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MULTIPLE SCATTERING OF ELECTROMAGNETIC WAVES IN AN INHOMOGENEOUS MEDIUM

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A general method is proposed for treating multiple scattering of electromagnetic waves in matter, which is based on a solution of Maxwell's equations and in which the radiant energy transfer equation is not used. The motion of a wave packet in a medium with inhomogeneities consisting of black spheres of radius $a \gg \lambda$ is considered as an illustration of the application of the method. The intensity distribution at a depth z is derived as a function of the coordinates or the direction of propagation.

MULTIPLE scattering of electromagnetic waves is usually considered with the aid of the classical radiant energy transfer equation. However, the solution of the kinetic equation is usually connected with serious mathematical difficulties, and various approximate methods are chosen in each considered region. Below we propose a more general, simpler method of solving the problem of propagation of electromagnetic waves in an inhomogeneous medium. The proposed method does not employ the kinetic equation, is based on a solution of Maxwell's equations, and assumes that the solution of the scattering problem by a single inhomogeneity and the average density of inhomogeneities in the medium is known.

The possibility of applying the proposed method is connected with the fact that in the cases considered the process of propagation of the electromagnetic wave in the medium constitutes a sequence of independent scattering acts by individual inhomogeneities. Two results follow directly from this physical assumption. First, the independence of the successive scattering acts indicates that one need not consider instances when the wave interacts simultaneously with two inhomogeneities. Secondly, the independence of the scattering acts indicates that in describing single-center scattering it is sufficient to know only the asymptotic form of the solution, i.e., the scattering amplitude and not the exact form of the wave near the center. Use of these assumptions makes it possible to simplify the solution considerably.

In the specific treatment we also made use of the additional assumption that the angular distribution of the scattering by an individual inhomogeneity is strongly anisotropic—scattering occurs predominantly in the forward direction at small angles to the initial direction of the wave.

1. THE ELECTROMAGNETIC FIELD IN AN INHOMOGENEOUS MEDIUM

As is well known, the electromagnetic field in an inhomogeneous medium with a dielectric permittivity $\epsilon(\mathbf{r}, \omega)$ satisfies the equation[1]

$$\Delta\mathbf{E}(\mathbf{r}, \omega) + \varepsilon\omega^2\mathbf{E}(\mathbf{r}, \omega) = \nabla(\nabla\mathbf{E}(\mathbf{r}, \omega)). \qquad (1.1)$$

We express the dielectric permittivity of an inhomogeneous medium in the form of a part ϵ_0 which is independent of the coordinates, and an inhomogeneous part $\epsilon_1(\mathbf{r}, \omega)$ due to the presence of randomly distributed inhomogeneities: $\epsilon(\mathbf{r}, \omega) = \epsilon_0(\omega) + \epsilon_1(\mathbf{r}, \omega)$. Then equation (1.1) takes on the form

$$\Delta\mathbf{E}(\mathbf{r}, \omega) + \varepsilon_0\omega^2\mathbf{E}(\mathbf{r}, \omega) = \nabla(\nabla\mathbf{E}(\mathbf{r}, \omega)) - \varepsilon_1\omega^2\mathbf{E}(\mathbf{r}, \omega). \qquad (1.2)$$

Let us consider the case when the wavelength of the radiation is small compared with the characteristic dimensions of the inhomogeneities a. It follows from Maxwell's equations that

$$\operatorname{div}\mathbf{E}(\mathbf{r}, \omega) = -(\varepsilon_0 + \varepsilon_1)^{-1}(\mathbf{E}(\mathbf{r}, \omega)\cdot\nabla\varepsilon_1(\mathbf{r}, \omega)),$$

so that $\operatorname{div}\mathbf{E} \sim a^{-1}\mathrm{E}$ and the term $\nabla(\nabla\mathbf{E})$ in (1.2) is small compared with all the remaining terms, since $\omega a \gg 1$. Therefore in the case considered below where $\omega a \gg 1$ we can confine ourselves to a consideration of the equation

$$\Delta\mathbf{E}(\mathbf{r}, \omega) + \varepsilon_0\omega^2\mathbf{E}(\mathbf{r}, \omega) = -\varepsilon_1(\mathbf{r}, \omega)\omega^2\mathbf{E}(\mathbf{r}, \omega). \qquad (1.3)$$

