ON PAIRING WITH AN ORBITAL MOMENTUM $l \neq 0$

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Two-particle excitations are investigated for the case of pairing with $l = 2$. Stability of the anisotropic state with respect to these excitations is proved. It is shown that the two-particle excitation spectrum satisfies the Landau superfluidity condition.

1. INTRODUCTION

COOPER pairs with orbital angular momentum l different from zero can be produced in a system of interacting Fermi particles. The usual methods of superconductivity theory lead in this case to solutions with anisotropic energy gap Δ (see, for example, [1]), with the exception of the case of pairing with $l = 1$, when the solutions may also include isotropic ones [2]. Gor'kov and Galitskiĭ [3] have proposed a new method for splitting the chain of equations for the Green's function and obtained a solution with Δ independent of the angles and with energy less than that obtained by Anderson and Morel [1]. However, this result cannot be regarded as definitely proved, especially because in the exactly solvable model (which takes into account only interaction of particles with opposite momenta) the energy gap is anisotropic for $l > 1$ (the existence of anisotropic solution was proved in [4,5] and its uniqueness was proved in [6]). It is of interest in this connection to investigate the stability of the anisotropic state in a more realistic model.

Any infinitesimally small change in the state of a system can apparently be represented as an aggregate of single-particle and two-particle elementary excitations. We shall consider two-particle excitations in the case of pairing with $l = 2$, which should probably be realized in liquid He[3].[1)] Stability against single-particle and two-particle excitations denotes that the investigated state can exist at least as a metastable one. We therefore regard the stability of the anisotropic state, which we shall prove below, as an important additional argument in favor of the theory with anisotropic gap.

The form of the spectrum of two-particle Bose excitations is essential for a clarification of superfluidity properties. The spectrum which we obtain below satisfies the Landau criterion.

We confine ourselves to the case of zero temperature.

2. FORMULATION OF THE PROBLEM AND EQUATIONS FOR TWO-PARTICLE GREEN'S FUNCTIONS

We start from the universally used model of a Fermi system with Hamiltonian

$$H = \sum_{\mathbf{p},\sigma} \zeta_{\mathbf{p}} a^{+}_{\mathbf{p}\sigma} a_{\mathbf{p}\sigma} + \frac{1}{2V} \sum_{\substack{\mathbf{p}+\mathbf{p}_1=\mathbf{p}'+\mathbf{p}_1' \\ \sigma,\sigma'}} V(\mathbf{p},\mathbf{p}')\, a^{+}_{\mathbf{p}\sigma} a^{+}_{\mathbf{p}_1\sigma'} a_{\mathbf{p}_1'\sigma'} a_{\mathbf{p}'\sigma}, \tag{1}$$

where ζ_{p} is the kinetic energy of the particle with momentum $\mathbf{p}$, reckoned from the Fermi level, $a^{+}_{\mathrm{p}\sigma}$ and $a_{\mathrm{p}\sigma}$ are the fermion creation and annihilation operators, σ and σ' the spin indices, and V the volume of the system. The interaction "potential" $V(\mathbf{p}, \mathbf{p}')$ is equal to zero in the model under consideration outside a narrow layer of thickness $2\tilde{\omega}$ near the Fermi surface, in which layer it depends only on the angle between momenta $\mathbf{p}$ and $\mathbf{p}'$, that is, $V(\mathbf{p},\mathbf{p}') = V(\mathbf{nn}')$, $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$. It is expanded in Legendre polynomials:

$$V(\mathbf{nn}') = \sum_{l} (2l+1) V_l P_l(\mathbf{nn}'). \tag{2}$$

We shall also assume that the particles interact with one another only in a state with relative angular momentum $l = 2$ (that is, $V_l = 0$ when $l \neq 2$), and this interaction is an attraction: $V_2 < 0$. In addition, we assume that the interaction is weak.

Anderson and Morel calculated the equation for the gap $\Delta(\mathbf{p})$,[1] which coincides with the

[1)]Two-particle excitations in pairing with $l = 2$ were investigated earlier by Vaks, Galitskii, and Larkin[7], but they did not prove the existence of a stable solution. The method proposed in their paper will be used by us.

ordinary equation of superconductivity theory[2)] where

$$\Delta(\mathbf{p}) = -\frac{\pi}{(2\pi)^4}\int d\mathbf{p}'\, V(\mathbf{p}, \mathbf{p}')\,\frac{\Delta(\mathbf{p}')}{\varepsilon_{\mathbf{p}'}}, \qquad (3)$$

$\epsilon_p = (\zeta_p^2 + |\Delta(\mathbf{p})|^2)^{1/2}$ is the energy of the quasi-particle. Some solutions of Eq. (30), for the case of pairing with $l = 2$, were investigated in detail in [1]. The solution corresponding to the lowest energy is[3)]

$$\Delta(\mathbf{n}) = \Gamma\cdot 2\tilde{\omega} e^{-1/\rho}\psi(\mathbf{n}),$$
$$\psi(\mathbf{n}) = \frac{1}{\sqrt{2}} Y_{20} + \frac{i}{2}(Y_{22} + Y_{2,-2}), \qquad (4)$$

where

$$\rho = \frac{mp_0|V_2|}{(2\pi)^2}, \qquad \ln\Gamma = -\int d\mathbf{n}\,|\psi(\mathbf{n})|^2 \ln|\psi(\mathbf{n})| = 1.154;$$

m is the particle mass and p_0 the Fermi momentum. However, since not all the solutions of (3) were obtained in [1], there is no complete assurance that this solution corresponds to the ground state. At the same time, the latter is perfectly probable, since in the states with

$$\Delta \sim \frac{1}{\sqrt{2}} Y_{20} + \frac{i}{2}(Y_{22} + Y_{2,-2})$$

the total angular momentum of the system is equal to zero and this state is stable, whereas states with lower energy are unstable[7].

