that in order to go over to $G^{(\sigma)}(\xi, \xi')$ it is necessary to carry out the following substitutions:

$$\eta \to i\sigma\eta, \quad \eta^2 \to -\eta^2 + i\sigma\delta \quad (\delta \to +0). \quad (35)$$

The angle χ between the unit vectors ξ and ξ'' is defined for $E < 0$ by the condition $(\xi\xi'') = \cos\chi$ and varies from 0 to π. In going over to $E > 0$ the variable χ passes into the complex plane, with $\chi \to \pm i\chi$ corresponding to the case when ξ and ξ'' lie on one sheet of the hyperboloid, and $\chi \to \pi \pm i\chi$ corresponding to the case when ξ and ξ'' lie on different sheets (cf., Fig. 3).

Carrying out in (13) the replacement

$$p_0^2 \to p_0^2 + i\sigma\delta, \quad \delta \to +0,$$

we obtain

$$\operatorname{Im}(\xi - \xi'')^2 = -\sigma(p^2 + p''^2 - 2p_0^2)\delta = -\sigma\sigma'\delta,$$

since the imaginary addition to $(\xi - \xi'')^2$ in (30) is significant only in the case when the points ξ, ξ'' belong to the same sheet of the hyperboloid. On the other hand, from the formula

$$(\xi - \xi'')^2 = 2(1 - \cos\chi)$$

it follows that for $0 < \operatorname{Re}\chi < \pi$ the sign of

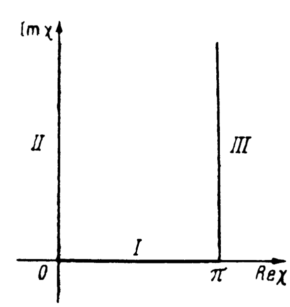

FIG. 3. The contour showing the variation of the angle χ – solid line; the region I corresponds to $E < 0$, the regions II and III correspond to $E > 0$. The scalar product of the vectors ξ, and ξ' is equal to: $(\xi, \xi') = \cos\chi$, $\cosh\chi$, $-\cosh\chi$ respectively in regions I, II, and III.

$\text{Im}(\xi - \xi'')^2$ agrees with the sign of $\text{Im}\,\chi$. From this results the following rule for the analytic continuation of the angle χ into the region $E > 0$:

$$\chi \to \begin{cases} -i\sigma\sigma'\chi & \text{when } \xi_0\xi_0'' > 0 \\ \pi \pm i\chi & \text{when } \xi_0\xi_0'' < 0 \end{cases}. \quad (36)$$

The hyperbolic angle χ on the right hand side of (36) varies from 0 to $+\infty$; the choice of the sign in the second case is unimportant. For $(\xi - \xi'')^2$ we obtain the following expressions:

$$(\xi - \xi'')^2 = \begin{cases} -4\,\text{sh}^2(\chi/2) & \text{for } \xi_0\xi_0'' > 0 \\ 4\,\text{ch}^2(\chi/2) & \text{for } \xi_0\xi_0'' < 0 \end{cases}.$$

It is convenient to carry out the explicit analytic continuation of $G(\xi, \xi')$ with the aid of the integral representation (B.5) for the difference of the Φ-functions appearing in (33). The final result has the form

$$G(\xi, \xi') = \delta(\xi - \xi') \mp \frac{\eta}{2\pi^2(\xi - \xi')^2} + \frac{\eta^2}{2\pi^2} F(\xi, \xi'), \quad (37)$$

where the sign $\mp$ agrees with the sign of E, while the function $F(\xi, \xi')$ has the following integral representations:

$$F(\xi, \xi') = \frac{\eta}{\sin\chi} \int_0^\infty \frac{\text{sh}(\pi - \chi)k}{\text{sh}\,\pi k} \times \frac{dk}{k^2 + \eta^2} \quad \text{for } E < 0 \quad (\cos\chi = (\xi\xi')); \quad (38a)$$