We shall denote the spatial position of an individual inhomogeneity by the radius vector of its center $\mathbf{R}_a$. The quantity $\epsilon_1(\mathbf{r}, \omega)$ is due to the presence of inhomogeneities and depends therefore on $\mathbf{R}_a$. The most general form of this dependence is

$$\varepsilon(\mathbf{r}, \omega) = \omega^{-2} \sum_a \alpha(\mathbf{r} - \mathbf{R}_a, \omega) + \omega^{-2} \sum_a \sum_{b \neq a} \beta(\mathbf{r} - \mathbf{R}_a, \mathbf{r} - \mathbf{R}_b, \omega) + \ldots, \quad (1.4)$$

where the first term corresponds to the independent contribution of each inhomogeneity to the dielectric permittivity, and the second term represents the difference in the action of two inhomogeneities from the action due to each of them separately. In the case of small wavelengths $\omega a \gg 1$ the second term should be negligibly small. Allowing for this, (1.3) can be transformed to the form

$$(\Delta + k_0^2)\mathbf{E}(\mathbf{r}, \omega) = -\sum_a \alpha(\omega, \mathbf{r} - \mathbf{R}_a)\mathbf{E}(\mathbf{r}, \omega) \quad (k_0^2 \equiv \varepsilon_0 \omega^2). \quad (1.3')$$

Equation (1.3′) describes the scattering of the electromagnetic field by the inhomogeneities. We shall seek the solution of this equation in the form

$$\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_0(\omega) \exp[i\mathbf{k}_0\mathbf{r} + s(\mathbf{k}_0, \mathbf{r})]. \quad (1.5)$$

The quantity $s(\mathbf{k}_0, \mathbf{r})$ describes the change of phase of the Fourier component of the field due to the scattering by the inhomogeneities and should also depend on their coordinates. In view of the fact that the wavelength of the radiation is small, it can be assumed that at each given instant the wave interacts with a single inhomogeneity center, i.e., the interaction with each inhomogeneity takes place independently. This means that the phase of the wave (1.5) is in first approximation simply a sum of the phases appearing in the interaction with each inhomogeneity center. The most general form of $s(\mathbf{k}_0, \mathbf{r})$ is

$$s(\mathbf{k}_0, \mathbf{r}) = \sum_a s_1(\mathbf{k}_0, \mathbf{r} - \mathbf{R}_a) + \sum_a \sum_{b \neq a} s_2(\mathbf{k}_0, \mathbf{r} - \mathbf{R}_a, \mathbf{r} - \mathbf{R}_b) + \ldots \quad (1.6)$$

It follows from the above that $s_1 \gg s_2$ and one can confine oneself to the first terms in (1.6). Each summation in (1.6) leads to the appearance of one power of the density of inhomogeneity centers n_0 in the final result. The corresponding dimensionless parameter $n_0\omega^{-3} \ll 1$, and it is therefore possible to neglect the double summations.

Retaining after substitution of (1.6) in (1.5) only terms with single summation, one readily obtains

$$\sum_a \{2i\mathbf{k}_0 \nabla s_1(\mathbf{k}_0, \mathbf{r} - \mathbf{R}_a) + \Delta s_1(\mathbf{k}_0, \mathbf{r} - \mathbf{R}_a) + (\nabla s_1(\mathbf{k}_0, \mathbf{r} - \mathbf{R}_a))^2 + \alpha(\omega, \mathbf{r} - \mathbf{R}_a)\} = 0. \quad (1.7)$$

This equation should be valid for arbitrary values of $\mathbf{R}_a$ and the summation may therefore be omitted. But in this case the equation coincides with the problem of the scattering of an electromagnetic field by one isolated inhomogeneity at the point $\mathbf{R}_a$. If such a solution is assumed to be known, then $s_1(\mathbf{k}_0, \mathbf{r} - \mathbf{R}_a)$ is a known function and the problem of determining the electromagnetic field in a system of randomly distributed inhomogeneous centers can be assumed solved.

Thus the solution of (1.3) can be written in the approximation under consideration in the form

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp\left[i\mathbf{k}_0\mathbf{r} - i\omega t + \sum_a s_1(\mathbf{k}_0, \mathbf{r} - \mathbf{R}_a)\right], \quad (1.8)$$

where $s_1(\mathbf{k}_0, \mathbf{r} - \mathbf{R}_a)$ is determined from the solution of the problem of scattering by an isolated inhomogeneity. The solution (1.8) corresponds to a boundary condition with which the field had until its interaction with the inhomogeneities the form $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp[i\mathbf{k}_0 \cdot \mathbf{r} - i\omega t]$. In order to obtain the solution of (1.3) with another boundary condition, one has to take the corresponding superposition of the solutions (1.8).