The quantity $|\Delta(\mathbf{n})|$ has cubic symmetry. In terms of the direction cosines (x, y, z) relative to the coordinate axes, the function $|\psi(\mathbf{n})|^2$ takes the form

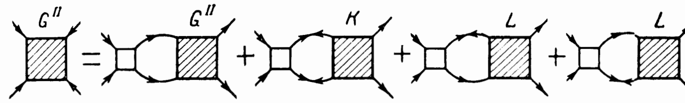
$$|\psi(\mathbf{n})|^2 = \frac{5}{8\pi}(x^4 + y^4 + z^4 - x^2y^2 - y^2z^2 - z^2x^2). \qquad (5)$$

It vanishes at the point $x^2 = y^2 = z^2 = 1/3$.

The frequencies of the two-particle excitations can be determined if one knows the poles of the Fourier components of the Green's function

$$G^{\text{II}}_{\alpha\beta;\,\gamma\delta}(x_1, x_2;\, x_3, x_4) = \langle T\psi_\alpha(x_1)\psi_\beta(x_2)\psi_\gamma^+(x_3)\psi_\delta^+(x_4)\rangle; \qquad (6)$$

ψ and ψ^+ are the second-quantization operators in the coordinate representation.



In the case of a small total momentum of the colliding particles, we can obtain a closed system of equations for this function by summing the so-called "ladder" diagrams. One such equation is shown graphically in the figure, where the light square denotes the bare vertex $\mathscr{T}^{(0)}$:

$$\overset{(0)}{\mathscr{T}}_{\alpha\beta;\,\gamma\delta}(p, -p;\, p', -p') \equiv \overset{(0)}{\mathscr{T}}_{\alpha\beta;\,\gamma\delta}(p, p')$$
$$= V(\mathbf{p}, \mathbf{p}')(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}),$$
$$\overset{(0)}{\mathscr{T}}_{\alpha\beta;\,\gamma\delta}(p, p';\, p, p') \equiv \overset{(0)}{\tilde{\mathscr{T}}}_{\alpha\beta;\,\gamma\delta}(p, p')$$
$$= V_0\delta_{\alpha\gamma}\delta_{\beta\delta} - V(\mathbf{p}, \mathbf{p}')\delta_{\alpha\delta}\delta_{\beta\gamma} \qquad (7)$$

($V_0 = V(\mathbf{nn})$) and new two-particle Green's functions have been introduced

$$K_{\alpha\beta;\,\gamma\delta}(x_1, x_2;\, x_3, x_4) = \langle T\psi_\alpha^+(x_1)\psi_\beta^+(x_2)\psi_\gamma^+(x_3)\psi_\delta^+(x_4)\rangle,$$
$$L_{\alpha\beta;\,\gamma\delta}(x_1, x_2;\, x_3, x_4) = \langle T\psi_\alpha^+(x_1)\psi_\beta(x_2)\psi_\gamma^+(x_3)\psi_\delta^+(x_4)\rangle, \qquad (8)$$

for which equations are set up in similar fashion and which form, together with the equation for G^{II}, a complete system.

We have left out of the equation shown in the figure the terms that represent products of the single-particle Green's functions. These terms are insignificant near the poles corresponding to the bound states. The latter can be obtained from the conditions for the solvability of the homogeneous system. We see from the figure that the second pair of indices ($x_3, \gamma;\, x_4, \delta$) of the functions G^{II}, K, and L play the role of parameters in the equations and can be omitted.

The system of equations for the Fourier components of the functions G^{II}, K, and L can be written in the explicit form

$$G_{\alpha\beta}^{\text{II}}(p_+, -p_-) = \frac{i}{2(2\pi)^4} G(p_+)G(-p_-)\int d^4p'\, \overset{(0)}{\mathscr{T}}_{\alpha\beta;\,\gamma\delta}(p, p')$$
$$\times G_{\gamma\delta}^{\text{II}}(p_+', -p_-') + \frac{i}{2(2\pi)^4} F(p_+)F(-p_-)\,\sigma_{\alpha\eta}^{(y)}\sigma_{\beta\rho}^{(y)}$$
$$\times \int d^4p'\, \overset{(0)}{\mathscr{T}}_{\gamma\delta;\,\eta\rho}(p, p')\, K_{\gamma\delta}(p_+', -p_-') - \frac{i}{(2\pi)^4}$$

