BEHAVIOR OF ORDERED SYSTEMS NEAR THE TRANSITION POINT

A. Z. PATASHINSKIĬ and V. L. POKROVSKIĬ

Semiconductor Physics Institute, Siberian Division, Academy of Sciences, U.S.S.R.

Submitted to JETP editor August 2, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) 50, 439–447 (February, 1966)

The form of the variation of the free energy with temperature and external field near the transition point is derived by estimating the correlation functions of a partially ordered system. The behavior of various thermodynamic quantities in the transition region is related to the binary correlation function. In particular, the magnetic susceptibility in a strong magnetic field is determined; for a plane Ising model it behaves as $H^{-14/15}$. The results of physical and mathematical experiments are discussed.

CONSIDER a system which is capable of ordering, and which is characterized by a conserved additive quantity (moment)

$$M = \sum_{\mathbf{r}} m(\mathbf{r}),$$

where $\mathbf{r}$ labels points in configuration space. Examples of such systems are ferromagnets, ferroelectrics etc. In the following we shall use the terminology appropriate to the magnetic systems. In the absence of an external field H these systems may, at a certain temperature T_c, undergo a transition to an ordered state in which $\langle M \rangle \neq 0$ (spontaneous magnetization). In the neighborhood of T_c one finds a number of characteristic phenomena: the correlation radius r_c of the quantities $m(\mathbf{r})$ increases, the magnetic susceptibility $\chi = \partial\langle M\rangle/\partial H$ increases, and the thermodynamic quantities have singularities at $T = T_c$, $H = 0$. All these phenomena are interconnected. The object of the present paper is to determine the quantitative relations between these variables.

The only known example for which rigorous results at $H = 0$ have been obtained is the plane Ising model.[1] When $H \neq 0$, no rigorous solution is available even for this system. For the three-dimensional Ising model there exist numerical calculation (see e.g. [2]) of χ, $\langle M\rangle$, and the specific heat C_H. About the Heisenberg model much less is known: numerical calculations have been done only for χ.[2] All these investigations are also limited to $H = 0$.

In this paper it will be shown that the singularities of all thermodynamic quantities are determined by two parameters, which determine the behavior of the correlation functions near the critical point in the absence of a magnetic field. These arguments will lead, in particular, to relations first obtained by Widom[3] and by Essam and Fisher.[4]

1. EXPANSION OF THE FREE ENERGY IN POWERS OF THE MAGNETIC FIELD

The Hamiltonian $\mathcal{H}$ of a magnetic system in the external magnetic field H has the form

$$\mathcal{H} = \mathcal{H}_0 - MH, \tag{1.1}$$

where $\mathcal{H}_0$ is the Hamiltonian of the system without the magnetic field. We shall measure the magnetic field in units of H_0, the molecular field at saturation. The n-th order correlation function $Q(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$ will be defined in terms of the mean products of the local spins,

$$K_n(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) = \langle m(\mathbf{r}_1) m(\mathbf{r}_2) \ldots m(\mathbf{r}_n)\rangle$$

by the relations $(T > T_c,\ H = 0)$

$$\begin{aligned}
&Q_2(\mathbf{r}_1, \mathbf{r}_2) = K_2(\mathbf{r}_1, \mathbf{r}_2),\\
&K_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = Q_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) + Q_2(\mathbf{r}_1, \mathbf{r}_2) Q_2(\mathbf{r}_3, \mathbf{r}_4)\\
&\quad + Q_2(\mathbf{r}_1, \mathbf{r}_3) Q_2(\mathbf{r}_2, \mathbf{r}_4) + Q_2(\mathbf{r}_1, \mathbf{r}_4) Q_2(\mathbf{r}_2, \mathbf{r}_3),\\
&K_6(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_6) = Q_6(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_6) + Q_2(\mathbf{r}_1, \mathbf{r}_2) Q_4(\mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_5, \mathbf{r}_6)\\
&\quad + Q_2(\mathbf{r}_1, \mathbf{r}_3) Q_4(\mathbf{r}_2, \mathbf{r}_4, \mathbf{r}_5, \mathbf{r}_6) + Q_2(\mathbf{r}_1, \mathbf{r}_4) Q_4(\mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_5, \mathbf{r}_6)\\
&\quad \ldots + Q_2(\mathbf{r}_1, r_2) Q_2(\mathbf{r}_3, \mathbf{r}_4) Q_2(\mathbf{r}_5, \mathbf{r}_6)\\
&\quad + Q_2(\mathbf{r}_1, \mathbf{r}_3) Q_2(\mathbf{r}_2, \mathbf{r}_4) Q_2(\mathbf{r}_5, \mathbf{r}_6)\\
&\quad \cdots\cdots\cdots
\end{aligned} \tag{1.2}$$

