CONSERVATION LAWS FOR FREE FIELDS

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It is shown that free field equations retain their form in nonlocal transformations forming a group. Correspondingly, all conservation laws for free fields can be generalized and result in conservation of the corresponding nonlocal quantities.

1. Recently Lipkin[1] has found a new relationship, having the form of a conservation law of a certain tensor composed from the electromagnetic fields. This conservation law has been afterwards generalized by Morgan,[2] who discussed tensors of the third and higher ranks bilinear in the fields and their derivatives.1) However, Maxwell's equations were used directly in the proof of these relations, and the group-theoretical nature of these new conservation laws remained unexplained.

In the present note it is shown that the usual equations of motion for the free fields can be obtained by variation of some nonlocal Lagrangian. The type of nonlocality remains to a large extent arbitrary. The relations of Lipkin and Morgan and some other relations appear as a consequence of the invariance of the action integral under the transformations of the group. In this formalism it is irrelevant if the mass of the field particle is zero or different from zero. The results obtained can be generalized easily to the ease of an arbitrary free field.

2. For definiteness let us first consider the case of the electromagnetic field and let us take the Lagrangian density of the system in the form

$$L_K(x) = \int L\left(x + \frac{z}{2}, x - \frac{z}{2}\right) K(z)\, dz, \tag{1}$$

$$L(x, y) = -\tfrac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(y), \tag{2}$$

where the function $K(z)$, which characterizes the nonlocality, falls off rapidly enough as z increases. Also, $K(z)$ has an inverse. In the particular case $K(z) = \delta(z)$, these conditions are satisfied and (1) gives the ordinary local expression for the Lagrangian.

Variation of the action integral defined with (1) leads to Maxwell's equations for the fields $F_{\mu\nu}(x)$ and $F'_{\mu\nu}(x)$, which are connected through the nonlocal transformation:

$$F_{\mu\nu}(x) \to F_{\mu\nu}'(x) = \int K(x - y) F_{\mu\nu}(y)\, dy. \tag{3}$$

These transformations form a group.

Let us consider the energy-momentum tensor

$$T'_{\mu\nu}(x) = -\tfrac{1}{2}[F'_{\mu\sigma}(x) F_\nu'^{\sigma}(x) + \tilde{F}'_{\mu\sigma}(x) \tilde{F}_\nu'^{\sigma}(x)], \tag{4}$$

where $\tilde{F}_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, $\epsilon_{0123} = 1$. It satisfies the conservation law

$$\partial^\nu T'_{\mu\nu}(x) = 0. \tag{5}$$

Inserting expression (3) in (4) we get from (5)

$$\int K(-y) K(-z)\, \partial_x{}^\nu T_{\mu\nu}(x, y, z)\, dy\, dz = 0, \tag{6}$$

$$T_{\mu\nu}(x, a, b) = -\tfrac{1}{2}[F_{\mu\sigma}(x + a) F_\nu{}^\sigma(x + b) + \tilde{F}_{\mu\sigma}(x + a) \tilde{F}_\nu{}^\sigma(x + b)]. \tag{7}$$

From the identity

$$\tilde{F}_{\mu\alpha}(y) \tilde{F}_\nu{}^\alpha(z) = F_{\mu\alpha}(z) F_\nu{}^\alpha(y) - \tfrac{1}{2} g_{\mu\nu} F_{\alpha\beta}(y) F^{\alpha\beta}(z) \tag{8}$$

and the definition (7) it follows that

$$T_{\mu\nu}(x, a, b) = T_{\mu\nu}(x, b, a) = T_{\nu\mu}(x, a, b). \tag{9}$$

If we substitute

$$K(x) = \delta(x + a) + \delta(x + b) \tag{10}$$

in (6) and use the symmetry properties (9) we have

$$\partial^\nu T_{\mu\nu}(x, a, b) = 0. \tag{11}$$

The nonlocal energy-momentum tensor (7) satisfies the conservation law (11) and an infinite number of conserved local quantities appear if it is expanded in a Taylor series with respect to the parameters a and b. Similarly, analyzing the orbital angular momentum and spin tensors we find that the nonlocal tensors

1)Feldman[3] obtained an expression for the conserved quantities found by Lipkin in the presence of a gravitational field, and also found the corresponding quantity for a gravitational field itself.