2. DISTRIBUTION OF THE ENERGY OF THE ELECTROMAGNETIC FIELD

Let us consider a substance placed in a layer between the planes $z = 0$ and $z = L$. Let a certain electromagnetic field $\mathbf{E}(\mathbf{r}, t)$ be incident on the surface of the material $z = 0$. We shall be interested in the distribution of the energy flow of this field in the inhomogeneous substance. During the time of observation the energy which will pass through the plane $z = z_0$ will be given by

$$(4\pi)^{-1} \int dt\, d^2\mathbf{r}_\perp [\mathbf{E}(\mathbf{r}, t)\mathbf{H}^*(\mathbf{r}, t)]_\parallel = (2\pi^2) \int d\omega\, d^2\mathbf{k}_\perp\, dk_\parallel\, dk_\parallel' \times \exp[i(k_\parallel - k_\parallel')z_0][\mathbf{E}(\mathbf{k}_\perp + k_\parallel, \omega)\mathbf{H}^*(\mathbf{k}_\perp + k_\parallel', \omega)]_\parallel, \quad (2.1)^*$$

where

$$\mathbf{E}(\mathbf{r}, t) = \int d\omega\, d^3\mathbf{k}\mathbf{E}(\mathbf{k}, \omega) \exp(i\mathbf{k}\mathbf{r} - i\omega t).$$

It follows from (2.1) that the energy passing through the plane $z = z_0$ in the frequency interval $d\omega$ and in the interval of transverse momenta $d^2\mathbf{k}_\perp$ is of the form

$$J(z_0, \mathbf{k}_\perp, \omega)\, d^2\mathbf{k}_\perp d\omega,$$

where

$$J(z_0, \mathbf{k}_\perp, \omega) = \int d^2\mathbf{r}_\perp\, d^2\mathbf{r}_\perp' \exp[i\mathbf{k}_\perp(\mathbf{r}_\perp - \mathbf{r}_\perp')] \times \langle[\mathbf{E}(\mathbf{r}, \omega)\mathbf{H}^*(\mathbf{r}', \omega)]\rangle_\parallel (8\pi^2)^{-1} + \text{c. c.} \quad (2.2)$$

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*$[\mathbf{E}(\mathbf{r}, t)\, \mathbf{H}^*(\mathbf{r}, t)] \equiv \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}^*(\mathbf{r}, t)$.

In the case of a medium with randomly distributed inhomogeneities it is sensible to consider only quantities averaged over the positions of the inhomogeneities, a fact denoted in (2.2) by the symbol $\langle \ldots \rangle$.

Let us consider initially the simplest case, assuming that a plane wave is incident in the positive z direction on the surface of the medium $z = 0$. The solution of Maxwell's equations in an inhomogeneous medium for $\omega a \gg 1$ is of the form (1.8), so that

$$\mathbf{E}(\mathbf{r},\omega) = \mathbf{E}_0 \exp\Big[i\mathbf{k}_0\mathbf{r} + \sum_a s_1(\mathbf{k}_0, \mathbf{r}-\mathbf{R}_a) \Big]\delta(\omega-\omega_0),$$
$$\mathbf{H}(\mathbf{r},\omega) = -i\omega^{-1}[\nabla\mathbf{E}(\mathbf{r},\omega)]. \qquad (2.3)$$

Substitution of (2.3) in (2.2) leads to the circumstance that $J(z, \mathbf{k}_\perp, \omega)$ turns out to be proportional to the total time of observation T and contains $\delta(\omega-\omega_0)$, so that the energy passing per unit time through the plane $z = z_0$ with a frequency $\omega = \omega_0$ and a value $\mathbf{k}_\perp$ in the interval $\mathbf{k}_\perp$, $\mathbf{k}_\perp + d\mathbf{k}_\perp$, has the form

$$w(z,\mathbf{k}_\perp) \equiv \frac{d\varepsilon(z,\mathbf{k}_\perp)}{dT\, d^2\mathbf{k}_\perp}$$
$$= (16\pi^3\omega)^{-1}\int d^2\mathbf{r}_\perp\, d^2\mathbf{r}_\perp{}'\exp[i(\mathbf{r}-\mathbf{r}')\mathbf{k}_\perp]$$
$$\Big\langle\Big[\mathbf{E}_0\Big[\Big(\mathbf{k}_0 + i\sum_a \nabla s_1(\mathbf{k}_0,\mathbf{r}-\mathbf{R}_a)\Big), \mathbf{E}_0\Big]\Big]_{\parallel}$$
$$\times\exp\sum_b\{s_1(\mathbf{k}_0,\mathbf{r}-\mathbf{R}_b) + s_1^*(\mathbf{k}_0,\mathbf{r}'-\mathbf{R}_b)\}\Big\rangle + \text{c. c.} \qquad (2.4)$$

Averaging of (2.4) over the positions of the inhomogeneities is readily carried out if the coordinates of different inhomogeneities are assumed independent. In this case one can assume that

$$\Big\langle \exp\sum_{a=1}^{N} Q_a\Big\rangle = \langle\exp Q_a\rangle^N$$
$$= \Big\{1 + \frac{1}{N}\Big\langle\sum_a(\exp Q_a - 1)\Big\rangle\Big\}^N.$$