There is a simple relation which determines the free energy, F(H), for $H \neq 0$, from F(0) and from the correlation functions:

$$F(H) = F(0) + kT \sum_{n=1}^{\infty} \frac{H^{2n}\Gamma_{2n}}{(2n)!}, \tag{1.3}$$

$$\Gamma_{2n} = \sum_{\mathbf{r}_j} Q_{2n}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{2n}). \tag{1.4}$$

By definition any $Q_{2n}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{2n})$ tends to zero if at least one of its arguments is moved far away from the others. We shall assume that all the Q_{2n} decrease sufficiently rapidly with increasing distance. Each Γ_{2n} is then proportional to the volume V of the system.

Below the transition point there is a spontaneous magnetization and therefore the average product K_{2n+1} of an odd number of spins is non-zero; consequently the definition (1.2) and the relations (1.3) and (1.4) are modified somewhat:

$$K_1 = Q_1 = \langle M \rangle / V, \; K_2(\mathbf{r}_1, \mathbf{r}_2) = Q_2(\mathbf{r}_1, \mathbf{r}_2) + Q_1(\mathbf{r}_1) Q_1(\mathbf{r}_2), \tag{1.2$'$}$$

$$K_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = Q_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + Q_1(\mathbf{r}_1) Q_2(\mathbf{r}_2, \mathbf{r}_3) + Q_1(\mathbf{r}_2) Q_2(\mathbf{r}_1, \mathbf{r}_3) + Q_1(\mathbf{r}_3) Q_2(\mathbf{r}_1, \mathbf{r}_2) + Q_1(\mathbf{r}_1) Q_1(\mathbf{r}_2) Q_1(\mathbf{r}_3) \text{ etc.}$$

$$F_H = F(H) - F(0) = kT \sum_{n=1}^{\infty} \frac{H^n \Gamma_n}{n!}, \tag{1.3$'$}$$

$$\Gamma_n = \sum_{\mathbf{r}_j} Q_n(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n). \tag{1.4$'$}$$

The equations (1.2′), (1.3′), and (1.4′) can also be looked at from a different point of view. Consider the system in a given non-zero field H_0 and add to it another field H. Then the equations (1.2) to (1.4) remain valid if we take the averages in (1.2) to mean averages in the given field H_0, and $F_H = F(H_0 + H) - F(H_0)$.

2. CORRELATION FUNCTIONS IN THE NEIGHBORHOOD OF THE TRANSITION POINT

Assume that, as $T \to T_c$, for $H = 0$, the correlation radius r_c increases as some negative power of $\tau = (T - T_c)/T_c$:

$$r_c(\tau) \sim \tau^{-\alpha}, \tag{2.1}$$

and that the behavior of the correlation function $Q_2(\mathbf{r}_1, \mathbf{r}_2)$ is given by

$$Q_2(0, \mathbf{r}) \sim r^{-\beta}, \quad 1 \ll r \ll r_c \tag{2.2}$$

(the lattice constant is taken as unity).

The physical state of the system above the transition point can be visualized in the following way. In each region of dimension r_c the spins are correlated, and the total moment $\mathfrak{M}$ of the region is non-zero. However, the total moments of different regions of this size are equally likely to have the same or opposite signs, so that the average moment of the whole system vanishes.