$$M_{\mu\sigma\nu}(x, a, b) = x_\sigma T_{\mu\nu}(x, a, b) - x_\nu T_{\mu\sigma}(x, a, b), \quad (12)$$

$$S^{\eta}_{\mu\nu\rho\sigma}(x, a, b) = F_{\mu\nu}(x+a)\partial^{\eta}F_{\rho\sigma}(x+b) - F_{\rho\sigma}(x+b)\partial^{\eta}F_{\mu\nu}(x+a) \quad (13)$$

satisfy the conservation laws

$$\partial^{\mu}M_{\mu\sigma\nu}(x, a, b) = 0, \qquad \partial_{\eta}S^{\eta}_{\mu\nu\rho\sigma}(x, a, b) = 0. \quad (14)$$

Maxwell's equations for the fields $F_{\mu\nu}(x+a)$ are invariant under rotations in the dual space, through an arbitrary angle θ that depends on a parameter a, $\theta = \theta(a)$. Applying the operation of dual rotation to the tensors $T_{\mu\nu}(x, a, b)$ and $M_{\mu\sigma\nu}(x, a, b)$ we get new tensors, which are also conserved.

Indeed, taking $\theta(a) = 0$ and $\theta(b) = \pi/2$ we see that the tensors

$$V_{\mu\nu}(x, a, b) = -{}^1\!/_2(F_{\mu\sigma}(x+a)\tilde{F}_{\nu}{}^{\sigma}(x+b) - \tilde{F}_{\mu\sigma}(x+a)F_{\nu}{}^{\sigma}(x+b)), \quad (15)$$

$$N_{\mu\sigma\nu}(x, a, b) = x_\sigma V_{\mu\nu}(x, a, b) - x_\nu V_{\mu\sigma}(x, a, b), \quad (16)$$

where

$$V_{\mu\nu}(x, a, b) = -V_{\mu\nu}(x, b, a) = V_{\nu\mu}(x, a, b),$$

are conserved:

$$\partial^{\nu}V_{\mu\nu}(x, a, b) = 0, \qquad \partial^{\mu}N_{\mu\sigma\nu}(x, a, b) = 0. \quad (17)$$

From the conservation of the nonlocal tensor $V_{\mu\nu}(x, a, b)$ follows the second of Morgan's conservation laws.[2]

3. It is clear that our method allows us to consider other free fields in a similar manner. For example, for a complex vector field we find that the non-local current vector

$$j_{\mu}(x, a, b) = -i[U_{\nu}^{*}(x+a)\partial_{\mu}U^{\nu}(x+b) - U^{\nu}(x+b)\partial_{\mu}U_{\nu}^{*}(x+a)] \quad (18)$$

is conserved, $\partial^{\mu}j_{\mu}(x, a, b) = 0$. The current (18) is different from zero also for hermitian fields.

4. To explain the physical meaning of the generalized conservation laws for the free fields it is convenient to use the method of second quantization. For example, from the expression for the ordinary local energy-momentum tensor in the framework of second quantization, it follows that

$$\sum_{\mathbf{k},\lambda} \omega N_{\mathbf{k}\lambda} = \text{const}, \qquad \sum_{\mathbf{k}\lambda} \mathbf{k}N_{\mathbf{k}\lambda} = \text{const}.$$

Similarly, the expression (7) for the nonlocal energy-momentum tensor yields

$$\sum_{\mathbf{k},\lambda} \omega N_{\mathbf{k}\lambda}(e^{ika} + e^{-ika}) = \text{const},$$

$$\sum_{\mathbf{k},\lambda} \mathbf{k}N_{\mathbf{k}\lambda}(e^{ika} + e^{-ika}) = \text{const}. \quad (19)$$

From (19) one can deduce

$$\sum_{\lambda} N_{\mathbf{k}\lambda} = \text{const}.$$

The existence of an infinite number of conserved expressions corresponds to the fact that the quantities conserved for the free fields are those related to each single field quantum.

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[1] D. M. Lipkin, J. Math. Phys. **5**, 696 (1964).
[2] T. A. Morgan, J. Math. Phys. **5**, 1659 (1964).
[3] K. S. Feldman, Nuovo Cimento **7**, 104 (1965).

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