DISPERSION OF FIRST AND SECOND SOUND IN SUPERFLUID HELIUM

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The problem of the propagation of sound in superfluid helium is studied for frequencies $\omega\tau \gtrless 1$ (ω is the frequency, τ some characteristic time). Expressions are found for the velocity and absorption coefficient of first sound and second sound, and also for the kinetic coefficients η_p, χ_p, η_1, η_2, η_3. It is shown that a significant dispersion of first sound appears when $\omega\tau \sim 1$, and for second sound when $\omega\tau \sim u_2/u_1$. Thus in both cases dispersion sets in when the wavelength of the corresponding sound is comparable with the mean free path of the phonon, that is, we are dealing with a spatial dispersion. In the region of high frequencies, $\omega\tau \gg u_2/u_1$, second sound, not being damped, propagates only by means of the roton gas with a velocity $u_{2\infty}$(2.4), which differs strongly from the equilibrium velocity u_{20} (2.8). The results of the investigations are compared with experimental data.

In a previous article[1] a set of equations was obtained describing the propagation of sound vibrations in superfluid helium, which is valid for all frequencies $\omega\tau \gtrless 1$, where τ is some characteristic time [Eqs. I(2.28)–I(2.31), I(2.33), I(2.34)].[1)] In the present article, by using these equations, we shall consider the phenomena of dispersion and absorption of first and second sound in superfluid helium.

We shall begin the investigation of the problem with first sound. Investigation, carried out previously,[2,3] of the problem of the absorption of first sound at high frequencies was based on an assumption of the relative slowness of the processes of establishing equilibrium in the number of phonons and rotons. Such a situation does not exist, so far as we know, in superfluid helium in the more interesting region of temperatures (below 1.2°K), and takes place only at very low temperatures.

1. FIRST SOUND

The region of temperatures from 0.9 to 1.2°K. This temperature region is characterized by the fact that the interaction of phonons with one another in it is negligibly small, and the principal role is played by the scattering of phonons by rotons. Equations I(2.30) and I(2.31), which describe the propagation of first sound with consideration of small terms having an order not larger than ρ_{np}/ρ have the form

$$-\tilde{\omega}\rho' + j_r - \frac{\rho_{np}}{\rho}\nu_1 = 0,$$

$$-\tilde{\omega}j_r + \frac{1}{c^2}\left(\frac{\partial\mathscr{P}}{\partial\rho}\right)_T\rho' + \frac{\rho_{np}}{\rho}[(\tilde{\omega} - \tilde{z}_{pr})\nu_1 - 3u\nu_0] = 0, \tag{1.1}$$

where ν_0 and ν_1 are, according to I(2.33) and I(2.34), equal to

$$\nu_0 = -\frac{\tilde{\omega}u\ln\tilde{a}\,\rho' + \tilde{z}_{pr}(-2 + \tilde{z}_{pr}\ln\tilde{a})j}{2 + (1-\beta)(\tilde{\omega} - \tilde{z}_{pr})\ln\tilde{a}} \tag{1.2}$$

$$\nu_1 = 3\tilde{\omega}u\rho' + j_r + 3[\tilde{\omega} - \beta(\tilde{\omega} - \hat{z}_{pr})]\nu_0; \tag{1.3}$$

$$\tilde{\omega} = \frac{\omega}{kc}, \quad \tilde{z}_{pr} = \tilde{\omega}\left(1 - \frac{1}{i\omega\tau_{pr}}\right), \quad \tilde{a} = \frac{\tilde{z}_{pr}+1}{\tilde{z}_{pr}-1}.$$

From the condition for the existence of a nontrivial solution of (1.1), we obtain the complex velocity of first sound:

$$\frac{\omega}{k} = u_{10} - \frac{1}{2}c\frac{\rho_{np}}{\rho}\varphi(z_{pr}), \tag{1.4}$$

where $u_{10} = (\partial\mathscr{P}/\partial\rho)_T^{1/2}$ is the velocity of propagation of first sound in the region of low frequencies $\omega\tau_{pr} \ll 1$, while the function $\varphi(z_{pr})$ is equal to[2)]

$$\varphi(z_{pr}) = z_{pr} - 3$$
$$\times\frac{u^2\ln a + \{2uz_{pr} + z_{pr}^2[1 - \beta(1 - z_{pr})]\}(-2 + z_{pr}\ln a)}{2 + (1-\beta)(1 - z_{pr})\ln a}$$

[1)]In what follows the references to formulas from [1] will be denoted by the Roman numeral I.

[2)]Inasmuch as the second component on the right side of (1.4) is much smaller than the first and $u_{10} \approx c$, we shall set $\tilde{\omega} = 1$ everywhere in the function $\varphi(z_{pr})$.

$$z_{pr}=1-\frac{1}{i\omega\tau_{pr}},\quad a=\frac{z_{pr}+1}{z_{pr}-1}. \tag{1.5}$$

The real part of (1.4) is the velocity of first sound

$$u_1=u_{10}-\frac{1}{2}c\frac{\rho_{np}}{\rho}\mathrm{Re}\,\varphi(z_{pr}). \tag{1.6}$$

The absorption coefficient of first sound α_1 is the imaginary part of the wave vector. According to (1.4), it is

$$\alpha_1=\frac{1}{2}\frac{\omega}{c}\frac{\rho_{np}}{\rho}\mathrm{Im}\,\varphi(z_{pr}). \tag{1.7}$$

Equations (1.6) and (1.7) determine the dispersion and absorption of ordinary sound, due to the comparatively slow process of scattering of phonons by rotons. Analysis of the function $\varphi(z_{pr})$ shows that the principal dispersion of ordinary sound sets in when the parameter $\omega\tau_{pr}$ becomes of the order of unity.

Let us consider the region of low frequencies satisfying the condition

$$\omega\tau_{pr}\ll 1.$$

In this region $1/z_{pr} \ll 1$, and the expression in (1.5) which contains ln a can be expanded in a power series in the quantity $1/z_{pr}$:

$$\ln a=\frac{2}{z_{pr}}\left(1+\frac{1}{3}\frac{1}{z_{pr}^2}\right),$$

$$-2+z_{pr}\ln a=\frac{2}{z_{pr}^2}\left(\frac{1}{3}+\frac{1}{5}\frac{1}{z_{pr}^2}\right). \tag{1.8}$$

Substituting (1.8) in (1.5) and keeping terms linear in $\omega\tau_{pr}$, we get

$$\varphi=i\omega\tau_{pr}\left[\frac{4}{15}+\frac{(3u+1)^2}{3\beta}\right],$$

whence, in accord with (1.6) and (1.7), it follows that in the region of low frequencies

$$u_1=u_{10},\quad \alpha_1=\frac{\omega^2\tau_{pr}}{c}\frac{\rho_{np}}{\rho}\left[\frac{2}{15}+\frac{(3u+1)^2}{6\beta}\right]. \tag{1.9}$$

The first term in the absorption of sound corresponds to the phonon part of the coefficient of ordinary viscosity

$$\eta_p={}^1/_5c^2\rho_{np}\tau_{pr}, \tag{1.10}$$

the second term to the coefficient of second viscosity[3)]

$$\zeta_2=\frac{1}{3\beta}(3u+1)^2c^2\rho_{np}\tau_{pr}. \tag{1.11}$$

As has already been noted above, the second viscosity is due to the fact that the establishment of the energy balance between the phonon and roton gases is made more difficult, while at the same time it exists in each of them separately.

Substitution in (1.9) of the numerical values of all the parameters shows that the second term is much greater in order of magnitude than the first in all regions of temperatures taken into consideration. Thus at low frequencies $\omega\tau_{pr} \ll 1$, the absorption of ordinary sound is chiefly due to the second viscosity.

In the region of high frequencies, satisfying the equation

$$\omega\tau_{pr}\gg 1,$$

z_{pr} becomes of the order of unity. Setting the quantity z_{pr} in (1.6) and (1.7) equal to unity, we get

$$u_1=u_{10}+c\frac{\rho_{np}}{\rho}\left[\frac{3}{4}(u+1)^2\ln(2\omega\tau_{pr})-3u-2\right], \tag{1.12}$$

$$\alpha_1=\frac{3}{8}\pi(u+1)^2\frac{\omega}{c}\frac{\rho_{np}}{\rho}. \tag{1.13}$$

Region of temperatures from 0.6 to 0.9°K. In this temperature region, in addition to the scattering of phonons by rotons, phonon-phonon scattering is also important. Therefore, we must keep the collision integral $J_{pp}(n)$ in Eq. I(2.16). Account of $J_{pp}(n)$ has an effect only on the form of the function φ, which now depends on the two parameters z_{pr} and $z_{pp} = 1 - 1/i\omega\tau_{pp}$:

$$\begin{aligned}\varphi(z_{pr},z_{pp}) &= z_{pr}-3\{u^2\ln a+[2uz_{pr}\\ &+z_{pr}^2[1-\beta(1-z_{pr})]+3u^2(1-z_{pp})]\\ &\times[-2+(z_{pr}+z_{pp}-1)\ln a]\}\{2+[1-z_{pp}\\ &+(1-\beta)(1-z_{pr})]\ln a+3(1-z_{pp})\\ &\times[1-\beta(1-z_{pr})][-2+(z_{pr}+z_{pp}-1)\ln a]\}^{-1},\end{aligned}$$

$$a=(z_{pr}+z_{pp})/(z_{pr}+z_{pp}-2). \tag{1.14}$$

The velocity and the absorption coefficient of the first sound are determined as before by Eqs. (1.6) and (1.7) but with the functions $\varphi(z_{pr}, z_{pp})$ from (1.14):

$$u_1=u_{10}-\frac{1}{2}c\frac{\rho_{np}}{\rho}\mathrm{Re}\,\varphi(z_{pr},z_{pp}), \tag{1.15}$$

$$\alpha_1=\frac{1}{2}\frac{\omega}{c}\frac{\rho_{np}}{\rho}\mathrm{Im}\,\varphi(z_{pr},z_{pp}). \tag{1.16}$$