We estimate the Γ_{2n} defined in (1.4) in the following manner. We break the sum over the $\mathbf{r}_j$ into parts corresponding to regions of dimensions of the order of r_c (we recall that we saw that only those configurations contribute substantially for which the distances between all the $\mathbf{r}_j$ do not exceed r_c). The summation over each such region gives a quantity of the order of $\mathfrak{M}^{2n}$. The number of regions is proportional to V/r_c^a (a is the number of space dimensions). This leads for Γ_{2n} to the estimate

$$\Gamma_{2n} \sim \mathfrak{M}^{2n} V / r_c^a. \tag{2.3}$$

In particular, for Γ_2

$$\Gamma_2 \sim \mathfrak{M}^2 V / r_c^a. \tag{2.4}$$

On the other hand, (2.2) gives the asymptotic estimate[1)]

$$\Gamma_2 \sim V r_c^{a-\beta}. \tag{2.5}$$

A comparison of (2.4) with (2.5) shows that

$$\mathfrak{M}^2 \sim r_c^{2a-\beta}. \tag{2.6}$$

This gives for Γ_{2n}

$$\Gamma_{2n} \sim V r_c^{(2a-\beta)n-a} \sim V \tau^{-\alpha[(2a-\beta)n-a]}. \tag{2.7}$$

Now assume that the system has a non-vanishing total moment $\langle M \rangle$, either because there is an external magnetic field, or because the temperature is below the transition point. This means that the function

$$K_2(0, \mathbf{r}) = \langle m(0) m(\mathbf{r}) \rangle$$

tends as $r \to \infty$ to the finite limit

$$\lim_{r \to \infty} K_2(0, \mathbf{r}) = m^2 = \langle M \rangle^2 / V^2.$$

The influence of the external magnetic field clearly becomes important at such distances $r \ll r_c$ for which $K_2 \sim r^{-\beta}$ is of the order of m^2. This relation defines a "magnetic correlation length:"

$$r_c(H) \sim m^{-2/\beta}.$$

The magnetic field has to be regarded as weak or as strong, according to whether the quantity $\mathfrak{M}H$ is less or greater than unity. In a weak magnetic field the magnetic correlation length $r_c(H)$ exceeds the "thermal" length $r_c(\tau)$. In this case the estimates for Γ_{2n} evidently remain unchanged. The quantities Γ_{2n+1} will now acquire non-zero values of the order $H\mathfrak{M}^{2n+1}V/r_c^a$. In strong mag-

1)An estimate of this kind was used first by M.E. Fisher[1] to calculate the magnetic susceptibility of the plane Ising model.

netic fields $r_c(H) \ll r_c(\tau)$. In that case all Γ_n are estimated by the single expression

$$\Gamma_n \sim Vm^n [r_c(H)]^{a(n-1)} \sim Vm^{n-2a(n-1)/\beta}. \quad (2.8)$$

3. THERMODYNAMICS OF AN ORDERED TRANSITION IN AN EXTERNAL FIELD NEAR THE TRANSITION POINT

Consider Eq. (1.3) for the free energy of the system in a magnetic field. By summing the series (1.3) we obtain an expression of the form

$$F(H) = F(0) + kT_c V\tau^{\alpha a} f(H^2 / \tau^{(2a-\beta)\alpha}), \quad (3.1)$$

where f(x) is some dimensionless function. For small x it can be expanded in a series of integral powers of its argument. We shall prove that the series for f(x) has a finite convergence radius x_0.