$$F(\xi, \xi') = \frac{1}{\text{sh}\,\chi} \int_0^\infty \frac{\sigma' k - \eta\,\text{cth}\,\pi k}{k^2 - \eta^2 + i\sigma\delta} \sin k\chi\, dk$$

$$\text{for } E > 0, \quad \xi_0\xi_0' > 0 \quad (\text{ch}\,\chi = (\xi\xi')); \quad (38b)^*$$

$$F(\xi, \xi') = \frac{\eta}{\text{sh}\,\chi} \int_0^\infty \frac{\sin k\chi}{\text{sh}\,\pi k\,(k^2 - \eta^2 + i\sigma\delta)} dk$$

$$\text{for } E > 0, \quad \xi_0\xi_0' < 0 \quad (\text{ch}\,\chi = -(\xi\xi')). \quad (38c)$$

7. EXPANSION OF THE COULOMB GREEN'S FUNCTION IN TERMS OF THE IRREDUCIBLE REPRESENTATIONS OF THE LORENTZ GROUP

Formula (37) defines $G(\xi, \xi')$ for arbitrary values of E. However, for many purposes it is convenient to have the expansion of $G(\xi, \xi')$ in terms of an orthonormal system of functions. For the case $E < 0$ such an expansion is given by formula (31). For $E > 0$ the dynamic symmetry of the hydrogen atom is described by the Lorentz group, and (31) should be replaced by the expansion of $G(\xi, \xi')$ in terms of the irreducible representations of the Lorentz group.

As follows from Sec. 2, such an expansion contains only the representations $D(0, \rho)$ of the principal series of infinite-dimensional unitary representations of the Lorentz group. The general form of this expansion is:

$$G(\xi, \xi') = \int_0^\infty d\rho\, g_{ij}(\rho) \sum_{lm} Y_{\rho lm}(\xi) Y^*_{\rho lm}(\xi'). \quad (39)$$

The indices i and j indicate the position of the points ξ, ξ' on the sheets of the hyperboloid; $i = 1(2)$, if ξ lies on the upper (lower) sheet; similarly, the index j is associated with the position of ξ'. Since $G(\xi, \xi') = G(\xi', \xi)$, we have $g_{12}(\rho) = g_{21}(\rho)$.

In order to determine the unknown functions $g_{ij}(\rho)$ we substitute (39) into the integral equation:

$$G^\pm(\xi, \xi') - \frac{\eta}{2\pi^2} \int \frac{d\xi''}{\xi_0''} \frac{G^\pm(\xi'', \xi')}{(\xi - \xi'')^2 - i\sigma\sigma'\delta} = \delta(\xi - \xi'). \quad (40)$$

The calculation is analogous to that in Sec. 3 for the wave function. We give the answer

$$g_{11}^\pm(\rho) = \left(1 - \frac{\eta}{p}\,\text{cth}\,\pi p\right) \frac{p^2}{p^2 - \eta^2 \pm i\delta},$$

$$g_{12}^\pm(\rho) = g_{21}^\pm(\rho) = \frac{\eta p}{\text{sh}\,\pi p} \frac{1}{p^2 - \eta^2 \pm i\delta},$$

$$g_{22}^\pm(\rho) = -\left(1 + \frac{\eta}{p}\,\text{cth}\,\pi p\right) \frac{p^2}{p^2 - \eta^2 \pm i\delta}. \quad (41)$$

Here $p = \rho/2$, $\delta \to +0$.

With the aid of the identity

$$\frac{p^2}{p^2 - \eta^2 \pm i\delta} = 1 + \frac{\eta^2}{p^2 - \eta^2 \pm i\delta}$$

it is possible to separate out from $G(\xi, \xi')$ its most singular terms. Finally, we obtain a formula analogous to (37):

$$G_{ij}(\xi, \xi') = \delta(\xi - \xi') + \frac{\eta}{2\pi^2(\xi - \xi')^2} + F_{ij}^\pm(\xi, \xi'). \quad (42)$$

We note that for the functions $G_{12}(\xi, \xi') = G_{21}(\xi, \xi')$ the delta-function in (42) is identically equal to zero.