In Eqs. (1.1) we took into account only terms of small order not larger than ρ_{np}/ρ. If we take the next terms of the expansion in (1.1), then a compo-

[3)]For more details on the coefficients η_p, κ_p, ξ_1, ξ_2, ξ_3, and ξ_4, see Sec. 5.

nent appears in the expressions for φ (1.5) and (1.14) containing the derivative $(\partial\rho/\partial T)_{\mathcal{P}}$ (that is, δ). Account of this component is important only in the region of low frequencies $\omega\tau_{pr}$, $\omega\tau_{pp} \ll 1$, and at temperatures below 0.9°K. The velocity and absorption coefficient in this case are equal to

$$u_1 = u_{10} + \frac{1}{2}c\delta^2 \frac{S_p}{C}\frac{\rho_{np}}{\rho}, \tag{1.17}$$

$$\alpha_1 = \frac{\omega^2\tau_{pr}}{c}\frac{\rho_{np}}{\rho}\left\{\frac{2/15}{1+\tau_{pr}/\tau_{pp}} + \frac{1}{6\beta}\left(3u+1+\delta\frac{C_p}{C}\right)^2 + \frac{1}{2}\delta^2\left(\frac{S_p}{C}\right)^2\left(1-\frac{TS}{\rho_n c^2}\right)^2\right\}. \tag{1.18}$$

As will be shown in Sec. 5, the first term in Eq. (1.18) is due to the phonon part of the first viscosity coefficient

$$\eta_p = \frac{1}{5}c^2\rho_{np}\frac{\tau_{pr}}{1+\tau_{pr}/\tau_{pp}}, \tag{1.19}$$

the second term to the coefficient of second viscosity

$$\zeta_2 = \frac{1}{3\beta}\left(3u+1+\delta\frac{C_p}{C}\right)^2 c^2\rho_{np}\tau_{pr}, \tag{1.20}$$

and the third to the phonon part of the coefficient of thermal conductivity

$$\varkappa_p = c^2 S_p \tau_{pr}(1 - TS/\rho_n c^2)^2. \tag{1.21}$$

For high frequencies, $\omega\tau_{pr}$, $\omega\tau_{pp} \gg 1$, in the temperature region under consideration, the velocity of first sound is equal to

$$u_1 = u_{10} + c\frac{\rho_{np}}{\rho}\left\{\frac{3}{4}(u+1)^2\ln\frac{2\omega\tau_{pr}}{1+\tau_{pr}/\tau_{pp}} - 3u - 2\right\},$$

while the absorption coefficient is determined as before by the expression (1.13).

2. SECOND SOUND

As will be shown below, the dispersion feature of second sound begins at frequencies satisfying the condition

$$\omega\tau_{pr} \sim u_2/u_1$$

(u_2 is the velocity of second sound). In spite of the smallness of the ratio u_2/u_1, up to the present time it has been regarded as insurmountably difficult to obtain temperature oscillations of such a frequency. However, according to Notarys and Pellam,[4] this range of frequencies can be obtained at the present time for second sound.

Region of temperatures from 0.9 to 1.2°K. Equations I(2.28) and I(2.29), which determine the propagation of second sound, have the following form, with account of terms of small order not higher than ρ_{np}/ρ,

$$-\tilde{\omega}T_r' + \frac{\rho_s}{\rho}\frac{S_r}{C_r}w_r + \frac{C_p}{C_r}\beta(\tilde{\omega}-\tilde{z}_{pr})(\nu_0+T_r') + \frac{S_r}{C_r}\frac{\rho_{np}}{\rho}(\nu_1+w_r) = 0,$$

$$-\tilde{\omega}w_r + \frac{TS_r}{\rho_{nr}c^2}T_r' + \frac{\rho_{np}}{\rho_{nr}}(\tilde{\omega}-\tilde{z}_{pr})(\nu_1+w_r) = 0, \tag{2.1}$$

where

$$\nu_0 + T_r' = \frac{1}{2+(1-\beta)(\tilde{\omega}-\tilde{z}_{pr})\ln\tilde{a}} \times\{[2+(\tilde{\omega}-\tilde{z}_{pr})\ln\tilde{a}]T_r' + [\tilde{\omega}-(\rho_s/\rho)\tilde{z}_{pr}](-2+\tilde{z}_{pr}\ln\tilde{a})w_r\},$$

$$\nu_1 + w_r = -3\tilde{\omega}T_r' + (\rho_s/\rho)w_r + 3[\tilde{\omega}-\beta(\tilde{\omega}-\tilde{z}_{pr})](\nu_0+T_r'). \tag{2.2}$$

It actually follows from Eq. (2.2) that the important dispersion of second sound begins when $\omega\tau_{pr}$ becomes of the order of u_2/u_1. This circumstance favors observation of the phenomenon described, since (inasmuch as $u_2/u_1 \lesssim 1$) the dispersion of the second sound begins at much lower frequencies than for first sound. The condition $\omega\tau_{pr} \sim u_2/u_1$ can be rewritten in the form $l_p \sim \lambda$, where l_p is the mean free path of the phonon and λ is the wavelength of second sound. Thus the dispersion of second sound begins (as should be expected) when the wavelength of the second sound is comparable with the mean free path of the phonons, that is, we are dealing with spatial dispersion.

The condition for the existence of a nontrivial solution of the system (2.1) is obtained by setting its determinant equal to zero. Expanding the determinant, we obtain an equation for the determination of the complex velocity of second sound ω/k. For frequencies of the order of $\omega\tau_{pr} \sim u_2/u_1$, this equation runs as follows:

$$\left(\frac{\omega}{k}\right)^2\left\{1-\beta\frac{C_p}{C_r}(1-z_{pr})\frac{2+(\tilde{\omega}-\tilde{z}_{pr})\ln\tilde{a}}{2+(1-\beta)(\tilde{\omega}-\tilde{z}_{pr})\ln\tilde{a}}\right\} = u_{2\infty}^2, \tag{2.3}$$

$$u_{2\infty} = \left[\left(1-\frac{\rho_{nr}}{\rho}\right)\frac{TS_r^2}{\rho_{nr}C_r}\right]^{1/2}, \tag{2.4}$$

($\tilde{a} = \tilde{z}_{pr} + 1)/(\tilde{z}_{pr} - 1)$). The expression for $u_{2\infty}$ is identical with the well-known expression for the velocity of second sound in which all the thermodynamic quantities for the roton gas have been substituted. In (2.3) we have omitted all terms the ratio of which to the last term does not exceed $\beta^{-1}(k\Delta/cP_0)^2 \ll 1$. (This inequality is satisfied for all temperatures above 0.3° K.)

Equation (2.3) for the frequencies $\omega \sim \tau_{pr}^{-1}u_2/u_1$ cannot be solved in algebraic form; its solution requires numerical calculations.

In the region of high frequencies,

$$\omega\tau_{pr} \gg u_2 / u_1,$$

it is materially simplified and one easily obtains

$$\frac{\omega}{k} = u_{2\infty}\left(1 - \frac{\beta C_p / C_r}{i\omega\tau_{pr}}\right)^{-1/2}. \tag{2.5}$$

It is seen from Eq. (2.5) in the case of very high frequencies $\omega\tau_{pr} \gg \beta C_p/C_r$ that second sound propagates only via the roton gas, with a velocity equal to $u_{2\infty}$ (2.4), while the absorption coefficient does not depend on the frequency and is equal to

$$\alpha_2 = \frac{1}{2}\frac{\beta C_p/C_r}{u_{2\infty}\tau_{pr}}. \tag{2.6}$$

In the region of low frequencies, satisfying the condition

$$\omega\tau_{pr} \ll u_2 / u_1,$$

$1/\tilde{z}_{pr} \ll 1$, and one can expand the expression in (2.2) containing $\ln \tilde{a}$ in a power series in the quantity $1/\tilde{z}_{pr}$ (1.8). As the result of lengthy calculations, the condition of consistency of the set of equations (2.1) takes the following form:

$$\left(\frac{\omega}{k}\right)^2 = u_{20}^2 - i\omega\tau_{pr}c^2\frac{\rho_s}{\rho_n}\frac{\rho_{np}}{\rho}\left\{\frac{4}{15} + \frac{\rho_n}{\rho_s}\frac{\rho c^2}{TC}\left(1 - \frac{TS}{\rho_n c^2}\right)^2 + \frac{1}{3\beta}\left(1 - 3\frac{S}{C}\right)^2\right\}, \tag{2.7}$$

$$u_{20} = \left(\frac{\rho_s}{\rho_n}\frac{TS^2}{\rho C}\right)^{1/2}. \tag{2.8}$$