It follows indeed from a theorem by Lee and Yang[6] that the free energy F is an analytic function of $z = e^{H/T}$, with a cut along the circle $|z| = 1$, except for an arc from $z = e^{-i\lambda}$ to $z = e^{i\lambda}$ through the point $z = 1$. As $T \to T_c$ from above ($\tau \to 0$, $\tau > 0$) the quantity λ tends to zero. Thus F is, for positive τ, an analytic function of $z - 1$ in a circle of radius $|e^{i\lambda} - 1|$. For small τ we may take the radius of convergence approximately equal to λ and replace $z - 1$ by H/T_c. To the same approximation we may take the cut in the H plane from $\pm i\lambda T_c$ to $\pm i\infty$. It follows that $F(\tau, H)$ is, for positive τ, an analytic function of H in the region $|H| < \lambda T_c$, and the same applies to the function $f(H^2/\tau^\nu)$ and its representation by the series (1.3). Since f(x) is a function only of its argument x, the radius of convergence of its expansion must be a dimensionless number of the order of unity. Hence the convergence radius of $F(\tau, H)$ as a function of H is $x_0^{1/2}\tau^{\nu/2}$. The function f(x) can be continued analytically beyond the convergence limit. This analytic continuation gives the behavior of $F(\tau, H)$ for $\tau > 0$ and $H \gg \tau^\nu$. From the second derivative of F(H) with respect to H we obtain the magnetic susceptibility in a weak magnetic field:

$$\chi \sim \tau^{-\alpha(a-\beta)}. \quad (3.2)$$

As $\tau \to 0$ the quantity F(H) must tend to a finite limit. This is possible only if $f(x) \sim x^{a/(2a-\beta)}$ as $x \to \infty$. Then

$$F(H) - F(0) \sim H^{2a/(2a-\beta)}. \quad (3.3)$$

This gives us the magnetization and the susceptibility in a strong magnetic field ($H \gg \tau^{(a-1/2\beta)\alpha}$):

$$M = -\partial F / \partial H \sim H^{\beta/(2a-\beta)}, \quad \chi \sim H^{2(a-\beta)/(2a-\beta)}. \quad (3.4)$$

The application of an external magnetic field leads to the appearance of long-range order, and, consequently, to a non-zero total moment, at any temperature. In this respect the system under consideration differs from a superconductor, in which the application of an external magnetic field destroys the ordered superconducting currents, and therefore intensifies the phase transition. In the systems we are considering the phase transition disappears when $H \neq 0$, since the only feature which can distinguish the two phases is the presence or absence of long-range order.

The assumption that there is no phase transition when $H \neq 0$ means that the free energy F(H) can be expanded in a series of integral powers of τ. This means, in particular, that the next term in the asymptotic expansion of f(x), following the one we have determined, must be proportional to $x^{(\alpha a-1)/(2a-\beta)\alpha}$. The corresponding term in F_H can, apart from a constant factor, be written as $\tau H^{2(a\alpha-1)/(2a-\beta)\alpha}$. The asymptotic form of f(x) must contain a term $f_0(x)$ of degree zero in x. The corresponding term in $F_H = F(H) - F(0)$, which is proportional to $\tau^{\alpha a} f_0(H^2/\tau^{(2a-\beta)\alpha})$, must cancel the singularity in F(0). This means that the quantity αa is related to the nature of the singularity of the specific heat C_H for $H = 0$ by

$$\alpha a = \mu + 2, \quad (3.5)$$

if $C_H \sim \tau^\mu$ as $\tau \to 0$. For $\mu = 0$ the singularity of the specific heat is logarithmic (unless there is a jump). In that case $f(x) \sim \ln|x|$. This is, indeed, the only case in which $f_0(x)$ can be represented as a sum of functions of τ and of H, respectively. If $\mu \neq 0$ (in that case μ is necessarily not an integer) $f_0(x)$ is a constant.