Formulas (39), (41) give the desired expansion of $G(\xi, \xi')$ in terms of the irreducible representations of the Lorentz group. We now show that the Green's function obtained above coincides with the expression (37) obtained by means of analytic continuation. In order to do this we note that the sum over l and m in (39) is an invariant of the representation $D(0, \rho)$. Utilizing for its evaluation the same method as in (32) we obtain

$$\sum_{lm} Y_{\rho lm}(\xi) Y^*_{\rho lm}(\xi') = \frac{\rho}{8\pi^2} \frac{\sin(\rho\chi/2)}{\text{sh}\,\chi} \quad (43)$$

*cth ≡ coth.

(the angle χ is defined in (38)). From here we obtain for $F_{ij}^{\pm}(\xi, \xi')$ a representation in the form of a Fourier integral:

$$F_{ij}^{\pm}(\xi, \xi') = \frac{\eta^2}{2\pi^2 \operatorname{sh}\chi} \int_0^\infty \frac{f_{ij}(k)}{k^2 - \eta^2 \pm i\delta} \sin k\chi \, dk,$$

$$f_{11}(k) = k - \eta \operatorname{cth} \pi k, \quad f_{12}(k) = f_{21}(k) = \eta / \operatorname{sh} \pi k,$$
$$f_{22}(k) = -(k + \eta \operatorname{cth} \pi k). \tag{44}$$

Comparison with (38) shows the identity of the two formulas.

We note that the expansion (39) cannot be obtained from (31) by means of analytic continuation with respect to E. From a comparison of formulas (31) and (41) it follows that the coefficients in the expansion of $G(\xi, \xi')$ in terms of the representations $D(0, \rho)$ of the Lorentz group (for $E > 0$) are not analytic continuations of the coefficients in the expansion of $G(\xi, \xi')$ in terms of the representations $D(j, j)$ of the O(4) group (for $E < 0$). And this is not surprising, since the wave functions of the continuous spectrum (18) are already not given by a simple analytic continuation of the wave functions $Y_{nlm}(\xi)$ of the discrete spectrum. In the final analysis the outcome is associated here with the fact that the surface $\xi^\mu \xi_\mu = 1$ has a different topology for $E < 0$ and $E > 0$.

There exists a different method of expanding the wave functions in terms of the unitary representations of the Lorentz group proposed by Shapiro [15] and developed further in a number of papers [16,24,25] As has been shown in [15], such an expansion of the scalar function $f(\xi)$ given over one of the sheets of the hyperboloid $\xi^\mu \xi_\mu = 1$ has the form

$$C_\rho(\mathbf{n}) = \left(\frac{1}{4\pi}\right)^{3/2} \int \frac{d^3\xi}{\xi_0} (\xi_0 - \boldsymbol{\xi}\mathbf{n})^{-1+i\rho/2} f(\xi),$$
$$f(\xi) = \left(\frac{1}{4\pi}\right)^{3/2} \int_0^\infty \rho^2 \, d\rho \int d\Omega_{\mathbf{n}} (\xi_0 - \boldsymbol{\xi}\mathbf{n})^{-1-i\rho/2} C_\rho(\mathbf{n}). \tag{45}$$

The amplitudes $C_\rho(\mathbf{n})$ for a given value of ρ transform in accordance with the representation $D(0, \rho)$ of the basic series with

$$\int \frac{d^3\xi}{\xi_0} |f(\xi)|^2 = \int_0^\infty \rho^2 \, d\rho \int \frac{d\Omega_{\mathbf{n}}}{4\pi} |C_\rho(\mathbf{n})|^2. \tag{46}$$