In the limit $N \to \infty$ we obtain the equation

$$\Big\langle\exp\sum_a Q_a\Big\rangle = \exp\Big\langle\sum_a(\exp Q_a - 1)\Big\rangle. \qquad (2.5)$$

It can also be readily shown that for $N \to \infty$

$$\Big\langle\sum_{b=1}^{N} R_b\exp\Big(\sum_{a=1}^{N} Q_a\Big)\Big\rangle = \Big\langle\sum_b R_b\exp Q_b\Big\rangle$$
$$\times\exp\Big\langle\sum_a(\exp Q_a - 1)\Big\rangle. \qquad (2.5')$$

Using the obtained equations, one can transform (2.4) to the form

$$w(z,\mathbf{k}_\perp) = (16\pi^2\omega)^{-1}\int d^2\mathbf{r}_\perp d^2\mathbf{r}_\perp{}'\exp[i(\mathbf{r}-\mathbf{r}')\mathbf{k}_\perp]$$
$$\times\Big\langle\Big\{[\mathbf{E}_0[\mathbf{k}_0\mathbf{E}_0]] + i(\mathbf{E}_0)^2\Big(\frac{\partial}{\partial\mathbf{r}} + \frac{\partial}{\partial\mathbf{r}'}\Big)$$
$$- i\mathbf{E}_0\Big(\mathbf{E}_0\Big(\frac{\partial}{\partial\mathbf{r}} + \frac{\partial}{\partial\mathbf{r}'}\Big)\Big)\Big\}\sum_a\exp[s_1(\mathbf{k}_0,\mathbf{r}-\mathbf{R}_a)$$
$$+ s_1^*(\mathbf{k}_0,\mathbf{r}'-\mathbf{R}_a)]\Big\rangle$$

or

$$w(z,\mathbf{k}_\perp) = (16\pi^3\omega)^{-1}\int d^2\mathbf{r}_\perp d^2\mathbf{r}_\perp{}'\exp[i(\mathbf{r}-\mathbf{r}')\mathbf{k}_\perp]$$
$$\times\Big\{[\mathbf{E}_0[\mathbf{k}_0\mathbf{E}_0]]\exp F(\mathbf{r},\mathbf{r}') + i\Big[\mathbf{E}_0\Big[\Big(\frac{\partial}{\partial\mathbf{r}} + \frac{\partial}{\partial\mathbf{r}'}\Big)$$
$$\times\exp F(\mathbf{r},\mathbf{r}')\mathbf{E}_0\Big]\Big]\Big\}. \qquad (2.6)$$

The calculation of the energy flow reduces thus to a calculation of the expression

$$F(\mathbf{r},\mathbf{r}') = \Big\langle\sum_a\{\exp(s_1(\mathbf{k}_0,\mathbf{r}-\mathbf{R}_a)$$
$$+ s_1^*(\mathbf{k}_0,\mathbf{r}'-\mathbf{R}_a)) - 1\}\Big\rangle. \qquad (2.7)$$

To carry out the averaging one must separate in $s_1(\mathbf{k}_0, \mathbf{r}-\mathbf{R}_a)$ the dependence on $\mathbf{R}_a$.

Under the assumptions made above it is sufficient to use the asymptotic form of $s_1(\mathbf{k}_0, \mathbf{r}-\mathbf{R}_a)$ at large distances from the corresponding inhomogeneity, and the accurate form of $s_1(\mathbf{k}_0, \mathbf{r}-\mathbf{R}_a)$ for $\mathbf{r} \to \mathbf{R}_a$ is not essential. The asymptotic form of $s_1(\mathbf{k}_0, \mathbf{r}-\mathbf{R}_a)$ can be found if it is taken into account that the quantity

$$\mathbf{E}_0\exp(i\mathbf{k}_0\mathbf{r}) + \mathbf{E}_0\exp(i\mathbf{k}_0\mathbf{r})(\exp s_1(\mathbf{k}_0,\mathbf{r}-\mathbf{R}_a) - 1)$$

is a solution of the problem of the scattering of an electromagnetic plane wave by one inhomogeneity at the point $\mathbf{R}_a$, so that at large distances

$$\exp(i\mathbf{k}_0\mathbf{r})[\exp s_1(\mathbf{k}_0,\mathbf{r}-\mathbf{R}_a) - 1]$$
$$= -f\Big(\mathbf{k}_0 - \frac{\mathbf{r}}{r}k_0\Big)\frac{\exp[ik_0|\mathbf{r}-\mathbf{R}_a|]}{|\mathbf{r}-\mathbf{R}_a|}, \qquad (2.8)$$

where $f(\mathbf{q})$ is the scattering amplitude as a function of the transferred momentum. Using a known transformation, one can obtain from (2.8)[2]

$$\exp s_1(\mathbf{k}_0,\mathbf{r}-\mathbf{R}_a) - 1 = \frac{1}{2\pi^2}\int d^3q f(\mathbf{q})(\mathbf{q}^2 - 2\mathbf{k}_0\mathbf{q} - i\delta)^{-1}$$
$$\times\exp[i\mathbf{q}(\mathbf{r}-\mathbf{R}_a)]. \qquad (2.8')$$