Consider now the behavior of $F(H, \tau)$ for negative τ. For this purpose we make use of the conclusions reached above about the analyticity of $F(H, \tau)$ as a function of τ for finite values of H. These imply that the function $\tau^{\alpha a}[f(H^2/\tau^\nu) - f_0]$ is an analytic function of τ for sufficiently small τ and finite H. The change from positive to negative values of τ ($|\tau|^\nu \gg H^2$) amounts to a rotation in the x plane through an angle of $\pm\pi\nu$ with $|x| \gg 1$. But it has been shown that the points $\pm ix_0$ are the only singularities of the function f(x). In the rotation by an angle of $\pm\pi\nu$ from the positive real x axis one must necessarily cross the cut. If we continue the function f(x) on the second sheet into the region of small x, at fixed arg x, we reach the region in which $\tau < 0$ and $H \ll \tau^\nu$. We see that changing the sign of τ at small H in (3.1) necessarily brings us from one branch of the many-valued function f(x) to another, which is obtained from the first by the procedure described above. We denote this branch by $\varphi(x)$. While we know that the first branch f(x) is analytic for $|x| < x_0$, we cannot assert this about $\varphi(x)$. On the contrary, it follows from the expansion (1.3) that this function can be represented as a power series in $\sqrt{x}$, and that the coefficient of $\sqrt{x}$ does not vanish, a fact which is related to the appearance of a spontaneous moment for $\tau < 0$. The magnitude of the spontaneous moment M_0 is found from the relation

$$M_s = -\left|\frac{\partial F}{\partial H}\right|_{H=0} = Vl\tau^{\alpha a-\gamma/2}, \quad l = -2\lim_{x\to 0}\sqrt{x}\,\varphi(x)kT_c.$$

In the presence of a non-zero magnetic field, the magnetic moment is, for $\tau < 0$, given by the relation

$$M = -\frac{\partial F}{\partial H} = -2kT_cV\tau^{-\alpha(a-\beta)}H\varphi'\left(\frac{H^2}{\tau^{(2a-\beta)\alpha}}\right). \quad (3.6)$$

By solving this equation for H we find

$$H = \tau^{(a-\beta/2)\alpha}g(m\tau^{-\alpha\beta/2}), \qquad m = M/V, \quad (3.7)$$

where $g^2(x)$ is the inverse function to $\sqrt{x}\,\varphi'(x)$.

The spontaneous magnetization m_s can be found as the root of the equation $H(m_s) = 0$. m_s is obviously connected with the root x_1 of the equation $g(x_1) = 0$ by the relation

$$m_s = x_1\tau^{\alpha\beta/2}. \quad (3.8)$$

It may be assumed that g(x) has no positive real roots. The nonvanishing roots of g(x) lie on the line $\arg x_1 = -\frac{1}{2}\pi\alpha\beta$. Note that g(x) is an odd function, so that for any root x_1 there always is another at $-x_1$.

The the behavior of all thermodynamic quantities is determined by two parameters, for which we may choose, for example, α and β.

For the value of the moment m in the case of a strong field,

$$\Phi(m) = F - H\partial F/\partial H = \Phi_{\text{reg}}(0) + Am^{2a/\beta} + B\tau m^{2(a\alpha-1)/\alpha\beta} + \ldots, \quad (3.9)$$

where $\Phi_{\text{reg}}(0)$ is the non-singular part of $\Phi(0)$. Equation (3.8) has the same form as the Landau[7] expansion in terms of the order parameter $\eta = m^{\alpha/\beta}$, but the exponents of the different terms in (3.9) have the relation suggested by Landau only if $\alpha a = 2$. If we use two terms of the series (3.9) to determine the spontaneous moment, we obtain the correct temperature dependence of $m_s(\tau)$, but for $m \approx m_s(\tau)$ all terms in the series (3.9) are of the same order, and one must not use a finite number of terms. Another difference from the Landau theory is that the term of order τ^2 in the expansion (3.9) may depend logarithmically on the moment.