Utilization of (45) leads to the following expansion of the wave functions of the continuous spectrum of the hydrogen atom:

$$\psi_{Elm}(\xi) = (p^2 - p_0^2)^2 \psi_{Elm}(\mathbf{p})$$
$$= \int_0^\infty \rho^2 \, d\rho \int d\Omega_{\mathbf{n}} |\xi_0 - \boldsymbol{\xi}\mathbf{n}|^{-1-i\rho/2} C_\rho(\mathbf{n}),$$
$$C_\rho(\mathbf{n}) = C_i \frac{\sqrt{2\pi}}{\rho} (-1)^{l+1} e^{i(\sigma_l - \sigma_0)} Y_{lm}(\mathbf{n}) \, \delta\left(\rho - \left(\frac{2E_C}{E}\right)^{1/2}\right). \tag{47}$$

The constants C_1 and C_2 are defined in (19) and correspond to the cases $\xi_0 > 1$ ($p > p_0$) and $\xi_0 < -1$ ($p < p_0$); σ_l is the Coulomb scattering phase. The corresponding expansion of $G(\xi, \xi')$ follows in a similar manner from (39) and has the form [6)]

$$G^\pm(\xi, \xi') = \frac{1}{4\pi^2} \int_0^\infty \rho^2 \, d\rho$$
$$\times \int d\Omega_{\mathbf{n}} |\xi_0 - \boldsymbol{\xi}\mathbf{n}|^{-1-i\rho/2} |\xi_0' - \boldsymbol{\xi}'\mathbf{n}|^{-1+i\rho/2} g_{ij}^{\pm}(\rho) \tag{48}$$

(the spectral densities $g_{ij}^{\pm}(\rho)$ are defined in (41)).

8. THE SPECIAL CASE: E = 0

As has been noted already in Sec. 4, a special case arises for $E = 0$ in an attractive field: the Lorentz group degenerates into the nonrelativistic Galilean group. In accordance with this there arises the expansion of the Coulomb Green's function for $E = 0$ in terms of the representations of the Galilean group. For the Green's function instead of (30) we have the equation

$$G_0(\xi, \xi') \pm \frac{1}{2\pi^2} \int d\xi'' \frac{G_0(\xi, \xi'')}{|\xi - \xi''|^2} = \delta(\xi - \xi'),$$
$$G_0(\xi, \xi') = -\frac{(pp')^4}{16 m p_C^3} G(\mathbf{p}, \mathbf{p}'; 0); \tag{49}$$

the signs ± correspond to repulsion (attraction), while the variables ξ are related to the momentum in accordance with (10a).

The solution of (49) has the form

$$G_0(\xi, \xi') = \frac{1}{(2\pi)^3} \int \left(1 \pm \frac{1}{k}\right)^{-1} \exp[i\mathbf{k}(\xi - \xi')] \, d\mathbf{k}. \tag{50}$$

The plane waves $e^{i\mathbf{k}\xi}$ generate the irreducible representations of the Galilean group. Picking out in (50) the most singular terms we obtain

$$G_0(\xi, \xi') = \delta(\xi - \xi') \mp \frac{1}{2\pi^2 (\xi - \xi')^2} + F_0(\xi, \xi'),$$
$$F_0(\xi, \xi') = \frac{1}{(2\pi)^3} \int \frac{\exp[i\mathbf{k}(\xi - \xi')]}{k(k \pm 1)} d\mathbf{k} \tag{51}$$

(the singularity in F_0 arises only in an attractive

6) I. S. Shapiro has drawn the attention of the authors to the fact that in the usual derivation of the transformation (45) it is assumed that the norm of (46) is finite. In the case of the continuous spectrum at present under consideration the amplitudes $C_\rho(\mathbf{n})$, as can be seen from (47), are not quadratically integrable, and yet the integrals in (45) converge. Therefore, in the derivation of formulas (47), and (48), it is in effect assumed that the transformation (45) due to Shapiro can be generalized to the case of functions with unbounded norm. A similar generalization of the Fourier transformation is constantly used in physics, and at the present time has also received a rigorous mathematical foundation.

field). The function $F_0(\xi, \xi')$ can be expressed in terms of the sine integral and the cosine integral.