For the frequencies considered, the imaginary terms on the right side of (2.7) are small in comparison with the real terms. In this case, the velocity of propagation of second sound is equal to u_{20} (2.8), while its absorption coefficient is determined by the expression

$$\alpha_2 = \frac{\omega^2 c^2 \tau_{pr}}{u_{20}^3}\frac{\rho_s}{\rho_n}\frac{\rho_{np}}{\rho}\left\{\frac{2}{15} + \frac{1}{2}\frac{\rho_n}{\rho_s}\frac{\rho c^2}{TC}\left(1 - \frac{TS}{\rho_n c^2}\right)^2 + \frac{1}{6\beta}\left(1 - 3\frac{S}{C}\right)^2\right\}. \tag{2.9}$$

The first two terms in (2.9) are due to the appearance respectively of the phonon part of the first viscosity coefficient η_p (1.10) and the phonon part of the coefficient of thermal conductivity κ_p (1.21).[4] The third term in (2.9) corresponds to a combination of the coefficients of second viscosity $(\zeta_2 + \rho^2\zeta_3 - 2\rho\zeta_1)$ entering into I(2). As will be shown in Sec. 5, the coefficients ζ_1 and ζ_3 are equal to

$$\rho\zeta_1 = \frac{1}{\beta}(3u + 1)\left(u + \frac{S}{C}\right)c^2\rho_{np}\tau_{pr},$$

$$\rho^2\zeta_3 = \frac{3}{\beta}\left(u + \frac{S}{C}\right)^2 c^2\rho_{np}\tau_{pr}.$$

Substitution of the numerical values of all parameters in (2.9) shows that the second term greatly exceeds the other two in magnitude. Thus, in the region of low frequencies ($\omega\tau_{pr} \ll u_2/u_1$), the absorption of second sound is essentially determined by the phonon part of the coefficient of thermal conductivity. The temperature dependences of the velocities u_{20} (4.8) and $u_{2\infty}$ (4.4) are drawn in Fig. 1. For a fixed temperature, as the frequency is increased, we gradually go over from the curve u_{20}, which describes the equilibrium second sound, to the curve $u_{2\infty}$, which describes the roton second sound.

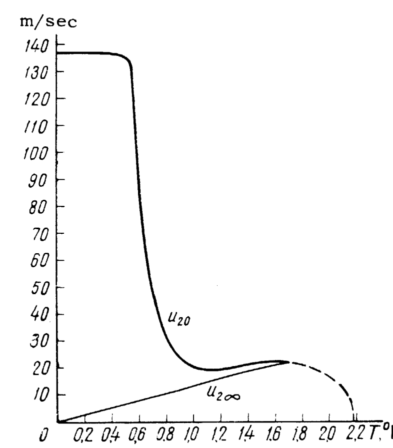

FIG. 1. Temperature dependence of the velocities u_{20} and $u_{2\infty}$.

<u>The region of temperatures from 0.6 to 0.9° K.</u> In this temperature region, phonon-phonon scattering becomes important and one must take into consideration in Eq. I(2.16) the collision integral $J_{pp}(n)$ which characterizes this scattering process. Consideration of $J_{pp}(n)$ transforms the equation

[4]Equation (1.21) for the phonon part of the coefficient of thermal conductivity is valid in the entire temperature range from 0.6 to 1.2°K (for more details see Sec. 5).

(2.3) to the form

$$\left(\frac{\omega}{k}\right)^2\Big[1-\beta\frac{C_p}{C_r}(1-z_{pr})\{2+(\tilde\omega-\tilde z_{pr})\ln\tilde a + 3\tilde\omega(\tilde\omega-\tilde z_{pp})[-2+(\tilde z_{pr}+\tilde z_{pp}-\tilde\omega)\ln\tilde a]\} \times\{2+[\tilde\omega-\tilde z_{pr}+(1-\beta)(\tilde\omega-\tilde z_{pr})\ln\tilde a] + 3(\tilde\omega-\tilde z_{pp})[\tilde\omega-\beta(\tilde\omega-\tilde z_{pr})][-2+(\tilde z_{pr}+\tilde z_{pp}-\tilde\omega)\ln\tilde a\}^{-1}\Big]=u_{2\infty}^2 \qquad (2.10)$$

(where $\tilde a = (\tilde z_{pr} + \tilde z_{pp})/(\tilde z_{pr} + \tilde z_{pp} - 2)$, $\tilde z_{pp} = \tilde\omega(1 - 1/i\omega\tau_{pp})$. As we have pointed out above, in the general case, this can be solved only numerically.

For high frequencies $\omega\tau_{pr}$, $\omega\tau_{pp} \ll 1$, Eq.(2.5) is again obtained from Eq. (2.10). For sufficiently low temperatures, the ratio $\beta C_p/C_r$ can become large and, as is seen from Eq. (2.5), the roton second sound will be strongly damped. At very high frequencies, $\omega\tau_{pr} \gg \beta C_p/C_r$, Eqs. (2.4) and (2.6) are valid as before.

In the region of low frequencies $\omega\tau_{pr}$, $\omega\tau_{pp} \ll 1$, the scattering of phonons by phonons does not give a significant contribution to the absorption of second sound, inasmuch as the account of the scattering mentioned has an effect only on the form of the term in Eq. (2.9) that is smallest in magnitude:

$$\alpha_2=\frac{\omega^2c^2\tau_{pr}}{u_{20}^3}\frac{\rho_s}{\rho_n}\frac{\rho_{np}}{\rho}\left\{\frac{2/15}{1+\tau_{pr}/\tau_{pp}}+\frac{1}{2}\frac{\rho_n}{\rho_s}\frac{\rho c^2}{TC}\left(1-\frac{TS}{\rho_n c^2}\right)^2 + \frac{1}{6\beta}\left(1-3\frac{S}{C}\right)^2\right\}, \qquad (2.11)$$

corresponding to η_p (1.19).

In the temperature region under consideration, for the frequencies $\omega\tau_{pr}$, $\omega\tau_{pp} \ll 1$, one must take into account in Eqs. (2.1) terms containing the derivative $(\partial\rho/\partial T)_{\mathscr{P}}(\sim\delta)$. Its account leads to the appearance of additional components in the expression for the velocity of second sound:

$$u_2=u_{20}\left(1-\frac{1}{2}\delta^2\frac{S_p}{C}\frac{\rho_{np}}{\rho}\right),$$

and also in the expressions for the coefficient η_2 of second viscosity (1.20),

$$\rho\zeta_1=\frac{1}{\beta}\left(3u+1+\delta\frac{C_p}{C}\right)\left(u+\frac{S}{C}+\delta\frac{S_p}{C}\right)c^2\rho_{np}\tau_{pr}$$

$$\rho^2\zeta_3=\frac{3}{\beta}\left(u+\frac{S}{C}+\delta\frac{S_p}{C}\right)^2c^2\rho_{np}\tau_{pr}. \qquad (2.12)$$

The fact that the derivative $(\partial\rho/\partial T)_{\mathscr{P}}$ does not enter into the expression for the absorption of second sound is connected with the fact that the combination of the coefficients of second viscosity (1.20), (2.12), entering into (2.11) remains the same as before for $T < 0.9$°K and is equal to

$$\zeta_2+\rho^2\zeta_3-2\rho\zeta_1=(1-3S/C)^2/3\beta.$$

3. REGION OF TEMPERATURES ABOVE 1.2°K

At temperatures above 1.2° K, the phonons which move in a given direction, as has already been pointed out in the work of [1], are described by the quasi-equilibrium distribution function I (1.14), which depends on the chemical potential α in this direction. The departure of n from its constant equilibrium value n_0 is from (1.15) equal to

$$n'=\frac{\partial n_0}{\partial\varepsilon}\left[\frac{\partial\varepsilon}{\partial\rho}\rho'+\varepsilon\nu(\cos\theta)+kT\alpha(\cos\theta)\right], \qquad (3.1)$$

where $\nu(\cos\theta)$ and $\alpha(\cos\theta)$ are unknown functions of the angle θ which will be found as a result of solution of the kinetic equation. The total energy of the phonons moving in a given direction and their number in small-angle scattering are conserved in a four-phonon process. In the region $T > 1.2$° K, a five-phonon process does not change the energy of the phonons moving in a given direction but significantly changes their number.

We substitute (3.1) into Eq. I (2.8) and integrate the right and left sides over all phonons and over all possible energies. Inasmuch as the integrals

$$\int J'_{pp}(n)\varepsilon p^2\,dp,\quad \int J'_{pp}(n)p^2\,dp,\quad \int J_{3\to2}(n)\varepsilon p^2\,dp$$

are equal to zero, according to what has been pointed out, we have[5)]

$$(\tilde\omega-\cos\theta)\left[\nu(\cos\theta)+\frac{27}{\pi^4}\alpha(\cos\theta)\right]+\tilde\omega u p'+\cos^2\theta v_s = -\frac{\tilde\omega}{i\omega}\int J_{pr}(n)\varepsilon p^2\,dp\Big/\int\frac{\partial n_0}{\partial\varepsilon}\varepsilon^2p^2\,dp,$$

$$(\tilde\omega-\cos\theta)\left[\nu(\cos\theta)+\frac{5\pi^2}{108}\alpha(\cos\theta)\right]+\omega u\rho'+\cos^2\theta v_s = -\frac{\tilde\omega}{i\omega}\left\{\int J_{pr}(n)p^2\,dp\Big/\int\frac{\partial n_0}{\partial\varepsilon}\varepsilon p^2\,dp + \int J_{3\to2}(n)p^2\,dp\Big/\int\frac{\partial n_0}{\partial\varepsilon}\varepsilon p^2\,dp\right\}. \qquad (3.2)$$

The collision integral $J_{pr}(n)$ computed from Eq. I(2.19) with the distribution function (3.1) is