Averaging over the coordinates of the inhomogeneities now reduces to averaging of expressions of the type

$$\Big\langle\sum_a\exp(i\mathbf{Q}\mathbf{R}_a)\Big\rangle = n_0(2\pi)^2\delta^{(2)}(\mathbf{q}_\perp)\int_0^L dR_z\exp(iQ_zR_z), \qquad (2.9)$$

where the average density of the medium is constant and equal to n_0 for $0 \le z \le L$. Using this, one readily obtains

$$F(\mathbf{r},\mathbf{r}') = n_0\pi^{-2}\int\int \frac{d^3q d^3q' f(\mathbf{q}) f^*(\mathbf{q}')\delta^{(2)}(\mathbf{q}_\perp - \mathbf{q}_\perp')}{(\mathbf{q}^2 - 2\mathbf{k}_0\mathbf{q} - i\delta)(\mathbf{q}'^2 - 2\mathbf{k}_0\mathbf{q}' + i\delta)}$$

$$\times \int_0^L d\xi \exp[i\xi(q_\| - q_\|')] + \frac{n_0}{2}\int dq_\| \Big[f(q_\|)(q_\|^2$$

$$- 2k_0 q_\| - i\delta)\int_0^L d\xi \exp[-iq_\|(\xi - z)] + \text{c. c.}\Big]. \quad (2.10)$$

We now integrate over $q_\|$ and $q'_\|$. Expressing $F(\mathbf{r},\mathbf{r}')$ in the form

$$F(\mathbf{r},\mathbf{r}') = \frac{n_0}{\pi^2}\int d^2\mathbf{q}_\perp \exp[i\mathbf{q}_\perp(\mathbf{r}_\perp - \mathbf{r}_\perp')] \quad (2.11)$$

$$\times \int_0^L d\xi \left| \int_{-\infty}^{\infty} \frac{dq_\| f(\mathbf{q}_\perp + q_\|)\exp(iq_\|(z-\xi))}{q_\|^2 + 2q_\| k_0 + q_\perp^2 - i\delta} \right|^2$$

$$+ 2n_0 \int_0^L d\xi \int_{-\infty}^{\infty} dq_\| \left\{ \frac{f(q_\|)\exp[iq_\|(z-\xi)]}{q_\|^2 + 2q_\| k_0 - i\delta} + \text{c. c.}\right\},$$

we can readily carry out the integration:

$$F(\mathbf{r},\mathbf{r}') \cong n_0 \int d^2q_\perp \exp[i\mathbf{q}_\perp(\mathbf{r}_\perp - \mathbf{r}_\perp')]\Big\{ z\frac{|f(\mathbf{q}_\perp + q_1)|^2}{k_0^2 - q_1^2}$$

$$+ n_0(2\pi)k_0^{-1}z[if(0) + \text{c. c.}]\Big\}, \quad (2.12)$$

where

$$q_1 = q_\perp^2/2k_0.$$

When $z > L$, $F(\mathbf{r},\mathbf{r}')$ coincides with its value at $z = L$.

3. PROPAGATION OF A PLANE WAVE IN A MEDIUM CONTAINING BLACK SPHERES

Let us illustrate the application of the theory developed above by an example in which the inhomogeneities are absolutely black spheres of radius a, large compared with the wavelength of the field, and $\omega a \gg 1$. In this limiting case the scattering amplitude by one sphere is[3]

$$f(\mathbf{q}_\perp) = kq_\perp^{-1}aJ_1(aq_\perp), \quad (3.1)$$

where $J_1(x)$ is the Bessel function, and $q_\perp$ is the momentum transferred in the transverse direction. The assumptions used in the derivation of (2.12) are valid, so that it follows from (2.12) and (3.1) that

$$F(\mathbf{r}_\perp,\mathbf{r}_\perp') = n_0 z\int d^2\mathbf{q}_\perp k^{-2}|f(\mathbf{q}_\perp)|^2 \exp[-i\mathbf{q}_\perp(\mathbf{r}_\perp - \mathbf{r}_\perp')]$$

$$- n_0 z\frac{4\pi}{k}\operatorname{Im} f(0).$$

The last term is transformed with the aid of the optical theorem to the form $-n_0z(\sigma_s + \sigma_a)$ and taking into account that the elastic scattering cross section σ_s is of the form

$$\sigma_s = \pi a^2 = a^2\int q_\perp^{-2}d^2\mathbf{q}_\perp J_1^2(aq_\perp),$$

the quantity $F(\mathbf{r},\mathbf{r}')$ will be written in the form (σ_a—inelastic scattering cross section)

$$F(\mathbf{r},\mathbf{r}') = -n_0z\sigma_a + n_0za^2\int d^2\mathbf{q}_\perp q_\perp^{-2}J_1^2(aq_\perp)$$

$$\times[\exp i\mathbf{q}_\perp(\mathbf{r}_\perp - \mathbf{r}_\perp') - 1]. \quad (3.2)$$

We recall that here and below we must replace z by L for $z \ge L$.