We next estimate the range of validity of (3.1). Its derivation was based on the method of summing the main terms in the series, neglecting in Γ_{2n} terms of the relative order τ. Their inclusion leads to an expression of the following form:

$$F(H) = F(0) + kT_cV\tau^{\alpha a}[f(x) + \tau f_1(x) + \tau^2 f_2(x) + \cdots] + AkT_cV\Psi(H), \quad (3.10)$$

where $f_1(x)$ and $f_2(x)$ are some functions of the argument $x = H^2/\tau^{(2a-\beta)\alpha}$. The last term on the right-hand side of (3.10) represents the contribution from short-range correlations which do not depend on τ. We note that $\Psi(H) \sim H^2$ as $H \to 0$. The condition for the validity of (3.1) is that $\tau \ll 1$ (in order to make the terms $\tau f_1(x)$, $\tau^2 f_2(x)$, etc., negligible compared to f(x)), and $H \ll 1$ (to make the non-singular term proportional to $\Psi(H)$ negligible).

From the equations of this section, a straightforward calculation gives the relations between the order of the singularities in the specific heat and in the weak-field susceptibility and the behavior of the spontaneous magnetization. They are the same as the relations found by Essam and Fisher,[4] who analyzed similar expansions under certain assumptions. Another relation connecting the behavior of the strong-field and weak-field susceptibility with that of the spontaneous magnetization was found by Widom.[3] The latter considered a lattice gas model and assumed that the critical isochore had a finite slope.

4. COMPARISON WITH THE RESULTS OF CALCULATIONS AND EXPERIMENTS

We choose the two-dimensional Ising model as an example. Here we use the values $\alpha = 1$, $\beta = \frac{1}{4}$, known from the rigorous solutions given by Onsager, Kaufman, and Yang. With these values we find from (3.5) the result $\mu = 0$ for the singularity in the specific heat at $\tau \to 0$ (logarithmic divergence), in agreement with the classical result of Onsager.

We also find from (3.8) that the spontaneous moment varies as $|\tau|^{1/8}$ (a result first obtained by Onsager and confirmed by Yang). (3.2) gives the behavior of the magnetic susceptibility for $\tau > 0$, $H \ll \tau^{15/8}$: $\chi \sim \tau^{-7/4}$ (this result was found by Fisher[5]).

In a strong magnetic field, $H \gg \tau^{15/8}$, the magnetic susceptibility has the form[3]

$$\chi \sim H^{-14/15}. \quad (4.1)$$

For large moments ($m \gg \tau^{1/8}$) the thermodynamic potential $\Phi = F - H\partial F/\partial H$ can be expanded in powers of τ. This expansion has a characteristic form, which was given by Landau[7] in his theory of phase transitions, and here the parameter η is the quantity $\eta = m^4$:

$$\Phi = \Phi_0 + A\tau\eta^2 + B\eta^4 + \ldots . \quad (4.2)$$

However, as we have already pointed out, the neglected terms in the expansion are not small near the minimum of Φ.

For the three-dimensional Ising model numerical calculations (cf. e.g. [2]) show that

$$m \sim |\tau|^{5/16} \quad (\tau < 0), \qquad \chi \sim \tau^{-5/4} \quad (\tau > 0).$$

The result for the singularity of the specific heat, which in our opinion is less reliable, is

$$C \sim \ln|\tau| \quad (\tau < 0), \qquad C \sim \tau^{-1/5} \quad (\tau > 0).$$

Since $C \sim \tau^{\mu}$ this behavior would suggest that $\mu = 0$ for $\tau < 0$, and $\mu = -1/5$ for $\tau > 0$. This variation of μ seems to us hard to understand, since, according to Sec. 3, $F(H) - F(0)$ behaves as

$$H^{(\mu+2)/(\mu+2-\gamma)} + \text{const}\cdot\tau H(\mu+1)/(\mu+2-\gamma),$$

where $\gamma = 1/2\,\alpha\beta$, and this expression should not depend on the sign of τ. The change in μ would mean that the phase transition persists for $H \neq 0$ without change in T_C.

If we accept the data on m and χ as evidence, we find the values $\alpha = 5/8$, $\beta = 1$, $\mu = -1/8$. The non-linear magnetic susceptibility varies as $H^{-4/5}$. We point out that the resulting value $\mu = -1/8$ does not agree with either of the values obtained from the numerical work.