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2)Recently Tareeva[8] obtained a different equation for the gap

$$\Delta_{\mathbf{p},\,\mathbf{q}-\mathbf{p}} = -\frac{1}{2V}\sum_{\mathbf{p}'} V(\mathbf{p}, \mathbf{p}')\,\Delta_{\mathbf{p}',\mathbf{q}-\mathbf{p}'}/\varepsilon_{\mathbf{p}'},$$

where the quantities $\Delta_{\mathbf{p},\,\mathbf{q}-\mathbf{p}}$ differ from zero only when

$$|\mathbf{p}| = |\mathbf{q} - \mathbf{p}|, \qquad |\Delta|^2 = \sum_{\mathbf{q}} |\Delta_{\mathbf{p},\,\mathbf{q}-\mathbf{p}}|^2.$$

It is stated in [8] that it is possible to obtain from this equation a solution with an isotropic gap in the case of pairing with $l \neq 0$. However, an error has crept into this paper and Tareeva's conclusions are therefore incorrect. The point is that the summation in the right side of the equation for $\Delta_{\mathbf{p},\,\mathbf{q}-\mathbf{p}}$ is carried out over the plane $\mathbf{pq} = |\mathbf{q}|^2/2$, and not over the volume. Such a two-dimensional summation gives rise to a factor $V^{2/3}$, which does not cancel out the factor 1/V in the right side. Therefore the equation for $\Delta_{\mathbf{q}-\mathbf{p}}$ has no nontrivial solutions when $|\mathbf{q}| \neq 0$. This factor becomes perfectly obvious by writing out the equation for $\Delta_{\mathbf{q}-\mathbf{p}}$ in the limit as $V \to \infty$ and going from summation with respect to $\mathbf{p}'$ to integration. On the other hand, if we put $|\mathbf{q}| = 0$, in this equation we obtain Eq. (3).

3)In [1] the function $\psi(\mathbf{n})$ was written in a different form:

$$\psi(\mathbf{n}) = 2^{-1/2}Y_{20} + {}^1/_2(Y_{22} - Y_{2,-2}).$$

The difference lies in the choice of coordinate axes. Both solutions go over into each other when the coordinate system is rotated through an angle $\pm\pi/4$ about the z axis.

$$\times G(p_+)F(-p_-)\sigma_{\beta\rho}^{(y)}\int d^4p'\,\tilde{\mathcal{T}}^{(0)}_{\alpha\gamma;\rho\delta}(p,p')\times L_{\gamma\delta}(p_+',-p_-')+\frac{i}{(2\pi)^4}F(p_+)G(-p_-)\sigma_{\alpha\rho}^{(y)}\times\int d^4p'\,\tilde{\mathcal{T}}^{(0)}_{\beta\gamma;\rho\delta}(p,p')\,L_{\gamma\delta}(p_+',-p_-');\quad(9)$$

$$K_{\alpha\beta}(p_+,-p_-)=\frac{i}{2(2\pi)^4}F^+(p_+)F^+(-p_-)\sigma_{\alpha\rho}^{(y)}\sigma_{\beta\eta}^{(y)}\times\int d^4p'\,\mathcal{T}^{(0)}_{\rho\eta;\gamma\delta}(p,p')G_{\gamma\delta}^{\mathrm{II}}(p_+',-p_-')+\frac{i}{2(2\pi)^4}G(-p_+)G(p_-)\int d^4p'\,\mathcal{T}^{(0)}_{\gamma\delta;\alpha\beta}(p,p')\times K_{\gamma\delta}(p_+',-p_-')+\frac{i}{(2\pi)^4}F^+(p_+)G(p_-)\sigma_{\alpha\rho}^{(y)}\times\int d^4p'\,\tilde{\mathcal{T}}^{(0)}_{\rho\gamma;\beta\delta}(p,p')L_{\gamma\delta}(p_+',-p_-')-\frac{i}{(2\pi)^4}G(-p_+)\times F^+(-p_-)\sigma_{\beta\rho}^{(y)}\int d^4p'\,\tilde{\mathcal{T}}^{(0)}_{\rho\gamma;\alpha\delta}(p,p')L_{\gamma\delta}(p_+',-p_-');\quad(10)$$

$$L_{\alpha\beta}(p_+,-p_-)=\frac{i}{2(2\pi)^4}F^+(p_+)G(-p_-)\sigma_{\alpha\rho}^{(y)}\times\int d^4p'\,\mathcal{T}^{(0)}_{\rho\beta;\gamma\delta}(p,p')\,G_{\gamma\delta}^{\mathrm{II}}(p_+'-p_-')-\frac{i}{2(2\pi)^4}\times G(-p_+)F(-p_-)\sigma_{\beta\rho}^{(y)}\int d^4p'\,\mathcal{T}^{(0)}_{\gamma\delta;\alpha\rho}(p,p')\times K_{\gamma\delta}(p_+',-p_-')-\frac{i}{(2\pi)^4}G(-p_+)G(-p_-)\times\int d^4p'\,\tilde{\mathcal{T}}^{(0)}_{\beta\gamma;\alpha\delta}(p,p')\,L_{\gamma\delta}(p_+',-p_-')-\frac{i}{(2\pi)^4}F^+(p_+)\times F(-p_-)\sigma_{\alpha\rho}^{(y)}\sigma_{\beta\eta}^{(y)}\int d^4p'\,\tilde{\mathcal{T}}^{(0)}_{\rho\gamma;\eta\delta}(p,p')\,L_{\gamma\delta}(p_+',-p_-').\quad(11)$$