5)We shall neglect the magnitude of the collision integral J_{pp} (n) in the region of temperatures above 1.2°K in comparison with J_{pr} (n).

equal to

$$J_{\mathrm{pr}}(n)=-\frac{1}{\tau_{\mathrm{pr}}(p)}\frac{\partial n_0}{\partial\varepsilon}\Big\{\varepsilon[\nu(\cos\theta)-\nu_0+\cos\theta w_{\mathrm{r}}]$$
$$+\frac{\varepsilon^2}{3\mu c^2}(\nu_0+T_{\mathrm{r}}')+kT[\alpha(\cos\theta)-\alpha_0]\Big\}, \qquad (3.3)$$

where α_0 is the coefficient for the zeroth harmonic in the expansion of the function $\alpha(\cos\theta)$ in spherical harmonics:

$$\alpha(\cos\theta)=\sum_{i=1}^{\infty}\alpha_i P_i(\cos\theta).$$

Integrating both sides of Eq. (3.3) over all phonons and over all possible energies, we obtain

$$\int J_{\mathrm{pr}}(n)\varepsilon p^2\,dp=-\frac{1}{\tau_{\mathrm{pr}}}\Big\{\nu(\cos\theta)-\nu_0+\cos\theta w_{\mathrm{r}}$$
$$+\beta(\nu_0+T_{\mathrm{r}}')+\frac{1}{8}[\alpha(\cos\theta)-\alpha_0]\Big\}\int\frac{\partial n_0}{\partial\varepsilon}\varepsilon^2p^2\,dp, \qquad (3.4)$$

$$\int J_{\mathrm{pr}}(n)p^2\,dp=-\frac{\pi^4}{216}\frac{1}{\tau_{\mathrm{pr}}}\Big\{\nu(\cos\theta)-\nu_0+\cos\theta w_{\mathrm{r}}$$
$$+\frac{8}{9}\beta(\nu_0+T_{\mathrm{r}}')+\frac{1}{7}[\alpha(\cos\theta)-\alpha_0]\Big\}\int\frac{\partial n_0}{\partial\varepsilon}\varepsilon p^2\,dp.$$

The value of the integral $\int J_{3\to2}(n)p^2dp$ has been previously computed:[2]

$$\int J_{3\to2}(n)p^2\,dp=\alpha(\cos\theta)\Gamma_{\mathrm{p}}.$$

With account of (1.13), this expression can be rewritten in the following form:

$$\int J_{3\to2}(n)p^2\,dp=-\frac{1}{\tau_{3\to2}}\alpha(\cos\theta)\int\frac{\partial n_0}{\partial\varepsilon}\varepsilon p^2\,dp, \qquad (3.5)$$

where $\tau_{3\to2}$ is the time characterizing the five phonon process,[3]

$$\frac{1}{\tau_{3\to2}}=\Lambda\frac{kT^{12}}{3N_{\mathrm{p}}},\qquad N_{\mathrm{p}}\approx2.4\cdot4\pi\left(\frac{kT}{2\pi\hbar c}\right)^3 \qquad (3.6)$$

(N_p is the number of phonons per unit volume of helium II).

Substituting (3.4) in (3.5) on the right side of Eq. (3.2), we finally obtain

$$(\tilde\omega-\cos\theta)\left[\nu(\cos\theta)+\frac{27}{\pi^4}\alpha(\cos\theta)\right]+\tilde\omega u\rho'+\cos^2\theta v_s$$
$$=\frac{\tilde\omega}{i\omega\tau_{\mathrm{pr}}}\Big\{\nu(\cos\theta)-\nu_0+\cos\theta w_{\mathrm{r}}+\beta(\nu_0+T_{\mathrm{r}}')$$
$$+\frac{1}{8}[\alpha(\cos\theta)-\alpha_0]\Big\},$$

$$(\tilde\omega-\cos\theta)\left[\nu(\cos\theta)+\frac{5\pi^2}{108}\alpha(\cos\theta)\right]+\tilde\omega u\rho'+\cos^2\theta v_s$$
$$=\frac{\pi^4}{216}\frac{\tilde\omega}{i\omega\tau_{\mathrm{pr}}}\Big\{\nu(\cos\theta)-\nu_0+\cos\theta w_{\mathrm{r}}+\frac{8}{9}\beta(\nu_0+T_{\mathrm{r}}')$$
$$+\frac{1}{7}[\alpha(\cos\theta)-\alpha_0]\Big\}+\frac{\tilde\omega}{i\omega\tau_{3\to2}}\alpha(\cos\theta). \qquad (3.7)$$

The further solution of the problem $T > 1.2^\circ$K is similar to the solution given in [1] for $T < 1.2^\circ$K. The computation of the velocity and the sound absorption in the general case of an arbitrary value of the parameter $\omega\tau_{\mathrm{pr}}$ is very complicated; therefore we shall limit ourselves to consideration only of low frequencies satisfying the inequality $\omega\tau_{\mathrm{pr}} \ll 1$ for first sound and $\omega\tau_{\mathrm{pr}} \ll u_2/u_1$ for second sound. For first sound, such a consideration practically exhausts all possible frequencies for temperatures above 1.2°K, inasmuch as the value of $\omega\tau_{\mathrm{pr}}$ is very much less than unity for $T > 1.2^\circ$K in accordance with I(1.18).

As the result of the lengthy calculations, Eqs. I(2.8)–I(2.11), which describe the propagation of first and second sound in the region of low frequencies with account only of terms linear in $\omega\tau_{\mathrm{pr}}$, take the following form:[6)]

$$-\tilde\omega T_{\mathrm{r}}'+\frac{\rho_s}{\rho}\frac{S}{C}w_{\mathrm{r}}+i\omega\tilde\tau_{\mathrm{pr}}\frac{S_{\mathrm{p}}}{C}\Big\{\frac{1}{\tilde\beta}\frac{\rho_s}{\rho}\left(1-3\frac{S}{C}\right)$$
$$+\left(1-\frac{\rho_n c^2}{TS}\right)\left(1-\frac{TS}{\rho c^2}\right)\Big\}w_{\mathrm{r}}=0, \qquad (3.8)$$

$$-\tilde\omega w_{\mathrm{r}}+\frac{TS}{\rho_n c^2}T_{\mathrm{r}}'-i\omega\tilde\tau_{\mathrm{pr}}\frac{\rho_{n\mathrm{p}}}{\rho_n}\Big\{\frac{1}{3\tilde\beta\tilde\omega}\frac{\rho_s}{\rho}\left(1-3\frac{S}{C}\right)$$
$$+\frac{4}{15}\frac{1}{\tilde\omega}\frac{\rho_s}{\rho}+\tilde\omega\left(1-\frac{\rho_n c^2}{TS}\right)\Big\}w_{\mathrm{r}}=0;$$

$$-\tilde\omega j_{\mathrm{r}}+\frac{1}{c^2}\left(\frac{\partial\mathscr{P}}{\partial\rho}\right)_T\rho'$$
$$-i\omega\tilde\tau_{\mathrm{pr}}\frac{\rho_{n\mathrm{p}}}{\rho}\Big\{\frac{4}{15}+\frac{1}{3\tilde\beta}(3u+1)^2\Big\}\rho'=0,$$

$$-\tilde\omega\rho'+j_{\mathrm{r}}=0. \qquad (3.9)$$

Here

$$\frac{1}{\tilde\beta}=\left[\frac{1}{\beta}+\left(\frac{27}{\pi^4}-\frac{1}{9}\right)\frac{\tau_{3\to2}}{\tau_{\mathrm{pr}}}\right]$$
$$\times\left[1+\frac{1}{8}\left(\frac{\pi^4}{216}-1\right)^2\Big/\frac{\pi^4}{216}\left(\frac{1}{56}\frac{\pi^4}{216}+\frac{\tau_{\mathrm{pr}}}{\tau_{3\to2}}\right)\right]^{-1},$$

$$\frac{1}{\tilde\tau_{\mathrm{pr}}}=\frac{1}{\tau_{\mathrm{pr}}}\left[1+\frac{1}{8}\left(\frac{\pi^4}{216}-1\right)^2\Big/\frac{\pi^4}{216}\left(\frac{1}{56}\frac{\pi^4}{216}+\frac{\tau_{\mathrm{pr}}}{\tau_{3\to2}}\right)\right]^{-1}. \qquad (3.10)$$

From the condition of consistency of the set of equations (3.8) and (3.9), we obtain the velocity and absorption coefficients of first and second sound:

$$u_1=u_{10},\quad \alpha_1=\frac{\omega^2\tilde\tau_{\mathrm{pr}}}{c}\frac{\rho_{n\mathrm{p}}}{\rho}\left[\frac{2}{15}+\frac{1}{6\tilde\beta}(3u+1)^2\right], \qquad (3.11)$$

---

6)In (3.8) and (3.9), we have omitted terms with $(\partial\rho/\partial T)_{\mathscr{P}}$, inasmuch as they are essential only at low temperatures (below 0.9°K), as has already been pointed out in Sec. 1.

$$u_2 = u_{20}, \quad \alpha_2 = \frac{\omega^2 c^2 \tilde{\tau}_{pr}}{u_{20}^3} \frac{\rho_s}{\rho_n} \frac{\rho_{nF}}{\rho}\left[\frac{2}{15} + \frac{1}{2}\frac{\rho_n}{\rho_s}\frac{\rho c^2}{TC}\right.$$

$$\left.\times\left(1 - \frac{TS}{\rho_n c^2}\right)^2 + \frac{1}{6\tilde{\beta}}\left(1 - 3\frac{S}{C}\right)^2\right]. \qquad (3.12)$$

The first terms in the square brackets in Eqs. (3.11) and (3.12) are due to the phonon part of the coefficient of ordinary viscosity

$$\eta_p = {}^1/_5 c^2 \rho_{np} \tilde{\tau}_{pr},$$

the second corresponds to the coefficient of second viscosity

$$\zeta_2 = (3u + 1)^2 c^2 \rho_{np} \tilde{\tau}_{pr} / 3\tilde{\beta}$$

and the phonon part of the coefficient of thermal conductivity

$$\varkappa_p = c^2 S_p \tilde{\tau}_{pr} (1 - TS / \rho_n c^2)^2.$$

The third term in (3.12) corresponds to the combination of coefficients of second viscosity $\zeta_2 + \rho^2\zeta_3 - 2\rho\zeta_1$ entering into I(2), where

$$\rho\zeta_1 = \tilde{\beta}^{-1}(3u + 1)(u + S / C) c^2 \rho_{np} \tilde{\tau}_{pr},$$

$$\rho^2\zeta_3 = 3\tilde{\beta}^{-1}(u + S / C)^2 c^2 \rho_{np} \tilde{\tau}_{pr}$$

(for more details, see Sec. 5).