Let us now find the distribution of electromagnetic waves over the directions, assuming the deviations from the initial direction to be small. In this case the total angle of deflection is accumulated from small deflections on each scatterer and the momentum q transferred to one scatterer is considerably less than the total momentum transferred. This allows one to consider the quantity $\mathbf{q}_\perp(\mathbf{r}_\perp - \mathbf{r}'_\perp)$ in $F(\mathbf{r},\mathbf{r}')$ small, so that

$$F(\mathbf{r},\mathbf{r}') = -n_0z\sigma_a - \frac{n_0za^2}{4}(2\pi)(\mathbf{r}_\perp - \mathbf{r}_\perp')^2\int_0^\infty J(q_\perp a)q_\perp dq_\perp \quad (3.3)$$

(the linear term of the expansion of the exponent is cancelled in the integration over the angles). Before we substitute (3.3) in (2.6), we note that the ratio of the terms $k_0 \exp F(\mathbf{r},\mathbf{r}')$ and $(\partial/\partial\mathbf{r} + \partial/\partial\mathbf{r}') \times \exp F(\mathbf{r},\mathbf{r}')$ in (2.6) is of the order of $k_0q^{-1} \gg 1$, and the terms with the derivatives can be omitted. In this case

$$w(z,\mathbf{k}_\perp) = (16\pi^3\omega)^{-1}\int d^2\mathbf{r}_\perp d^2\mathbf{r}_\perp' \exp\{i\mathbf{k}_\perp(\mathbf{r}_\perp - \mathbf{r}_\perp')\}$$

$$\times[\mathbf{E}_0[\mathbf{k}_0\mathbf{E}_0]]\exp\Big\{-n_0\sigma_a z - n_0za^2\frac{\pi}{2}(\mathbf{r} - \mathbf{r}')^2$$

$$\times\int_0^\infty J_1(q_\perp a)q_\perp dq_\perp\Big\}.$$

Hence, one can obtain by integration

$$w(z,\mathbf{k}_\perp) = w_0\exp(-n_0\sigma_a z)\exp\Big[-\frac{k_\perp^2}{4k_0^2\langle\theta_0^2\rangle n_0za^2}\Big]$$

$$\times\frac{(2\pi)^{-3}}{2k_0^2a^2\langle\theta_0^2\rangle n_0z}, \quad (3.4)$$

where $\langle\theta_0^2\rangle$ is the average angle of diffraction by one sphere:

$$\langle\theta_0^2\rangle \sim (k^2a^2)^{-1}.$$

The average width of the beam is, as one should have expected, determined by the relation

$$n_0za^2\langle\theta_0^2\rangle \sim n_0z/k^2. \quad (3.5)$$

4. PROPAGATION OF A WAVE PACKET IN AN INHOMOGENEOUS MEDIUM

To describe the passage of a light pulse in an inhomogeneous medium, one must solve the problem of the propagation of a wave packet in such a medium. In the case considered $\omega a \gg 1$ the polarization of the wave practically does not change and one can assume that $\mathbf{E}(\mathbf{r},\omega) \cong \mathbf{E}_0\,\psi(\mathbf{r},\omega)$, where ψ is a scalar function satisfying the equation

$$(\Delta + k_0^2)\psi(\mathbf{r},\omega) = -\sum_a \alpha(\omega, \mathbf{r}-\mathbf{R}_a)\psi(\mathbf{r},\omega), \qquad (4.1)$$

the intensity distribution of the field over the coordinates being given by $\mathbf{E}_0^2|\psi(\mathbf{r},\omega)|^2$. We introduce also the quantity $\psi(\mathbf{r}, t)$ by means of the relation $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}, t)$ and the quantity

$$J(\mathbf{r},\mathbf{p},t) = (2\pi)^{-3}E_0^2\int d^3\mathbf{r}'\exp(-i\mathbf{p}\mathbf{r}')\left\langle\psi\left(\mathbf{r}+\frac{\mathbf{r}'}{2},t\right)\times\psi^*\left(\mathbf{r}-\frac{\mathbf{r}'}{2},t\right)\right\rangle, \qquad (4.2)$$

which we shall call the intensity distribution function. We note that Eq. (4.1) coincides with the Schrödinger equation for a particle in a potential field

$$U(\mathbf{r}) = -\frac{1}{2m}\sum_a \alpha(\omega, \mathbf{r}-\mathbf{R}_a).$$

The quantity (4.2) coincides within a factor E_0^2 with the quantum mechanical distribution function in the mixed representation.[4, 5]