It seems to us that the results indicate a bad convergence of the methods of approximation in either of the regions.

In the Heisenberg model we know of numerical results only for $\chi \sim \tau^{-4/3}$ (cf. [2]), so that we cannot at present apply our theory. Measurements of the specific heat of iron (Kraftmakher[8]) are well described by a logarithmic law ($\mu = 0$). Experiments on the scattering of neutrons in iron (Jacrot et al.[9]) show that the correlation function between the spins decreases, for $\tau = 0$, as r^{-1} ($\beta = 1$). Hence, from our relations $\alpha = 2/3$, $\chi \sim \tau^{-4/3}$ in a weak field, $\chi \sim H^{1/5}$ in a strong field, and $m_S \sim |\tau|^{1/3}$. Unfortunately we do not know of any sufficiently accurate measurements of these quantities for iron.

Measurements of the magnetic susceptibility of nickel near the Curie point were made by Weiss and Forrer,[10] and analyzed in a recent paper by Kouvel and Fisher.[11] Their results agree with the empirical formulae: $\chi \sim \tau^{-1.37}$ (weak field) and $\chi \sim H^{0.237}$ (strong field), which is not in bad agreement with the values given above.

One may hope that the phase transition is in all ferromagnets described by the unique values $\mu = 0$ and $\beta = 1$, and that these values will also be obtained from the Heisenberg model.

5. EQUATION OF STATE AND THERMODYNAMIC FUNCTIONS OF A LATTICE GAS NEAR THE CRITICAL POINT

Using the well-known analogy between the Ising model and the lattice gas (cf. e.g. [12]) we can derive from the results of Sec. 3 an equation of state for the lattice gas in an approximation which is valid near the critical point. It has the form

$$\frac{P - P_c - b\tau}{\tau^{2+\mu-\gamma}} = g(x), \quad \tilde{g}(x) = g(x) - xg'(x),$$
$$x = \frac{\rho - \rho_c}{\tau^{\gamma}}, \quad (5.1)$$

where P is the pressure, ρ the density, P_C and ρ_C their values at the critical point, b is a constant, $\gamma = 1/2\,\alpha\beta$, and the function g(x) is defined by (3.8). For the two-dimensional lattice gas, in particular,

$$\frac{P - P_c - b\tau}{\tau^{15/18}} = \tilde{g}\left(\frac{\rho - \rho_c}{\tau^{1/8}}\right). \quad (5.2)$$

We know the asymptotic behavior of the function $\tilde{g}(x)$ ($\tau > 0$):

$$\tilde{g}(x) \sim x^{(\mu+2-\gamma)/\gamma} \quad \text{for} \quad x \to \infty, \qquad \tilde{g}(x) \sim x^2 \quad \text{for} \quad x \to 0.$$

The equation for the critical isotherm takes the form

$$P - P_c \sim (\rho - \rho_c)^{(2+\mu-\gamma)/\gamma}. \quad (5.3)$$

For the plane lattice gas the exponent of the power of $\rho - \rho_C$ in (5.3) is 15.[3] For $\tau < 0$ the asymptotic behavior of $\tilde{g}(x)$ for $x \to \infty$ remains the same. For $\tau < 0$ the function g(x) vanishes for two opposite and equal non-zero values $\pm x_1$ (see equation (3.9)). The curve on which g(x) vanishes for $\tau < 0$ is the boundary of the two-phase region. Its equation is

$$\rho - \rho_c = \pm x_1\tau^{\gamma} \quad \text{for} \quad P - P_c = b(|\rho - \rho_c|/x_1)^{1/\gamma}. \quad (5.4)$$

This equation is known for the plane case (cf. [11]).