9. GENERALIZATION TO THE n-DIMENSIONAL CASE

In conclusion we shall say a few words regarding the generalization of the problem considered above to the case of n-dimensional space[7] (such a generalization is not only of methodological interest, but, apparently, can find an application in the many-body problem). The Schrödinger equation with a "Coulomb" interaction has in the n-dimensional case the form

$$\left(\frac{p^2}{2m} - E + \frac{Z_1 Z_2 e^2}{r}\right)\psi = 0 \qquad (52)$$

in the x-representation and

$$\left(\frac{p^2}{2m} - E\right)\psi(p) + \frac{Z_1 Z_2 e^2}{2\pi^{(n+1)/2}}\Gamma\left(\frac{n-1}{2}\right)\int \frac{\psi(p')\,d^n p'}{|p-p'|^{n-1}} = 0 \qquad (53)$$

in the p-representation.

Let $E < 0$; going over to the new function

$$\psi(\xi) = \text{const}\cdot(p^2 + p_0^2)^{(n+1)/2}\psi(p) \qquad (54)$$

and taking into account the equations

$$\frac{1}{|p-p'|^{n-1}} = \left[\frac{4p_0^2}{(p^2+p_0^2)(p'^2+p_0^2)}\right]^{(n-1)/2}\frac{1}{[(\xi-\xi')^2]^{(n-1)/2}},$$

$$d^n p = \left(\frac{p^2+p_0^2}{2p_0}\right)^n \frac{d^n\xi}{\xi_0},$$

we obtain the integral equation

$$\psi(\xi) + \frac{\eta}{2\pi^{(n+1)/2}}\Gamma\left(\frac{n-1}{2}\right)\int \frac{d^n\xi'}{\xi_0'}\frac{\psi(\xi')}{(\xi-\xi')^{n-1}} = 0 \qquad (55)$$

(here, as before, $p_0 = (2m|E|)^{1/2}$, $\eta = Z_1 Z_2 e^2 m/p_0$).

The solutions of this equation are the spherical harmonics in $n+1$-dimensional space $Y_{N\nu}(\xi)$; the functions $f_N = |\xi|^N Y_{N\nu}(\xi)$ are homogeneous polynomials of degree N in the variables $\xi_1, \xi_2, \ldots, \xi_{n+1}$, satisfying the $(n+1)$-dimensional Laplace equation $\Delta f = 0$. The number N (the degree of the homogeneous polynomial) plays in the n-dimensional problem the same role as the principal quantum number in the three-dimensional case: all the states with the same value of N have the same energy

$$E_N = -E_C/2[N + (n-1)/2]^2,$$

with the degree of degeneracy of the level of energy E_N being equal to

$$d_N = (N+n-2)!\,(2N+n-1)/(n-1)!\,N!$$

Equation (55) is invariant with respect to inversion ($\xi_\mu \to -\xi_\mu$) and as a result of this its solutions have definite parity: $Y_{N\nu}(-\xi) = (-1)^N Y_{N\nu}(\xi)$. Taking (53) into account we obtain from this the following symmetry relation for the wave functions of the discrete spectrum belonging to the level E_N:

$$\frac{\psi(\mathbf{p}_1)}{\psi(\mathbf{p}_2)} = (-)^N \left|\frac{p_1^2+p_0^2}{p_2^2+p_0^2}\right|^{-(n+1)/2},$$

$$\mathbf{p}_1 = p_1\mathbf{n}, \quad \mathbf{p}_2 = -p_2\mathbf{n}, \quad p_1 p_2 = p_0^2. \qquad (56)$$

This equation is a generalization of (20a) to the n-dimensional case[8]. It is of interest to note that the symmetry (56) follows directly from the invariance of the Schrödinger equation (55) with respect to the reflection $\xi_\mu \to -\xi_\mu$.