The wave properties of the electromagnetic field lead to the circumstance that the quantity $J(\mathbf{r}, \mathbf{p}, t)$ is not a directly observable quantity. In fact, the precise measurement of the field intensity at a given point with a given wave vector $\mathbf{p}$ is impossible. However, with the aid of (4.2) one can calculate the intensity distribution, for example, along the longitudinal coordinate and transverse wave vector:

$$J(r_\parallel, \mathbf{p}_\perp, t) = \int d^2\mathbf{r}_\perp dp_\parallel J(\mathbf{r},\mathbf{p},t).$$

In analogy with the inequality which the quantum mechanical distribution must obey, the quantity $J(\mathbf{r}, \mathbf{p}, t)$ should satisfy the inequality

$$E_0^{-2}\int d^3\mathbf{r}d^3\mathbf{p}J(\mathbf{r},\mathbf{p},t)J(\mathbf{r},\mathbf{p},t) \leqslant 1. \qquad (4.3)$$

The experimentally determined intensity is the result of averaging of $J(\mathbf{r}, \mathbf{p}, t)$ over the sensitivity intervals of the measuring device, i.e., over a large $(\Delta\mathbf{r}\cdot\Delta\mathbf{p} \gg 1)$ phase volume of the detector.

Let us now consider the propagation of a wave packet in a medium, the inhomogeneities of which are black spheres. Let the initial state of the wave packet before entering the medium have a certain frequency spread described by a function $\eta_1(\omega-\omega_0)$, and a certain spread of the transverse component of the wave vector characterized by the function $\eta_2(\mathbf{k}_\perp)$. This corresponds to a signal limited in time also in the transverse direction. The solution of Eq. (4.1) in the medium will be described by a superposition of plane waves.

$$\psi(\mathbf{r},t) = \int d\omega\eta_1(\omega-\omega_0)\int d^2\mathbf{k}_\perp\eta_2(\mathbf{k}_\perp)\exp\left[i\mathbf{k}\mathbf{r}-i\omega t + \sum_a s_1(\mathbf{k},\mathbf{r}-\mathbf{R}_a)\right]. \qquad (4.4)$$

Substituting this expression in (3.2) and averaging over the distribution of the atoms with the aid of (2.5), one can obtain the intensity distribution function for the problem under consideration in the form

$$J(\mathbf{R},\mathbf{p},t) = E_0^2(2\pi)^{-3}\int d^3\mathbf{r}d\omega\, d\omega' d^2\mathbf{k}_\perp d^2\mathbf{k}_\perp'\exp(-i\mathbf{r}\mathbf{p})\times\eta_1(\omega-\omega_0)\eta_1^*(\omega'-\omega_0)\eta_2(\mathbf{k}_\perp)\eta_2^*(\mathbf{k}_\perp')\times\exp[L_1+L_2+L_3]. \qquad (4.5)$$

The transformation of the expressions for L_1, L_2, and L_3, is carried out in the same way as in the preceding section and yields ($\sigma_t = \sigma_a + \sigma_s$)

$$L_1 = -{}^1\!/_2 n_0\sigma_t(k)\tilde{z}_+, \qquad \tilde{z}_+ = \min(L, R_z + {}^1\!/_2 r_z);$$
$$L_2 = -{}^1\!/_2 n_0\sigma_t(k')\tilde{z}_-, \qquad \tilde{z}_- = \min(L, R_z - {}^1\!/_2 r_z);$$
$$L_3 = n_0(kk')^{-1}\int d^2q_\perp\exp(i\mathbf{q}_\perp\mathbf{r}_\perp)f(\mathbf{q}_\perp,k)f^*(\mathbf{q}_\perp,k')\times\int_0^{\tilde{z}} d\xi\exp\left[iq_\parallel\left(R_z+\frac{r_z}{2}-\xi\right)-iq_\parallel'\left(R_z-\frac{r_z}{2}-\xi\right)\right],$$
$$q_\parallel = -(q_\perp^2 + 2\mathbf{k}_\perp\mathbf{q}_\perp)/2k, \quad q_\parallel' = -(q_\perp^2+2\mathbf{k}_\perp'\mathbf{q}_\perp)/2k',$$
$$\tilde{z} = \min(\tilde{z}_+, \tilde{z}_-). \qquad (4.6)$$

Substitution of (3.1) for the scattering amplitude leads to the formulas

$$L_1 = -n_0\pi a^2\tilde{z}_+, \quad L_2 = -n_0\pi a^2\tilde{z}_-,$$
$$L_3 = n_0a^2\int q_\perp^{-2}d^2\mathbf{q}_\perp J_1^2(aq_\perp)\exp(i\mathbf{q}_\perp\mathbf{r}_\perp)\times\int_0^{\tilde{z}} d\xi\exp\left[iq_\parallel\left(R_z+\frac{r_z}{2}-\xi\right)-iq_\parallel'\left(R_z-\frac{r_z}{2}-\xi\right)\right]. \qquad (4.7)$$

Formulas (4.5)–(4.7) provide a general expression for the intensity distribution function (4.2).