Here $p_\pm = p \pm k/2$, $k = \{\mathbf{k}, \omega\}$, $\sigma^{(y)}$ is a Pauli matrix, and G, F, and F^+ are single-particle Green's functions ($p = \{\mathbf{p}, \omega\}$):

$$G(p)=\frac{\omega+\zeta_{\mathbf{p}}}{(\omega-\varepsilon_{\mathbf{p}}+i\delta)(\omega+\varepsilon_{\mathbf{p}}-i\delta)},\quad F(p)=-\frac{i\Delta(\mathbf{n})}{(\omega-\varepsilon_{\mathbf{p}}+i\delta)(\omega+\varepsilon_{\mathbf{p}}-i\delta)},\quad F^+(p)=\frac{i\Delta^*(\mathbf{n})}{(\omega-\varepsilon_{\mathbf{p}}+i\delta)(\omega+\varepsilon_{\mathbf{p}}-i\delta)}.\quad(12)$$

The function G^{II} has poles also when the momentum transfer is small. The corresponding equations can be readily written out, but there is no need for them, since these poles coincide with the functions L, which are determined from (9)—(11).

In order to disclose the spin structure of the excitations, we shall expand the functions G^{II}, K, and L in a complete system of linearly independent two-row matrices:

$$G_{\alpha\beta}^{\mathrm{II}}=\bar{\chi}_0\delta_{\alpha\beta}+\bar{\chi}_i\sigma_{\alpha\beta}^{(i)},\quad K_{\alpha\beta}=\bar{\varphi}_0\delta_{\alpha\beta}+\bar{\varphi}_i\sigma_{\alpha\beta}^{(i)},\quad L_{\alpha\beta}=\bar{\nu}_0\delta_{\alpha\beta}+\bar{\nu}_i\sigma_{\alpha\beta}^{(i)}.\quad(13)$$

In our case the excitations should have zero spin, that is, the poles corresponding to the bound states are possessed only by the components $\bar{\chi}_2$, $\bar{\varphi}_2$, and $\bar{\nu}_0$. It is convenient to consider in lieu of these quantities the following:

$$\chi(\mathbf{n})=\int d^4p'\,V(\mathbf{nn}')\bar{\chi}_2(p_+',-p_-'),\quad \varphi(\mathbf{n})=\int d^4p'\,V(\mathbf{nn}')\bar{\varphi}_2(p_+'\ -p_-'),\quad \nu(\mathbf{n})=\int d^4p'\,[2V_0-V(\mathbf{nn}')]\bar{\nu}_0(p_+'\ -p_-').\quad(14)$$

The system of equations for the functions χ, φ, and ν takes the form

$$\chi(\mathbf{n})=\frac{i}{(2\pi)^4}\int d^4p'\,V(\mathbf{nn}')\{G(p_+')G(-p_-')\chi(\mathbf{n}')-F(p_+')\times F(-p_-')\varphi(\mathbf{n}')+[G(p_+')F(-p_-')+F(p_+')G(-p_-')]\nu(\mathbf{n}')\},\quad(15)$$

$$\varphi(\mathbf{n})=\frac{i}{(2\pi)^4}\int d^4p'\,V(\mathbf{nn}')\{-F^+(p_+')F^+(-p_-')\chi(\mathbf{n}')+G(-p_+')\,G(p_-')\varphi(\mathbf{n}')+[F^+(p_+')G(p_-')+G(-p_+')\times F^+(-p_-')]\nu(\mathbf{n}')\},\quad(16)$$

$$\nu(\mathbf{n})=\frac{i}{(2\pi)^4}\int d^4p'\,[2V_0-V(\mathbf{nn}')]\{F^+(p_+')G(-p_-')\chi(\mathbf{n}')+G(-p_+')F(-p_-')\varphi(\mathbf{n}')-[G(-p_+')G(-p_-')-F^+(p_+')F(-p_-')]\nu(\mathbf{n}')\}.\quad(17)$$

3. TWO-PARTICLE EXCITATION FREQUENCIES IN THE CASE $|\mathbf{k}| = 0$

We shall show first that excitations with $\omega = 0$ exist when $|\mathbf{k}| = 0$. To this end we integrate in (15)—(17) over the frequency component of the momentum p' and put $|\mathbf{k}| = \omega = 0$, after which the system (15)—(17) is greatly simplified:

$$\chi(\mathbf{n})=-\frac{\pi}{2(2\pi)^4}\int d\mathbf{p}'\,V(\mathbf{nn}')\left[\left(2-\frac{|\Delta|^2}{\varepsilon^2}\right)\frac{\chi(\mathbf{n}')}{\varepsilon}+\frac{\Delta^2}{\varepsilon^3}\varphi(\mathbf{n}')\right],\quad(18)$$

$$\varphi(\mathbf{n})=-\frac{\pi}{2(2\pi)^4}\int d\mathbf{p}'\,V(\mathbf{nn}')\left[\frac{\Delta^{*2}}{\varepsilon^3}\chi(\mathbf{n}')+\left(2-\frac{|\Delta|^2}{\varepsilon^2}\right)\frac{\varphi(\mathbf{n}')}{\varepsilon}\right],\quad(19)$$

$$\nu(\mathbf{n})=-\frac{\pi}{(2\pi)^4}\int d\mathbf{p}'\,[2V_0-V(\mathbf{nn}')]\frac{|\Delta|^2\nu(\mathbf{n}')}{\varepsilon^3}.\quad(20)$$