The spherical harmonics $Y_{N\nu}(\xi)$ enter into the expansion of the so-called Poisson kernel[26]:

$$K(\xi,\xi';\rho) = \frac{1}{2\pi^{(n+1)/2}}\Gamma\left(\frac{n+1}{2}\right)\frac{1-\rho^2}{[(1-\rho)^2+\rho(\xi-\xi')^2]^{(n+1)/2}},$$

$$K(\xi,\xi';\rho) = \sum_{N\nu}\rho^N Y_{N\nu}(\xi)Y^*_{N\nu}(\xi'), \qquad (57)$$

$$\lim_{\rho\to 1} K(\xi,\xi';\rho) = \delta(\xi-\xi')$$

(the letter ν denotes the set of several indices required for the complete specification of a spherical harmonic, thus, in the case of the hydrogen atom considered above $N = n-1$, n is the principal quantum number, while the role of ν is played by the pair of indices l and m). We consider the function

$$D(\xi,\xi';\rho) = \frac{1}{2\pi^{(n+1)/2}}\Gamma\left(\frac{n-1}{2}\right) \times [(1-\rho)^2+\rho(\xi-\xi')^2]^{-(n-1)/2}. \qquad (58)$$

---

[7] The energies and the wave functions of bound states for the n-dimensional Kepler problem have been obtained by Alliluev[2].

[8] The symmetry relation for the wave functions of the continuous spectrum can be obtained from (56) if we take into account that in the n-dimensional case $\eta = i[N+(n-1)/2]$; it has the form

$$\frac{\psi(\mathbf{p}_1)}{\psi(\mathbf{p}_2)} = i^{n-1}e^{-\pi\eta}\left|\frac{p_1^2-p_0^2}{p_2^2-p_0^2}\right|^{-(n+1)/2}$$

(the points $\mathbf{p}_1$ and $\mathbf{p}_2$ are symmetric with respect to the hypersphere $|\mathbf{p}| = p_0$, with $p_1 > p_2$). For n = 3 we obtain (20) from this equation.

It can be easily verified that

$$K(\xi, \xi'; \rho) = \frac{1}{\rho^{(n-3)/2}} \frac{\partial}{\partial \rho} (\rho^{(n-1)/2} D(\xi, \xi'; \rho)).$$

Utilizing (57) we have

$$D(\xi, \xi'; \rho) = \sum_{N=0} \sum_{\nu} \frac{\rho^N}{N + (n-1)/2} Y_{N\nu}(\xi) Y_{N\nu}^*(\xi'). \quad (59)$$

Setting here $\rho = 1$ we obtain the expansion of the kernel of the integral equation (55) in terms of spherical harmonics. Utilizing further the same method which was used in [11] for the case $n = 3$, we obtain the desired expansion of the Green's function:

$$G(\xi, \xi') = \sum_{N=0}^{\infty} \sum_{\nu} \frac{Y_{N\nu}(\xi) Y_{N\nu}^*(\xi')}{1 + \eta/[N + (n-1)/2]}. \quad (60)$$

For $n = 3$ this expression goes over into formula (31).

A generalization of formula (32) to spherical harmonics in $(n+1)$-dimensional space is given by the following formula:

$$\sum_{\nu} Y_{N\nu}(\xi) Y_{N\nu}^*(\xi') = \frac{1}{2\pi^{(n+1)/2}} \Gamma\left(\frac{n-1}{2}\right)\left(N + \frac{n-1}{2}\right) C_N^{(n-1)/2}(\cos\chi), \quad (61)$$

where $C_N^{(n-1)/2}(z)$ is a Gegenbauer polynomial, $\cos \chi = (\xi\xi')$. With the aid of (59)–(61) we can represent $G(\xi, \xi')$ in the following forms which are equivalent to one another:

$$G(\xi, \xi') = \delta(\xi - \xi') - \eta D(\xi, \xi'; 1) + \frac{\eta^2}{2\pi^{(n+1)/2}} \Gamma\left(\frac{n-1}{2}\right) \Delta(\xi, \xi'), \quad (62)$$

where

$$\Delta(\xi, \xi') = \begin{cases} 2\pi^{(n+1)/2} \Gamma^{-1}\left(\frac{n-1}{2}\right) \sum_{N\nu} \frac{Y_{N\nu}(\xi) Y_{N\nu}^*(\xi')}{[N + (n-1)/2]\,[N + (n-1)/2 + \eta]} \\ \sum_{N=0}^{\infty} \frac{C_N^{(n-1)/2}(\cos\chi)}{N + (n-1)/2 + \eta} = \int_0^1 \frac{x^{(n-3)/2+\eta}\, dx}{[(1-x)^2 + x(\xi - \xi')^2]^{(n-1)/2}} \end{cases} \quad (63)$$

The last form of $G(\xi, \xi')$ generalizes Schwinger's result[11] to the n-dimensional case.