The value of the numerical coefficient Λ in (3.6) can be computed from experimental values of the coefficient of absorption of first sound in helium II. According to the data of Atkins and Chase,[5] $\Lambda = 3.4 \times 10^{43}$. Substitution in (3.11) and (3.12) of the numerical values of all the parameters shows that the absorption of first sound is determined by the second viscosity and second sound by the thermal conductivity.

4. REGION OF TEMPERATURES BELOW 0.6°K

In the region of temperatures below 0.6°K, the contribution of rotons to all the phenomena becomes unimportant and one can consider a purely phonon gas. The equations described in the propagation of sound in a phonon gas, according to I(2.8), I(2.10) and I(2.11), have the form

$$-\tilde{\omega}\rho' + j = 0,$$

$$-\tilde{\omega} j + \frac{1}{c^2}\left(\frac{\partial \mathscr{P}}{\partial \rho}\right)_T \rho' + \frac{\rho_{np}}{\rho}(3uT' + \tilde{\omega} w) = 0,$$

$$-\tilde{\omega} T' + \frac{1}{3}\frac{\rho_{sp}}{\rho} w + \tilde{\omega}\frac{3u + 1}{3}\rho' = 0,$$

$$-[2 + (\tilde{\omega} - \tilde{z}_{pp})\ln\tilde{a}]\, T'$$

$$-\left(\tilde{\omega} - \frac{\rho_{sp}}{\rho}\tilde{z}_{pp}\right)(-2 + \tilde{z}_{pp}\ln\tilde{a})\, w$$

$$+ u\tilde{\omega}\ln\tilde{a}\rho' + \tilde{z}_{pp}(-2 + \tilde{z}_{pp}\ln\tilde{a})\, j = 0 \qquad (4.1)$$

($\tilde{a} = (\tilde{z}_{pp} + 1)/(\tilde{z}_{pp} - 1)$). In (4.1), we have introduced the notation

$$j = \left|\frac{\rho_{sp}}{\rho}\mathbf{v}_s + \frac{\rho_{np}}{\rho}\mathbf{v}_n\right|,$$

$$\frac{\rho_{sp}}{\rho} = 1 - \frac{\rho_{np}}{\rho}, \qquad \nu_0 = -T', \qquad \nu_1 = -w$$

(T' and $|\mathbf{v}_n - \mathbf{v}_s|$ are the ratios of the departure of the temperature and the relative velocity, determined for the equilibrium state of helium II, to T_0 and c, respectively).

The relations $\nu_0 = -T'$, $\nu_1 = -w$ follow from the requirements that

$$\int \varepsilon(n - n^0)\, d\tau_r = 0, \qquad \int \mathbf{p}(n - n^0)\, d\tau_r = 0,$$

that is, that the nonequilibrium distribution function of the phonons n must lead to the same value for the total energy and the total momentum as the equilibrium function

$$n^0 = \left\{\exp\left[\frac{\varepsilon - \mathbf{p}(\mathbf{v}_n - \mathbf{v}_s)}{kT}\right] - 1\right\}^{-1}. \qquad (4.2)$$

The condition for the consistency of the set (4.1) is the vanishing of its determinant. By expanding the determinant, we get an equation which, with account of only the terms linear in ρ_{np}/ρ, splits into two equations[7)] ($a = (z_{pp} + 1)/(z_{pp} - 1)$):

$$\left(\frac{\omega}{k}\right)^2 - u_{10}^2 + c^2\frac{\rho_{np}}{\rho}\varphi(z_{pp}) = 0, \qquad (4.3)$$

$$\varphi(z_{pp}) = 1 - 3$$

$$\times\frac{u^2\ln a + \{2u + 1 + 3u^2(1 - z_{pp})\}(-2 + z_{pp}\ln a)}{2 + (1 - z_{pp})\ln a + 3(1 - z_{pp})(-2 + z_{pp}\ln a)};$$

$$2 + (\tilde{\omega} - \tilde{z}_{pp})\ln\tilde{a} + 3\tilde{\omega}(\tilde{\omega} - \tilde{z}_{pp})(-2 + \tilde{z}_{pp}\ln\tilde{a})$$

$$-3\frac{\rho_{np}}{\rho}\frac{u^2 + 2u\tilde{\omega}^2 + \tilde{\omega}^2}{\tilde{\omega}^2 - 1}(-2 + \tilde{z}_{pp}\ln\tilde{a}) = 0. \qquad (4.4)$$

Equation (4.3) determines the complex velocity of ordinary sound, the velocity and the coefficient of absorption of which are equal to

$$u_1 = u_{10} - \frac{1}{2}c\frac{\rho_{np}}{\rho}\operatorname{Re}\varphi(z_{pp}), \qquad (4.5)$$

$$\alpha_1 = \frac{1}{2}\frac{\omega}{c}\frac{\rho_{np}}{\rho}\operatorname{Im}\varphi(z_{pp}). \qquad (4.6)$$

It is not difficult to establish the fact that the resultant expressions for u_1 and α_1, as also (1.6)

7)Inasmuch as the last component in Eq. (4.3) is much smaller than the first and $u_{10} \approx c$, we can everywhere set $\tilde{\omega} = 1$ in the function $\varphi(z_{pp})$.

and (1.7), are the result of the more general formulas (1.15) and (1.16).

Equation (4.4) has an undamped acoustic solution only in the region of low frequencies satisfying the condition $\omega\tau_{pp} \ll 1/\sqrt{3}$. In this region of frequencies, one can expand $\ln\tilde{a}$ and $-2 + z_{pp}\ln a$ in power series in the function $1/z_{pp}$ (1.8), as a result of which (4.4) takes the form

$$\left(\frac{\omega}{k}\right)^2 = \frac{c^2}{3}\left[1 - \frac{3}{2}(3u^2 + 2u + 1)\frac{\rho_{np}}{\rho} - \frac{4}{5}i\omega\tau_{pp}\right].$$

The root of this equation determines the complex viscosity and consequently the velocity and the absorption coefficient of second sound:

$$u_2 = \frac{c}{\sqrt{3}}\left[1 - \frac{3}{4}(3u^2 + 2u + 1)\frac{\rho_{np}}{\rho}\right], \qquad \alpha_2 = \frac{2}{5\sqrt{3}}\frac{\omega^2\tau_{pp}}{c}.$$

As was to be expected, in a purely phonon gas, the quantity u_2 at low temperatures approaches the limit $c/\sqrt{3}$, while its absorption depends only on the coefficient of ordinary viscosity

$$\eta_p = \frac{1}{5}c^2\rho_{np}\tau_{pp}. \tag{4.7}$$

In the region of low frequencies, $\omega\tau_{pp} \ll 1$, the velocity and absorption coefficient of first sound, in accord with (4.5) and (4.6), are equal to

$$u_1 = u_{10} + c\frac{1}{4}(3u + 1)^2\frac{\rho_{np}}{\rho},$$

$$\alpha_1 = \frac{3}{10}(u + 1)^2\frac{\omega^2\tau_{pp}}{c}\frac{\rho_{np}}{\rho}.$$

In a purely phonon gas,

$$u_{10} = c\left[1 - \frac{3}{2}\left(u^2 - \frac{1}{4}\frac{\rho^2}{c}\frac{\partial^2 c}{\partial\rho^2}\right)\frac{\rho_{np}}{\rho}\right]$$

and consequently for $T = 0$, $u_1 = c$, as it must be. The absorption of ordinary sound is connected with the coefficient of ordinary viscosity (4.7).[8] The term $(u + 1)^2$ arises because of the fact that we have taken into account the derivative term $(\partial\rho/\partial T)_{\mathscr{P}}$ $(\delta = -3u - 1)$ in Eqs. (4.1).