The equation for $\nu(\mathbf{n})$ has separated, and we can readily see that it has no nonvanishing solution. On the other hand, the choice of χ and φ in the form

$$\chi_1=-\varphi_1^*=i\Delta\quad(21)$$

satisfies Eqs. (18) and (19). To verify this, it is necessary to compare these equations with Eq. (3) for $\Delta(n)$. We shall make use in addition of the fact that (3) is invariant against spatial rotations. At the same time, the solution $\Delta(n)$ is not invariant. The invariance of (3) signifies that the new Δ, obtained as a result of rotation, should satisfy the same equation as the initial one. Making an infinitesimally small transformation $\mathbf{n} \to \mathbf{n} + \epsilon \times \mathbf{n}$, we find that the quantity $\operatorname{var}\Delta = [\epsilon \times \mathbf{n}]\partial\Delta/\partial\mathbf{n}$ satisfies the equation

$$\operatorname{var}\Delta = -\frac{\pi}{(2\pi)^4}\int d\mathbf{p}' V(\mathbf{nn}')\left[\left(1-\frac{|\Delta|^2}{2\varepsilon^2}\right)\frac{\operatorname{var}\Delta}{\varepsilon} - \frac{\Delta^2}{2\varepsilon^2}(\operatorname{var}\Delta)^*\right]. \quad (22)$$

Comparing it with (18) and (19) we conclude that there are three other solutions

$$\chi_{234} = -\varphi_{234}^* = \operatorname{var}\Delta \sim \begin{cases}(\sqrt{3}+i)yz,\\ (\sqrt{3}-i)xz,\\ 2ixy.\end{cases} \quad (23)$$

By virtue of the linearity of the equations for χ and φ, it is obvious that we can choose as a solution any linear combination of the solutions (21) and (23). The level $\omega = 0$ has fourfold degeneracy. The degeneracy is lifted when $|\mathbf{k}| \neq 0$.

We shall show further that in the case when $|\mathbf{k}| = 0$ there are no poles at $\omega \neq 0$. These poles must be investigated to prove the stability of the state under consideration. In our model the unstable states are the normal state and the state with $\Delta \sim Y_{2m}$, as manifest by the presence of a pole at the imaginary frequency.

In the case $|\mathbf{k}| = 0$, $\omega \neq 0$ the equation for $\nu(\mathbf{n})$ no longer separates

$$\nu(\mathbf{n}) = -\frac{\pi}{(2\pi)^4}\int d\mathbf{p}'[2V_0 - V(\mathbf{nn}')]\frac{1}{\varepsilon[\varepsilon^2-(\omega/2)^2-i\delta]} \times\left\{\frac{i\omega}{4}[\Delta^*\chi(\mathbf{n}')-\Delta\varphi(\mathbf{n}')]+|\Delta|^2\nu(\mathbf{n}')\right\}, \quad (24)$$

but ν is small compared with χ and φ, so that we can put henceforth $\nu = 0$. This means that we neglect in (15) and (16) the terms of order ρ^2 (see (4)). Thus, the system of equations which we must investigate is of the form

$$\chi(\mathbf{n}) = -\frac{\pi}{(2\pi)^4}\int d\mathbf{p}' V(\mathbf{nn}')\frac{\varepsilon}{\varepsilon^2-(\omega/2)^2-i\delta} \times\left[\left(1-\frac{|\Delta|^2}{2\varepsilon^2}\right)\chi(\mathbf{n}')+\frac{\Delta^2}{2\varepsilon^2}\varphi(\mathbf{n}')\right], \quad (25)$$

$$\varphi(\mathbf{n}) = -\frac{\pi}{(2\pi)^4}\int d\mathbf{p}' V(\mathbf{nn}')\frac{\varepsilon}{\varepsilon^2-(\omega/2)^2-i\delta} \times\left[\frac{\Delta^{*2}}{2\varepsilon^2}\chi(\mathbf{n}')+\left(1-\frac{|\Delta|^2}{2\varepsilon^2}\right)\varphi(\mathbf{n}')\right]. \quad (26)$$

We shall find useful the following relations which are valid for any function $f(\mathbf{n})$ that has the same symmetry as $|\Delta|$:

$$\int d\mathbf{n} f(\mathbf{n})\Delta = \int d\mathbf{n} f(\mathbf{n})\operatorname{var}\Delta = 0,$$

$$\int d\mathbf{n} f(\mathbf{n})\Delta^2 = \int d\mathbf{n} f(\mathbf{n})\Delta^*\operatorname{var}\Delta = \int d\mathbf{n} f(\mathbf{n})\Delta\operatorname{var}\Delta = 0,$$

$$\int d\mathbf{n} f(\mathbf{n})(\operatorname{var}\Delta)^*\Delta^3 = \int d\mathbf{n} f(\mathbf{n})\operatorname{Im}\Delta^4 = 0,$$