We emphasize that all the formulas obtained in this section for $G(\xi, \xi')$ refer directly to the simplest case $E < 0$; the corresponding expressions for $E \geq 0$ can be obtained by means of analytic continuation. As can be seen from Sec. 3, this operation is by no means trivial.

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APPENDIX A

We assume that on the unit hyperboloid $\xi_0^2 - \boldsymbol{\xi}^2 = 1$ a scalar function $f(\xi, \xi')$ is given which depends only on the square of the four-dimensional interval:

$$f(\xi, \xi') = f((\xi - \xi')^2). \quad (A.1)$$

We expand it is terms of the irreducible representations $D(0, \rho)$ of the Lorentz group:

$$f((\xi - \xi')^2) = \int_0^{\infty} d\rho\, F(\rho) \sum_{lm} Y_{\rho lm}(\xi) Y_{\rho lm}^*(\xi'). \quad (A.2)$$

In order to obtain the "spectral density" $F(\rho)$ we utilize the invariance of $f((\xi - \xi')^2)$ under Lorentz transformations and go over to the rest system of the vector ξ. From (17) it follows that

$$\Pi_{\rho l}(0) = -\frac{\rho}{2\sqrt{\pi}} \delta_{l0}, \qquad \Pi_{\rho 0}(\alpha) = -\frac{1}{\sqrt{\pi}} \frac{\sin(\rho\alpha/2)}{\operatorname{sh}\alpha};$$

the sum over l and m in (A.2) reduces to one term with $l = m = 0$, from where we have

$$\sum_{lm} Y_{\rho lm}(\xi) Y_{\rho lm}^*(\xi') = \frac{\rho}{8\pi^2} \frac{\sin(\rho\chi/2)}{\operatorname{sh}\chi}. \quad (A.3)$$

The angle χ introduced here is the "hyperbolic angle" between the points ξ and ξ':

$$(\xi\xi') = \pm \operatorname{ch}\chi, \qquad (\xi - \xi')^2 = \mp 2(\operatorname{ch}\chi \mp 1)$$

(the upper (lower) signs correspond to the case when the points ξ and ξ' lie on the same (on different) sheets of the hyperboloid).

In the former case, $\xi_0\xi_0' > 0$ we obtain the equation

$$f\left(-4\operatorname{sh}^2\frac{\chi}{2}\right) = \frac{1}{8\pi^2 \operatorname{sh}\chi} \int_0^{\infty} d\rho\, F(\rho)\, \rho \sin\frac{\rho\chi}{2}, \quad (A.4)$$

which is a Fourier sine-transform. Its solution has the form

$$F_1(\rho) = \frac{8\pi}{\rho} \int_0^{\infty} f(2(1 - \operatorname{ch}\chi)) \sin\frac{\rho\chi}{2} \operatorname{sh}\chi\, d\chi = \frac{4\pi}{\rho} \int_0^{\infty} f(-t) \sin\left[\rho \ln\left(\sqrt{\frac{t}{4}} + \sqrt{\frac{t}{4} + 1}\right)\right] dt. \quad (A.5)$$

A similar solution is also obtained for $\xi_0\xi_0' < 0$:

$$F_2(\rho) = \frac{8\pi}{\rho} \int_0^{\infty} f(2(1 + \operatorname{ch}\chi)) \sin\frac{\rho\chi}{2} \operatorname{sh}\chi\, d\chi = \frac{4\pi}{\rho} \int_4^{\infty} f(t) \sin\left[\rho \ln\left(\sqrt{\frac{t}{4}} + \sqrt{\frac{t}{4} - 1}\right)\right] dt. \quad (A.6)$$