For high frequencies, $\omega\tau_{pp} \gg 1$

$$u_1 = u_{10} + c\frac{\rho_{np}}{\rho}\left\{\frac{3}{4}(u + 1)^2\ln(2\omega\tau_{pp}) - 3u - 2\right\}, \tag{4.8}$$

$$\alpha_1 = \frac{3}{8}\pi(u + 1)^2\frac{\omega}{c}\frac{\rho_{np}}{\rho}. \tag{4.9}$$

As has already been pointed out previously,[1] in the region of very high frequencies,

$$\frac{1}{\omega\tau_{pp}} \ll 3\gamma\left(2\pi\frac{kT}{c}\right)^2\frac{B_3}{B_2},$$

terms cubic in the momentum must be taken into consideration in the expression for the energy ϵ I(1.2) entering into Eqs. I(2.8), I(2.10) and I(2.11). Account of γp^2 gives the following formulas for u_1 and α_1:[9]

$$u_1 = u_{10} - c\frac{\rho_{np}}{\rho}\left\{\frac{3}{4}(u + 1)^2\ln\left[\frac{3}{2}\gamma\left(2\pi\frac{kT}{c}\right)^2\frac{B_3}{B_2}\right] + 3u + 2\right\}, \tag{4.10}$$

$$\alpha_1 = \frac{3}{4}(u + 1)^2\frac{\rho_{np}}{\rho}\left[3\gamma\left(2\pi\frac{kT}{c}\right)^2\frac{B_3}{B_2}c\tau_{pp}\right]^{-1}. \tag{4.11}$$

5. THE KINETIC COEFFICIENTS

In the previous sections on the calculation of the absorption coefficients of first and second sound at low frequencies, we have obtained certain linear combinations of the kinetic coefficients. In this section, we shall concern ourselves with deriving formulas for each of these coefficients separately.

Equations I(2.28)–I(2.31), I(2.33) and I(2.34), which described the propagation of sound in helium II in the region of low frequencies[10] have the form

$$-\frac{\omega}{k}\rho' + j_p - ikc^2\rho_{np}\tau_{pr}\left(1 - \frac{TS}{\rho_n c^2}\right)\frac{T_r'}{T} = 0,$$

$$-\frac{\omega}{k}j_r + \left(\frac{\partial\mathscr{P}}{\partial\rho}\right)_T\rho' + \left(\frac{\partial\mathscr{P}}{\partial T}\right)_\rho T_r' - \frac{4}{15}\frac{1}{1 + \tau_{pr}/\tau_{pp}}ikc^2\rho_{np}\tau_{pr}v_{nr} + i\omega c^2\rho_{np}\tau_{pr}\left(1 - \frac{TS}{\rho_n c^2}\right)\frac{T_r'}{T}$$
$$- ikc^2\rho_{np}\tau_{pr}\left\{\frac{1}{\beta\rho}\left(3u + 1 + \delta\frac{C_p}{C}\right)\left(u + \frac{S}{C} + \delta\frac{S_p}{C}\right)\times(j_r - \rho v_{nr}) + \frac{1}{3\beta}\left(3u + 1 + \delta\frac{C_p}{C}\right)\times\left(3u + 1 + \delta\frac{C_p}{C}\right)v_{nr}\right\} = 0;$$

$$-\frac{\omega}{k}v_s + \left(\frac{\partial\mu}{\partial\rho}\right)_T\rho' + \left(\frac{\partial\mu}{\partial T}\right)_\rho T_r' - ikc^2\rho_{np}\tau_{pr}\left\{\frac{3u}{\rho^2\beta}\left(u + \frac{S}{C} + \delta\frac{S_p}{C}\right)(j_r - \rho v_{nr}) + \frac{u}{\beta\rho}\left(3u + 1 + \delta\frac{C_p}{C}\right)v_{nr}\right\} = 0.$$

[8] As we shall see in Sec. 5, all kinetic coefficients in a phonon gas are equal to zero except η_p (4.7).

[9] In the research of Andreev and one of the authors,[6] the case of the high frequency limit was considered for which $\omega t_{pp} \gg 1$.

[10] We recall that the low frequencies for first sound satisfy the conditions ωt_{pr}, $\omega t_{pp} \ll 1$, and for second sound the conditions ωt_{pr}, $\omega t_{pp} \ll u_2/u_1$.

$$-\frac{\omega}{k}\left(\frac{\partial S}{\partial \rho}\right)_T \rho' - \frac{\omega}{k}\left(\frac{\partial S}{\partial T}\right)_\rho T_r' + S v_{nr}$$

$$- ik\tau_{pr} c^2 S_p \left(1 - \frac{TS}{\rho_n c^2}\right)\frac{T_r'}{T}$$

$$+ i\omega\tau_{pr} C_p \left\{\frac{1}{\beta\rho}\left(u + \frac{S}{C} + \delta\frac{S_p}{C}\right)(j_r - \rho v_{nr})\right.$$

$$\left. + \frac{1}{3\beta}\left(3u + 1 + \delta\frac{C_p}{C}\right) v_{nr}\right\} = 0, \quad (5.1)$$

where T_r' and v_{nr} are the temperature and velocity of the normal part of the roton gas, while $j_r = |\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_{nr}|$. The values of ρ' and $\mathbf{v}_s$, as has already been pointed out in [1], play the role of external conditions for the excitation gas.

We convert the system (5.1) to the form of the hydrodynamic equations. For this purpose, we use the conditions

$$\int \varepsilon (n - n^0)\, d\tau_r + \int E (N - N^0)\, d\tau_r,$$
$$\int \mathbf{p} (n - n^0)\, d\tau_r + \int \mathbf{P} (N - N^0)\, d\tau_r, \quad (5.2)$$

which follow from the requirement that the non-equilibrium functions n and N must lead to the same values of the total energy and total momentum as the equilibrium function n^0 (4.2), and

$$N^0 = \exp\left[-\frac{E - \mathbf{P}(\mathbf{v}_n - \mathbf{v}_s)}{kT}\right]$$

(here the temperature T and the velocity of the normal part of the liquid $\mathbf{v}_n$ refer to the equilibrium state of helium II). Deviations of the equilibrium functions n^0 and N^0 from their values in the motionless liquid, n_0 and N_0, are equal to[11]

$$n^{0\prime} = \frac{\partial n_0}{\partial \varepsilon}\left[\frac{\partial \varepsilon}{\partial \rho}\rho' - \varepsilon\frac{T'}{T} + \cos\theta\, p\,(v_n - v_s)\right],$$

$$N^{0\prime} = \frac{\partial N_0}{\partial E}\left[\frac{\partial E}{\partial \rho}\rho' - E\frac{T'}{T} + \cos\vartheta\, P\,(v_n - v_s)\right]. \quad (5.3)$$

Substituting in (5.2) the values n, N, n^0 and N^0 in the form of sums of the constant equilibrium values n_0, N_0 and the small corrections n' I(2.12), N' I(2.13) and $N^{0\prime}$, $N^{0\prime}$ (5.13), we get

$$\rho_n v_{nr} = \rho_n v_n + ikc^2 \rho_{np}\tau_{pr}\left(1 - \frac{TS}{\rho_n c^2}\right)\frac{T'}{T},$$

$$\left(\frac{\partial S}{\partial T}\right)_\rho T_r' = \left(\frac{\partial S}{\partial T}\right)_\rho T' + ikS_p\tau_{pr}\left\{\frac{3}{\rho\beta}\left(u + \frac{S}{C} + \delta\frac{S_p}{C}\right)\right.$$

$$\left.\times (j_r - \rho v_n) + \frac{1}{\beta}\left(3u + 1 + \delta\frac{C_p}{C}\right)\right\} v_n. \quad (5.4)$$

With the help of Eqs. (5.4), the set of equations (5.1), after small transformations, reduces to the form

$$-\omega\rho' / k + j = 0, \quad (5.5)$$

$$-\frac{\omega}{k} j + \mathcal{P}' = ikc^2\rho_{np}\tau_{pr}\left\{\frac{1}{\beta\rho}\left(3u + 1 + \delta\frac{C_p}{C}\right)\right.$$

$$\times\left(u + \frac{S}{C} + \delta\frac{S_p}{C}\right)(j - \rho v_n)$$

$$\left.\times\left[\frac{4}{15}\frac{1}{1 + \tau_{pr}/\tau_{pp}} + \frac{1}{3\beta}\left(3u + 1 + \delta\frac{C_p}{C}\right)^2\right] v_n\right\}, \quad (5.6)$$

$$-\frac{\omega}{k} v_s + \mu' = ikc^2\rho_{np}\tau_{pr}\left\{\frac{3}{\beta\rho^2}\left(u + \frac{S}{C} + \delta\frac{S_p}{C}\right)^2 (j - \rho v_n)\right.$$

$$\left. + \frac{1}{\beta\rho}\left(3u + 1 + \delta\frac{C_p}{C}\right)\left(u + \frac{S}{C} + \delta\frac{S_p}{C}\right) v_n\right\}, \quad (5.7)$$

$$-\frac{\omega}{k} S' + S v_n = ikc^2 S_p\tau_{pr}\left(1 - \frac{TS}{\rho_n c^2}\right)^2\frac{T'}{T}, \quad (5.8)$$

where S', $\mathcal{P}'$, μ', and also $j = |\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n|$ refer to the equilibrium state of helium II.