$$\int d\mathbf{n} f(\mathbf{n})\operatorname{Im}(\Delta^*\operatorname{var}\Delta)^2 = 0,$$

$$\int d\mathbf{n}' V(\mathbf{nn}')f(\mathbf{n}')\Delta(\mathbf{n}') \sim \Delta(\mathbf{n}),$$

$$\int d\mathbf{n}' V(\mathbf{nn}')\Delta(\mathbf{n}') = -4\pi|V_2|\Delta(\mathbf{n}),$$

$$\int d\mathbf{n}' V(\mathbf{nn}')f(\mathbf{n}')\operatorname{var}\Delta(\mathbf{n}') \sim \operatorname{var}\Delta(\mathbf{n}),$$

$$\int d\mathbf{n}' V(\mathbf{nn}')\operatorname{var}\Delta(\mathbf{n}') = -4\pi|V_2|\operatorname{var}\Delta(\mathbf{n}),$$

$$\int d\mathbf{n}' V(\mathbf{nn}')f(\mathbf{n}')\Delta^3(\mathbf{n}') \sim \Delta^*(\mathbf{n}),$$

$$\int d\mathbf{n}' V(\mathbf{nn}')f(\mathbf{n}')\Delta^2(\mathbf{n}')[\operatorname{var}\Delta(\mathbf{n}')]^* \sim \operatorname{var}\Delta(\mathbf{n}). \quad (27)$$

On the basis of these relations we can seek solutions in the form

$$\chi_1 = -\varphi_1^* = i\Delta, \quad (28)$$

$$\chi_{234} = -\varphi_{234}^* = \operatorname{var}\Delta, \quad (29)$$

$$\chi_{56} = \pm\varphi_{56}^* = \Delta^*, \quad (30)$$

$$\chi_7 = -\varphi_7^* = \Delta, \quad (31)$$

$$\chi_{8910} = \varphi_{8910}^* = \operatorname{var}\Delta. \quad (32)$$

We note that by expanding the functions χ and φ in terms of the harmonics Y_{2m}, we would obtain in lieu of (25) and (26) a system of ten linear homogeneous equations with respect to the coefficients in the expansion. By equating to zero the determinant of this system, we would get just as many equations with respect to ω. On the other hand, by substituting in (25) and (26) the derived relations between χ and φ, we obtain the same ten equations with respect to ω. Thus, formulas (28)—(32) exhaust all the possible relations between χ and φ.

Solutions of the form (28) and (29) with $\omega = 0$ have already been found, but it is not excluded, generally speaking, that such solutions can exist also for a nonzero frequency.

Substituting $\chi = -\varphi^* = i\Delta$, we get

$$\Delta(\mathbf{n}) = -\frac{\pi}{(2\pi)^4}\int d\mathbf{p}' V(\mathbf{nn}')\frac{\varepsilon}{\varepsilon^2-(\omega/2)^2-i\delta}\Delta(\mathbf{n}').$$

The right side in this equation is proportional to $\Delta(\mathbf{n})$, on the basis of one of their relations (27). Multiplying both parts of this equation by $\Delta^*(\mathbf{n})$ and integrating with respect to the angles, we obtain

$$\int d\mathbf{n}\,|\Delta(\mathbf{n})|^2 = \frac{|V_2|}{(2\pi)^2}\int dp\,\frac{\varepsilon|\Delta|^2}{\varepsilon^2-(\omega/2)^2-i\delta}. \qquad (28')$$

We know that this equation has a solution for $\omega = 0$. If ω^2 is complex or positive, then the imaginary part on the right side differs from zero, that is, Eq. (28′) cannot be satisfied for these values of ω. On the other hand, if $\omega^2 < 0$, then the right side decreases with increasing $|\omega|^2$ and cannot be equal to the left side, since the equality is satisfied when $\omega = 0$.

Substituting χ and φ in accordance with (29)—(32), we obtain in similar fashion

$$\int d\mathbf{n}\,|\mathrm{var}\,\Delta|^2 = \frac{|V_2|}{(2\pi)^2}\int dp\,\frac{\varepsilon}{\varepsilon^2-(\omega/2)^2-i\delta}\Big[|\mathrm{var}\,\Delta|^2 - \frac{|\Delta\,\mathrm{var}\,\Delta|^2+\mathrm{Re}(\Delta^*\,\mathrm{var}\,\Delta)^2}{2\varepsilon^2}\Big], \qquad (29')$$

$$\int d\mathbf{n}\,|\Delta|^2 = \frac{|V_2|}{(2\pi)^2}\int dp\,\frac{\varepsilon}{\varepsilon^2-(\omega/2)^2-i\delta}\Big[|\Delta|^2 - \frac{|\Delta|^4 \mp \mathrm{Re}\,\Delta^4}{2\varepsilon^2}\Big] \qquad (30')$$

$$\int d\mathbf{n}\,|\Delta|^2 = \frac{|V_2|}{(2\pi)^2}\int dp\,\frac{\varepsilon}{\varepsilon^2-(\omega/2)^2-i\delta}\Big(1-\frac{|\Delta|^2}{\varepsilon^2}\Big)|\Delta|^2, \qquad (31')$$

$$\int d\mathbf{n}\,|\mathrm{var}\,\Delta|^2 = \frac{|V_2|}{(2\pi)^2}\int dp\,\frac{\varepsilon}{\varepsilon^2-(\omega/2)^2-i\delta}\Big[|\mathrm{var}\,\Delta|^2 - \frac{|\Delta\,\mathrm{var}\,\Delta|^2-\mathrm{Re}(\Delta^*\,\mathrm{var}\,\Delta)^2}{2\varepsilon^2}\Big] \equiv f(\omega^2). \qquad (32')$$