From here in the special case $f((\xi-\xi')^2) = [2\pi^2(\xi-\xi')^2]^{-1}$ we obtain

$$F_1(\rho)=-\frac{2}{\rho}\operatorname{cth}\frac{\pi\rho}{2},\quad F_2(\rho)=\frac{2}{\rho}\operatorname{sh}^{-1}\frac{\pi\rho}{2}. \tag{A.7}$$

Thus, we have

$$\frac{1}{2\pi^2(\xi-\xi')^2}=\int_0^\infty d\rho\, F(\rho)\sum_{lm} Y_{\rho lm}(\xi)Y^*_{\rho lm}(\xi'), \tag{A.8}$$

where

$$F(\rho)=\begin{cases}-\frac{2}{\rho}\operatorname{cth}\frac{\pi\rho}{2}, & \text{if } \xi_0\xi_0'>0\\ \frac{2}{\rho}\operatorname{sh}^{-1}\frac{\pi\rho}{2}, & \text{if } \xi_0\xi_0'<0\end{cases}.$$

APPENDIX B

The function $\Phi(z,\eta)$ introduced in this paper (cf., Sec. 5) is a special case of the function $\Phi(z,s,\eta)$ (cf., [27], p. 27) for $s=1$. Since we do not encounter other values of the parameter s we have introduced the abbreviation $\Phi(z,1,\eta)=\Phi(z,\eta)$.

$\Phi(z,\eta)$ can be expressed in terms of the hypergeometric function:

$$\Phi(z,\eta)=\eta^{-1}F(1,\eta;1+\eta;z). \tag{B.1}$$

For $z\to 1$ it has a logarithmic singularity:

$$\Phi(z,\eta)=\ln\frac{1}{1-z}-[\psi(\eta)-\psi(1)]+\ldots, \tag{B.2}$$

where $\psi(\eta)$ is the logarithmic derivative of the Γ-function. The discontinuity of $\Phi(z,\eta)$ at the cut $1<z<\infty$ has a simple form:

$$-{}^1\!/_2 i[\Phi(z+i\varepsilon,\eta)-\Phi(z-i\varepsilon,\eta)]=\pi/z^\eta. \tag{B.3}$$

For certain values of η $\Phi(z,\eta)$ is expressed in terms of elementary functions, thus, for example,

$$\Phi(z,1)=\frac{1}{z}\ln\frac{1}{1-z},\quad \Phi\left(z,\frac{1}{2}\right)=\frac{1}{\sqrt{z}}\ln\frac{1+\sqrt{z}}{1-\sqrt{z}}.$$

As can be seen from (33), the following combination of the functions $\Phi(z,\eta)$ appears in $G(\xi,\xi')$:

$$H(z,\eta)=-{}^1\!/_2 i[\Phi(z,\eta)-\Phi(z^{-1},\eta)] \tag{B.4}$$

for $z=e^{i\chi}$. With the aid of the Sommerfeld-Watson transformation we can easily obtain for the integral representation:

$$H(z,\eta)=\frac{1}{4}\int_{-\infty}^{\infty}\frac{(-z)^{i\nu}-(-z^{-1})^{i\nu}}{(\eta+i\nu)\operatorname{sh}\pi\nu}d\nu, \tag{B.5}$$

which is valid under the following conditions:

$$\operatorname{Re}\nu>0,\quad |\arg(-z)|<\pi,\quad |\arg(-z^{-1})|<\pi.$$

In the case I, $0\le\chi\le\pi$, $-z=e^{i(\chi-\pi)}$ (cf., Fig. 3) and from (B.5) it follows that

$$H(e^{i\chi},\eta)=\eta\int_0^\infty\frac{\operatorname{sh}(\pi-\chi)k}{(k^2+\eta^2)\operatorname{sh}\pi k}dk,$$

and this gives the formula (38a). In a similar manner from (B.5) one obtains the representations (38b) and (38c).

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