The equations written in this form allow us to determine the kinetic coefficients which enter into the hydrodynamic equation[12]

$$\dot{\rho}' + \operatorname{div}\mathbf{j} = 0, \quad (5.9)$$

$$\frac{\partial \mathbf{j}}{\partial t} + \nabla\mathcal{P}' = \frac{4}{3}\eta\Delta\mathbf{v}_n$$

$$+ \nabla\{\zeta_1 \operatorname{div}(\mathbf{j} - \rho\mathbf{v}_n) + \zeta_2 \operatorname{div}\mathbf{v}_n\}, \quad (5.10)$$

$$\dot{\mathbf{v}}_s + \nabla\mu' = \nabla\{\zeta_3 \operatorname{div}(\mathbf{j} - \rho\mathbf{v}_n) + \zeta_4 \operatorname{div}\mathbf{v}_n\}, \quad (5.11)$$

$$\dot{S}' + S\operatorname{div}\mathbf{v}_n = \varkappa\Delta T' / T. \quad (5.12)$$

Thus, from the comparison of the right side of Eqs. (5.8) and (5.12), and also (5.7) and (5.11), we find the coefficient of the thermal conductivity[13]

$$\varkappa_p = c^2 S_p\tau_{pr}(1 - TS/\rho_n c^2)^2 \quad (5.13)$$

and the coefficient of second viscosity

$$\rho^2\zeta_3 = \frac{3}{\beta}\left(u + \frac{S}{C} + \delta\frac{S_p}{C}\right)^2 c^2\rho_{np}\tau_{pr}, \quad (5.14)$$

$$\rho\zeta_4 = \frac{1}{\beta}\left(3u + 1 + \delta\frac{C_p}{C}\right)\left(u + \frac{S}{C} + \delta\frac{S_p}{C}\right) c^2\rho_{np}\tau_{pp}, \quad (5.15)$$

In order to distinguish the coefficient of ordinary viscosity η_p from the coefficient of second viscosity ζ_2 in (5.6), we note that the second vis-

[11] The index 0 is omitted for the constant equilibrium values of the thermodynamic quantities.

[12] The set of equations (5.9) – (5.12) is written in the linear approximation.

[13] Naturally, in this way we find only the phonon part of the coefficient of thermal conductivity and the phonon part of the coefficient of ordinary viscosity.

cosity (as has already been said) is due to the comparatively slow processes of establishing the energy equilibrium between the phonon and roton gases; therefore, in Eq. (5.6), the parameter β should contain only the coefficients of second viscosity ζ_1 and ζ_2. Taking this into account, we get from (5.6) and (5.10)

$$\eta_p = \frac{1}{5}\frac{1}{1+\tau_{pr}/\tau_{pp}} c^2\rho_{np}\tau_{pr}, \qquad (5.16)$$

$$\rho\zeta_1 = \frac{1}{\beta}\left(3u+1+\delta\frac{C_p}{C}\right)\left(u+\frac{S}{C}+\delta\frac{S_p}{C}\right)c^2\rho_{np}\tau_{pr}, \qquad (5.17)$$

$$\zeta_2 = \frac{1}{3\beta}\left(3u+1+\delta\frac{C_p}{C}\right)^2 c^2\rho_{np}\tau_{pr}. \qquad (5.18)$$

The quantities ζ_1 and ζ_2 are thus seen to be equal, as should be expected from Onsager's symmetry principle for kinetic coefficients. In the approximations that we have considered, there ex ists another relation between the coefficients o. second viscosity:

$$\zeta_1^2 = \zeta_2\zeta_3. \qquad (5.19)$$

While $\zeta_1 = \zeta_2$ is a rigorous equation which follows from Onsager's symmetry principle, the resultant equation (5.19) follows from the fact that we have taken into account in the theory only slow processes of establishing equilibrium, which take place in the phonon gas. A curious situation is obtained, viz., if

$$\operatorname{div}\mathbf{v}_n = -\frac{3(u+S/C+\delta S_p/C)}{3u+1+\delta C_p/C}\operatorname{div}(\mathbf{j}-\rho\mathbf{v}_n),$$

then such a motion is not accompanied by dissipation of energy. In a certain sense, this situation is reminiscent of the situation in a monatomic gas, when a second viscosity is strictly equal to zero.

The formulas obtained above for the kinetic coefficients are valid in the region of temperatures from 0.6 to 0.9° K. In the temperature region from 0.9 to 1.3° K, as has already been pointed out, the terms containing the collision integral $J_{pp}(n)$ and the derivative $(\partial\rho/\partial T)_{\mathcal{P}}$ ($\sim\delta$) must be omitted in the equations which describe the propagation of sound in helium II. Taking this into account, it is easy to obtain the result that

$$\eta_p = {}^1/_5 c^2\rho_{np}\tau_{pr},$$

$$\rho\zeta_1 = \rho\zeta_4 = \frac{1}{\beta}(3u+1)\left(u+\frac{S}{C}\right)c^2\rho_{np}\tau_{pr},$$

$$\rho^2\zeta_3 = \frac{3}{\beta}\left(u+\frac{S}{C}\right)^2 c^2\rho_{np}\tau_{pr},$$

$$\zeta_2 = \frac{1}{3\beta}(u+1)^2 c^2\rho_{np}\tau_{pr}, \quad \zeta_1^2 = \zeta_2\zeta_3,$$

and κ_p is defined as before by Eq. (5.13).

In a purely phonon gas (T < 0.6° K), Eqs. (4.1), which describe the propagation of sound, are transformed by means of (5.2) ($\nu_0 = -T'/T$, $\nu_1 = -w/c$) in the region of low frequencies to the form

$$-\frac{\omega}{k}\rho'+j=0, \quad -\frac{\omega}{k}j+\mathcal{P}' = ik\frac{4}{15}c^2\rho_{np}\tau_{pp}v_n,$$

$$-\frac{\omega}{k}v_s+\mu'=0, \quad -\frac{\omega}{k}S'+Sv_n=0,$$

whence it is evident that for T < 0.6° K, all the kinetic coefficients with the exception of $\eta_p = {}^1/_5 c^2\rho_{np}\tau_{pp}$ are equal to zero.[14)]

In the region of relatively high temperatures, above 1.2° K, as has already been said, in addition to the scattering of phonons by rotons one should also take into account a five-phonon process. Account of these two slow scattering processes gives the following connection between the equilibrium and roton values of the temperature and the relative velocity:

$$\rho_n w_p = \rho_n w + ikc^2\rho_{np}\tilde{\tau}_{pr}\left(1-\frac{TS}{\rho_n c^2}\right)\frac{T'}{T},$$

$$\left(\frac{\partial S}{\partial T}\right)_\rho T_r' = \left(\frac{\partial S}{\partial T}\right)_\rho T' + ikS_p\tilde{\tau}_{pr}\left\{\frac{3}{\tilde{\beta}\rho}\left(u+\frac{S}{C}\right)(j-\rho v_n) + \frac{1}{\tilde{\beta}}(3u+1)v_n\right\}. \qquad (5.20)$$

Here $\tilde{\tau}_{pr}$ and $\tilde{\beta}$ are determined by Eqs. (3.10). With the help of (5.20), Eqs. (3.8) and (3.9) can be transformed to the form (5.9)–(5.12), and it is easy to obtain the result that

$$\varkappa_p = c^2 S_p\tilde{\tau}_{pr}\left(1-\frac{TS}{\rho_n c^2}\right)^2, \quad \eta_p = \frac{1}{5}c^2\rho_{np}\tilde{\tau}_{pr},$$

$$\rho\zeta_1 = \rho\zeta_4 = \frac{1}{\tilde{\beta}}(3u+1)\left(u+\frac{S}{C}\right)c^2\rho_{np}\tilde{\tau}_{pr},$$

$$\zeta_2 = \frac{1}{3\tilde{\beta}}(3u+1)^2 c^2\rho_{np}\tilde{\tau}_{pr},$$

$$\rho^2\zeta_3 = \frac{3}{\tilde{\beta}}\left(u+\frac{S}{C}\right)^2 c^2\rho_{np}\tilde{\tau}_{pr}, \quad \zeta_1^2 = \zeta_2\zeta_3. \qquad (5.21)$$

Substitution of numerical values of the parameter shows that one can neglect the term $TS/\rho_n c^2$ in the coefficient for thermal conductivity for all temperatures above 0.8° K.

The formulas obtained above for the phonon part of the coefficients of ordinary viscosity and

14)The fact that the coefficients of thermal conductivity and second viscosity are equal to zero in a purely phonon gas also follows from Eqs. (5.13) – (5.18), inasmuch as the quantity $\delta = -3u-1$ for T < 0.6°K.

thermal conductivity in all the regions of temperature are identical with those which were calculated earlier.[2,7]

6. COMPARISON WITH EXPERIMENT

The calculation of the coefficient of absorption of first sound has been carried out by us for the frequency 14.4 Mc/sec and for the temperature range from 0.6 to 1.6° K. For temperatures above 0.7° K, the absorption coefficient α_1 was computed from Eqs. (1.16) and (1.14) for $T \le 0.9$° K, from (1.7) and (1.5) for $T = 0.9-1.2$° K, and from (3.11) for $T \ge 1.2$° K. For the temperatures 0.6–0.7° K, and for the frequency chosen by us, one of the conditions I(1.18) is violated, namely $\omega\tau_{rr} \ll 1$. However, for the frequency 14.4 Mc/sec, for the temperatures mentioned, as is seen from Eq. (1.13) ($\omega\tau_{pr}$, $\omega\tau_{pp} \ll 1$), the rotons do not in practice make a contribution to the absorption of first sound and α_1 can be computed from Eq. (4.9) for the phonon gas. The temperature dependence of the absorption and coefficient of first sound computed in this fashion is represented by this continuous curve in Fig. 2. For comparison, the experimental data of Atkins and Chase[5] are indicated in the same drawing by the circles. The values of α_1 measured by Atkins and Chase agreed excellently with the theoretical values in the range of temperatures from 0.6 to 1.6° K.