Taking account of the fact that the following relations hold

$$\int d\mathbf{n}\,|\Delta|^2 = \frac{|V_2|}{(2\pi)^2}\int dp\,\frac{|\Delta|^2}{\varepsilon}, \qquad (33)$$

$$\int d\mathbf{n}\,|\mathrm{var}\,\Delta|^2 = \frac{|V_2|}{(2\pi)^2}\int dp\,\frac{1}{\varepsilon}\Big[|\mathrm{var}\,\Delta|^2 - \frac{|\Delta\,\mathrm{var}\,\Delta|^2+\mathrm{Re}(\Delta^*\,\mathrm{var}\,\Delta)^2}{2\varepsilon^2}\Big] \qquad (34)$$

we can easily see that the right side in (29′)—(31′) is smaller than the left side when $\omega^2 < 0$ and is complex for other values of ω. In the case of $\omega = 0$, only Eq. (29′) is satisfied. Thus, the equations under consideration do not yield any new solutions.

Finally, let us investigate Eq. (32′). In this case, as in the preceding ones, we can readily prove the absence of solutions for positive or complex values of ω^2, but the question of the existence of a solution for $\omega^2 < 0$ is not so simple to solve as before, since the quantity $\mathrm{Re}(\Delta^*\,\mathrm{var}\,\Delta)^2$ reverses sign in the integration region. At the same time it is obvious, as before, that the right side of (32′), which we have designated $f(\omega^2)$, decreases with increasing $|\omega|^2$ when $\omega^2 < 0$. Therefore, in order for a solution to exist, the quantity $f(0)$ must satisfy the inequality

$$f(0) \geqslant \int d\mathbf{n}\,|\mathrm{var}\,\Delta|^2. \qquad (35)$$

Adding the expression for $f(0)$ to the right side in (34), we obtain

$$f(0) + \int d\mathbf{n}\,|\mathrm{var}\,\Delta|^2 = \frac{2|V_2|}{(2\pi)^2}\int dp\,\frac{1}{\varepsilon}\Big(1-\frac{|\Delta|^2}{2\varepsilon^2}\Big)|\mathrm{var}\,\Delta|^2. \qquad (36)$$

It follows from this that the inequality (35) is equivalent to

$$\int d\mathbf{n}\,|\mathrm{var}\,\Delta|^2 \leqslant \frac{|V_2|}{(2\pi)^2}\int dp\,\frac{1}{\varepsilon}\Big(1-\frac{|\Delta|^2}{2\varepsilon^2}\Big)|\mathrm{var}\,\Delta|^2. \qquad (37)$$

We put

$$\mathrm{var}\,\Delta \sim \psi'(\mathbf{n}), \qquad \int d\mathbf{n}\,|\psi'(\mathbf{n})|^2 = 1 \qquad (38)$$

and integrate over the energy in (37). Then the condition of the existence of the sought solution will take the form

$$1 + 2\ln\Gamma + M \leqslant 0, \qquad M = \int d\mathbf{n}\,|\psi'(\mathbf{n})|^2 \ln|\psi'(\mathbf{n})|^2, \qquad (39)$$

where Γ and $\psi(\mathbf{n})$ are defined by relations (4).

Owing to the cubic symmetry of the functions $\psi(\mathbf{n})$, the integral in (39) does not depend on the choice of var Δ, according to the formulas in (23). It will be convenient in the calculation of M to take half the sum of the integrands for $M_2(\mathrm{var}\,\Delta \sim yz)$ and $M_3(\mathrm{var}\,\Delta \sim xz)$. Then

$$M = \frac{15}{8\pi}\int_{-1}^{+1} dx \int_0^{2\pi} d\varphi\, x^2(1-x^2)\ln\Big\{\frac{5}{32\pi}\big[(1-3x^2)^2 + 3(1-x^2)^2\cos^2 2\varphi\big]\Big\}. \qquad (40)$$

After integration with respect to φ we can easily obtain the value of M by Simpson's rule. It turns out that

$$1 + 2\ln\Gamma + M = 0.6, \qquad (41)$$

which contradicts the condition (39), so that there are no solutions of the form $\chi = \varphi^* = \mathrm{var}\,\Delta$.

Consequently, two-particle excitations have no gap in our system.