The compressibility $\beta = V^{-1}(\partial V/\partial P)_{T,N}$ is obtained from the thermodynamic identity

$$\beta = -\rho^{-2}(\partial\rho/\partial\mu)_{T,V}. \quad (5.5)$$

The derivative $(\partial\rho/\partial\mu)_{T,V}$ for the lattice gas is identical with the quantity $\frac{1}{4}(\partial m/\partial H)_{T,V}$ for the Ising model. Therefore we obtain from (3.8) and (5.5)

$$\beta = -\left[\tau^{\mu+2-2\gamma} g'\left(\frac{\rho-\rho_c}{\tau^{\gamma}}\right)\right]^{-1}. \qquad (5.6)$$

In the region $|\rho - \rho_c| \gg \tau^{\gamma}$ in particular

$$\beta \sim -|\rho - \rho_c|^{-(\mu+2-2\gamma)/\gamma}. \qquad (5.7)$$

For the plane case the exponent on the right-hand side of (5.7) is -14.

The specific heat $C_{V,N}$ of the lattice gas in the single-phase region coincides with the specific heat $C_{V,m}$ of the Ising model. The calculation gives the following results:

$$C_{V,N} = A(\rho-\rho_c)^{\mu/\gamma} \quad (\text{or } A\gamma^{-1}\ln(\rho-\rho_c) \text{ for } \mu=0)$$

for $|\rho - \rho_c| \gg \tau^{\gamma}$,

$$C_{V,N} = A\tau^{\mu} \quad (\text{or } A\ln\tau \text{ for } \mu=0) \text{ for } |\rho-\rho_c| \ll \tau^{\gamma}. \qquad (5.8)$$

The specific heat $C_{P,N}$ is found by means of the familiar relation

$$C_{P,N} = C_{V,N} - T(\partial P/\partial T)^2_{V,N}(\partial P/\partial V)^{-1}_{T,N}. \qquad (5.9)$$

A study of the quantity $(\partial P/\partial T)_{V,N}$ shows that this may be treated as a constant, say, b, if $1 + \mu - \gamma > 0$. We shall restrict ourselves to this case. Then

$$C_{P,N} - C_{V,N} \approx -Tb^2(\partial V/\partial P)_{T,N}. \qquad (5.10)$$

We are indebted to A. S. Borovik-Romanov, I. Z. Fisher, Yu. M. Kagan, L. Pal, Ya. Gordon and other members of the seminar on the theory of condensed matter at Dubna, and also to G. V. Ryazanov for interesting discussions.

[1] L. Onsager, Phys. Rev. **65**, 117 (1944); B. Kaufman and L. Onsager, Phys. Rev. **76**, 1244 (1949); C. N. Yang, Phys. Rev. **85**, 809 (1952).

[2] C. Domb and A. R. Miedema, Progr. Low Temp. Phys. **4**, 296 (1964).

[3] B. Widom, J. Chem. Phys. **41**, 1633 (1964).

[4] J. W. Essam and M. E. Fisher, J. Chem. Phys. **38**, 802 (1963); D. S. Gaunt, M. E. Fisher, M. F. Sykes, and J. W. Essam, Phys. Rev. Lett. **13**, 713 (1964).

[5] M. E. Fisher, Physica **25**, 521 (1959).

[6] T. D. Lee and C. N. Yang, Phys. Rev. **87**, 410 (1952).

[7] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika, (Statistical Physics) Nauka, 1964.

[8] Ya. A. Kraftmakher and T. Yu. Romashina, PTT **7**, 2532 (1965), Soviet Phys. Solid State **7**, 2040 (1966).

[9] B. Jacrot, J. Konstantinovic, G. Parette, and D. Gribier, Symposium on Inelastic Scattering of Neutrons in Solids and Liquids, Chalk River, 1962.

[10] P. Weiss and R. Forrer, Ann. de Physique **5**, 153 (1926).

[11] J. S. Kouvel and M. E. Fisher, Phys. Rev. **136**, A1626 (1964).

[12] T. L. Hill, Statistical Mechanics, McGraw-Hill, 1956. (Russ. Transl. IIL, 1960).

Translated by R. E. Peierls
58