The temperature part of the velocity of first sound $u_1(T) - u_1(0)$ ($u_1(0)$ is the velocity of first sound at absolute zero, $u_1(0) = c$) was computed by us for the frequency 1 Mc/sec and for the most interesting region of temperatures for this frequency, namely, from 0.5 to 0.8° K. In accord with (1.15) and (4.5),

$$u_1(T) = u_{10}(T) - \frac{1}{2}u_1(0)\,\frac{\rho_{np}}{\rho}\,\mathrm{Re}\,\varphi,$$

where the function φ is determined by Eqs. (1.14)

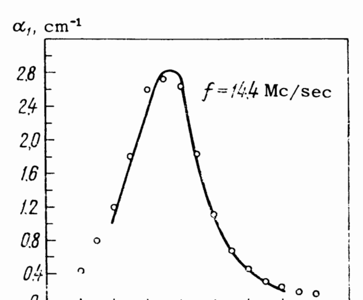

FIG. 2. Absorption coefficient of first sound in helium II: the continuous curve gives the theoretical values, the points are the experimental values of Atkins and Chase.

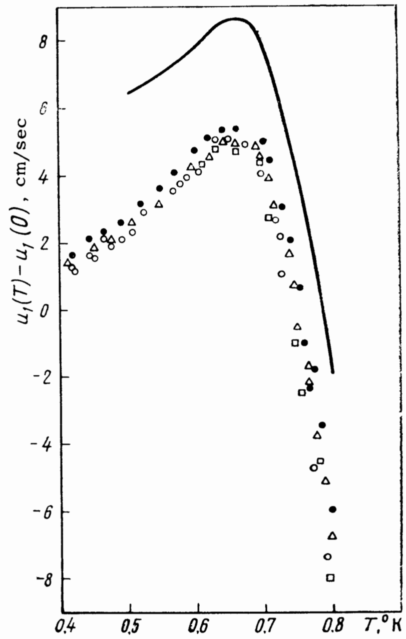

FIG. 3. Velocity of first sound in helium II: the continuous curve gives the theoretical values, the points are the experimental data of Whitney and Chase.[10]

($T \ge 0.6$° K) and (4.3) ($T < 0.6$° K) while $u_{10}(T)$ is equal to

$$u_{10}(T) = \left[\rho\frac{\partial}{\partial\rho}\left(\mu_0 + \int\frac{\partial\varepsilon}{\partial\rho}n_0 d\tau_r + \int\frac{\partial E}{\partial\rho}N_0\,d\tau_r\right)\right]_T^{1/2}. \quad (6.1)$$

The third term on the right side of (4.1), which takes into account the contribution of rotons in the temperature range from 0.5 to 0.8° K is unimportant in comparison with the first two terms and can be neglected. Taking this into consideration and carrying out the integration over all p and differentiating with respect to the density, we obtain[15)]

$$u_{10}(T) = u_1(0)\left[1 - \frac{3}{2}\left(u^2 - \frac{1}{4}\frac{\rho^2}{c}\frac{\partial^2 c}{\partial\rho^2}\right)\right].$$

Thus the sought temperature differences are finally equal to

$$u_1(T) - u_1(0)$$

$$= -\frac{1}{2}u_1(0)\,\frac{\rho_{np}}{\rho}\left[\mathrm{Re}\,\varphi + 3\left(u^2 - \frac{1}{4}\frac{\rho^2}{c}\frac{\partial^2 c}{\partial\rho^2}\right)\right]. \quad (6.2)$$

The values of $u_1(T) - u_1(0)$ computed from Eq. (6.2) are represented by the continuous curve in Fig. 3. For comparison, we have plotted the experimental values of $u_1(T) - u_1(0)$ obtained by

15)The value of the second derivative of c with respect to the density as well as the first was computed from the data of Atkins and Stasior[8] and the data of Keesom and Miss Keesom:[9] $(\rho^2/c)(\partial^2 c/\partial\rho^2) = 5.5$.

Whitney and Chase[10] in the same figure. Whitney and Chase measured the velocity $u_1(T)$ directly. The velocity $u_1(0)$ was obtained by them by interpolation of the curve $u_1(T)$ to absolute zero; it is equal to 238.27 ± 0.1 m/sec. Comparing the curve $u_1(T) - u_1(0)$, obtained by Whitney and Chase with that computed by Eq. (6.2), we reached the conclusion that if we select $u_1(0) = 238.23$ m/sec, which does not fall outside the limits of accuracy of the measurement by Whitney and Chase of the value of the velocity $u_1(0)$, then both curves agree very well.

Strictly speaking, for the calculation of the temperature part of the velocity of first sound in the region of temperatures below 0.6° K, it is necessary to make use of the expression (4.10), and not (6.2), inasmuch as, for $T < 0.6°$ K,

$$\frac{1}{\omega\tau_{pp}} \ll 3\gamma\left(2\pi\frac{kT}{c}\right)^2\frac{B_3}{B_2}.$$

For the same reason, the absorption coefficient of first sound at temperatures below 0.7° K must be computed from Eq. (4.11), and not from (4.8). However, the fact that the theoretical values of α_1 and $u_1(T) - u_1(0)$ agree excellently with the experimental values indicates that the value of γ is, apparently, $\ll 3 \times 10^{37}$.

For second sound unfortunately, there is no such quantity of experimental data as for first sound. Experiments on second sound occupy principally the region of low frequencies ($\omega \sim 10^{+4}$ cps), for which the value of the parameter $\omega\tau_{pr}$ is much smaller than unity in a wide range of temperatures. In this region of frequencies, the values of the absorption coefficient of second sound computed from Eqs. (2.9) and (3.12) are in excellent agreement with the experimental data, as is seen in Fig. 4.

At the present time the region of high frequencies has already been achieved for second sound (≈ 25 Mc/sec)[4] and it can be hoped that measurements of the sound velocity and absorption of second sound for high frequencies will be carried out.

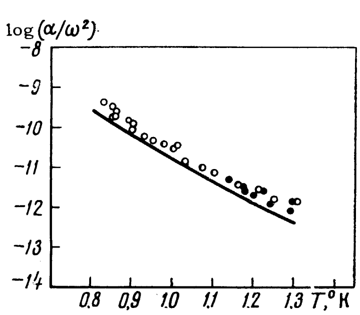

FIG. 4. Absorption coefficient of second sound in helium II: o – experimental values of Zinov'eva[11] ● – Atkins and Hart[12], the continuous curve gives the theoretical values.

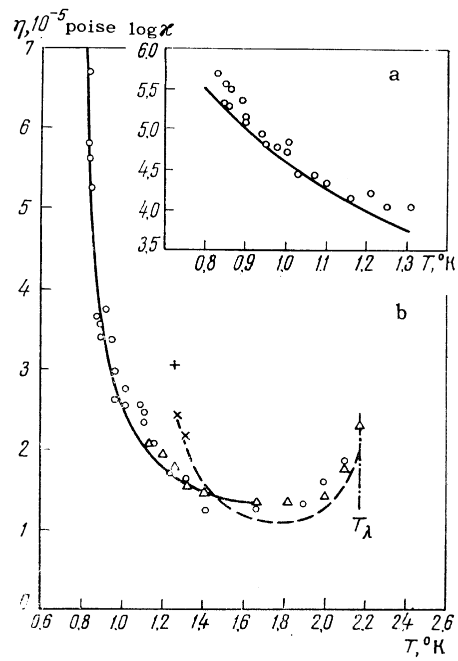

FIG. 5. a – dependence of the coefficient of thermal conductivity κ (erg/cm–sec–deg) in helium II on the temperature: o – experimental data of Zinov'eva,[11] continuous curve – theoretical values; b – temperature dependence of the coefficient of first viscosity in helium II: o – data of Zinov'eva,[11] Λ – Heikkila and Hollis-Hallet,[13] curve – Andronikashvili,[14] continuous curve – theoretical values, + – deFroyer and Van Itterbeek[15].

APPENDIX

In the preceding sections we have used for the calculation of the velocities and absorption coefficients of first and second sound the values of the parameters Δ, P_0, μ, $\partial\Delta/\partial\rho$, $\partial^2\Delta/\partial\rho^2$, $\partial P_0/\partial\rho$, given in [1]. It is of interest to compare with the experimental data the values of the coefficients of first viscosity η and thermal conductivity κ computed with account of these new values of the parameters. According to I(1.12),

$$\eta = \eta_r + \eta_p, \quad \varkappa = \varkappa_r + \varkappa_p,$$

where η_r and κ_r and the roton parts of the coefficients in η and κ

$$\eta_r = \frac{\hbar^4 P_0}{60\mu^2|V_0|^2}, \quad \varkappa_r = \frac{5\Delta^2}{P_0^2 T}\eta_r.$$

Computation of κ_p was carried out by us from Eqs. (5.13) and (5.21). In the calculation of η_p here we used an exact value rather than the value of the cross section σ_{pr} averaged over the angles of the incident and scattered phonons and rotons (see [7]). We also want to draw attention again to the fact that in the calculation of α_1 and α_2, substitution for the exact value of σ_{pr} in the kinetic equations by the average (over the angles) value did not have a significant effect on the accuracy of the results, inasmuch as such a substitution affects only the very small term containing η_p. The values of the coefficients of first viscosity and thermal conductivity that have been computed agree with the experiments in [11-15] within the limits of accuracy of the experiment (Fig. 5).[16)]

16)The calculation of α_1, α_2, η, and κ has been carried out by us for temperatures below 1.6°K, inasmuch as for T > 1.6°K, it is necessary to take into account the nonideal nature of the roton gas.

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