4. SPECTRUM OF TWO-PARTICLE EXCITATIONS FOR SMALL $|\mathbf{k}|$

We have noted earlier that the level $\omega = 0$ is degenerate. For small $|\mathbf{k}|$ we can find the frequencies $\omega(\mathbf{k})$ and the "regular" zeroth-approximation eigenfunctions $\tilde{\chi}$ and $\tilde{\varphi}$ from the condition of solvability of the next higher approximation. We put

$$\tilde{\chi} = \sum_{s=1}^{s=4} c_s \chi_s, \quad \tilde{\varphi} = \sum_{s=1}^{s=4} c_s \varphi_s. \tag{42}$$

Expanding in (15)—(17) the quantities that depend on **k** and ω in a series, we obtain the indicated condition (for a detailed derivation, see [7]):

$$\sum_s c_s \int d\mathbf{n} \frac{\omega^2 - (\mathbf{kv})^2}{|\Delta|^2} \Big\{ \chi_r^* \chi_s + \varphi_r^* \varphi_s + \frac{1}{2} \Big[\frac{\Delta^2}{|\Delta|^2} \chi_r^* \varphi_s + \frac{\Delta^{*2}}{|\Delta|^2} \varphi_r^* \chi_s \Big] \Big\} = 0, \tag{43}$$

where $\mathbf{v} = v\mathbf{n}$ is the velocity on the Fermi surface. We see then that ω^2 is positive.

The functions χ_r and φ_r were chosen by us in such a way that $\chi_r = -\varphi_r^*$. Therefore relations (43) can be rewritten in the form:

$$\sum_s A_{rs} c_s = 0,$$

$$A_{rs} = A_{sr} = \int d\mathbf{n} \frac{\omega^2 - (\mathbf{kv})^2}{|\Delta|^2} \operatorname{Re}\Big(2\chi_r^* \chi_s - \frac{\Delta^{*2}}{|\Delta|^2} \chi_r \chi_s \Big). \tag{44}$$

The equation for the frequencies $\omega(k)$ is obtained by equating to zero the determinant made up of the coefficients A_{rs}. We note that in the case when $r > 1$ and $s > 1$ the coefficients A_{rs} contain the logarithmically diverging integral[4)] $I = \int d\mathbf{n}/|\Delta|^2$:

$$A_{rs} = A_{rs}' I + \ldots,$$
$$A_{23}' = -{}^8/_{27} k_x k_y v^2, \quad A_{24}' = -{}^8/_{27} k_x k_z v^2,$$
$$A_{22}' = A_{33}' = A_{44}' = {}^8/_9 (\omega^2 - {}^1/_3 |k|^2 v^2),$$
$$A_{34}' = {}^8/_{27} k_y k_z v^2. \tag{45}$$

The presence of the divergence is connected with a fact that our series expansion cannot be used in the vicinity of points at which $|\Delta|$ vanishes. Were we to calculate the kernel in (44) more accurately, then we would find that when $|\mathbf{k}| \to 0$ and $\omega \to 0$

$$I_{rs} = I(\mathbf{k}, \omega) \to \infty \quad (r > 1,\ s > 1).$$

It is important that $I(\mathbf{k}, \omega)$ does not depend on the indices r and s. Therefore, in the limit as $|\mathbf{k}| \to 0$ and $\omega \to 0$ we have

$$\det A_{rs} \approx I^3 A_{11} \det A_{rs}' = 0. \tag{46}$$

Putting $A_{11} = 0$ we obtain the frequency of the acoustic excitations

$$\omega_1 = \frac{1}{\sqrt{3}} |\mathbf{k}| v. \tag{47}$$

In the remaining excitations, the directions of the chosen axis oscillate. By equating to zero the determinant made up of A_{rs}', we obtain the cubic equation

$$x^3 - {}^1/_9 (k_x^2 k_y^2 + k_y^2 k_z^2 + k_x^2 k_z^2) x + {}^2/_{27} k_x^2 k_y^2 k_z^2 = 0,$$
$$x = (\omega / v)^2 - {}^1/_3 |\mathbf{k}|^2, \tag{48}$$

from which we can get the frequencies of the indicated excitations. They are equal to

$$\omega_{n+1} = \frac{1}{\sqrt{3}} v \Big[|\mathbf{k}|^2 + \frac{2}{\sqrt{3}} (k_x^2 k_y^2 + k_y^2 k_z^2 + k_x^2 k_z^2)^{1/2} \cos \frac{\alpha + 2\pi n}{3} \Big]^{1/2}, \quad n = 1, 2, 3,$$

$$\cos \alpha = -\frac{3\sqrt{3}\, k_x^2 k_y^2 k_z^2}{(k_x^2 k_y^2 + k_y^2 k_z^2 + k_x^2 k_z^2)^{3/2}}. \tag{49}$$

It is essential that the velocity of these excitations never vanishes (this, generally speaking, is not a trivial matter [7]). Therefore the spectrum of the two-particle excitations satisfies the Landau superfluidity criterion.

In conclusion we are grateful to A. I. Larkin for numerous discussions of the questions touched upon here, and L. P. Pitaevskiĭ for discussion of the results.

[4)]In calculating the coefficients A_{rs} it is necessary to discard in the integrand the terms that contain odd powers of x, y, or z. Further, by separating the diverging integrals, we can greatly simplify all the factors which contain even powers of x, y, and z and which do not vanish identically either at infinity or at the points where $|D| = 0$. Namely, it is necessary to put in these factors $x^2 = y^2 = z^2 = 1/3$ (these are the values of these quantities at the points $|\Delta| = 0$). It then becomes obvious that the second term in (44), proportional to Δ^{*2}, makes no contribution to the coefficient of I (see (27)). We thus obtain formulas (45).

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