The intensity distribution function $J(\mathbf{r}, \mathbf{p}, t)$ allows one to investigate the effect of multiple scattering of light in an inhomogeneous medium on the motion of a wave packet in space. For this purpose we will consider an expression $J_0(\mathbf{r}, t)$ defined by the relation

$$J_0(\mathbf{r}, t) = \int J(\mathbf{r}, \mathbf{q}, t)\, d^3\mathbf{q}. \qquad (4.8)$$

Utilizing expressions (4.5)–(4.7) and integrating over $\mathbf{q}$, we find that

$$J_0(\mathbf{r}, t) = I_0 \int d\omega\, d\omega' d^2\mathbf{k}_\perp d^2\mathbf{k}_\perp' \eta_1(\omega-\omega_0)\eta_1^*(\omega'-\omega_0)$$
$$\times \eta_2(\mathbf{k}_\perp)\eta_2^*(\mathbf{k}_\perp') \exp[-i(\omega-\omega')t + i(\mathbf{k}-\mathbf{k}')\mathbf{r}$$
$$+ L_1 + L_2 + L_3], \qquad (4.9)$$

where

$$L_1 + L_2 = -n_0(\sigma_a + \sigma_s)\tilde{L}, \quad \tilde{L} = \min(L, z),$$

$$L_3 = n_0 a^2 \int q_\perp^{-2} d^2\mathbf{q}_\perp J_1^2(aq_\perp) \int_0^{\tilde{L}} \exp[i(q_\parallel - q_\parallel')(z-\xi)]\, d\xi,$$

$$q_\parallel - q_\parallel' = \frac{q_\perp^2}{2k_0^2}(k_\parallel - k_\parallel') - \mathbf{q}_\perp\left(\frac{\mathbf{k}_\perp}{k_\parallel} - \frac{\mathbf{k}_\perp'}{k_\parallel'}\right).$$

For not too large z ($n_0\sigma_s\langle\theta_0^2\rangle z \ll 1$) expression (4.9) can be simplified considerably:

$$J_0(\mathbf{r}, t) = I_0 \int d\omega\, d\omega'\, d^2\mathbf{k}_\perp d^2\mathbf{k}_\perp' \eta_1(\omega-\omega_0)\eta_1^*(\omega'-\omega_0)$$
$$\times \eta_2(\mathbf{k}_\perp)\eta_2^*(\mathbf{k}_\perp') \exp\{-i(\omega-\omega')t$$
$$+ i(k_\parallel - k_\parallel')(z+\varkappa) - n_0\sigma_a\tilde{L} - \alpha(\mathbf{k}_\perp - \mathbf{k}_\perp')^2$$
$$+ \alpha(\mathbf{k}_\perp^2 - \mathbf{k}_\perp'^2)(k_\parallel - k_\parallel')/k_0\}, \qquad (4.10)$$

where

$$\alpha = n_0\sigma_s \frac{\langle\theta_0^2\rangle}{4}\int_0^{\tilde{L}} (z-\xi)^2\, d\xi, \qquad \varkappa = n_0\sigma_s\frac{\langle\theta_0^2\rangle}{2}\int_0^{\tilde{L}} (z-\xi)\, d\xi.$$

In order to determine the characteristics of the motion of a wave packet in space, one must know the initial form of the intensity distribution. However, using the rather general assumptions that $\eta_1(\omega-\omega_0)$ differs appreciably from zero only in a small frequency interval near ω_0 and that $\eta_2(\mathbf{k}_\perp)$ is a gaussian function with a half-width r_0, one can readily obtain from (4.10) the equation of motion of the center of gravity of the wave packet:

$$-\left(\frac{\partial\omega}{\partial k}\right)_{\omega=\omega_0} t + (z+\varkappa)$$
$$+ \frac{r_\perp^2\alpha(z+\varkappa)}{8r_0^2k_0^2[\alpha + r_0^2/2 + z^2/8r_0^2k_0^2]^2} = 0. \qquad (4.11)$$

It is apparent from (4.11) that the presence of optical inhomogeneities leads to a decrease in the group velocity of the packet. The quantity κ determines the velocity decrease on the axis of the ray, whereas the slowing down of the packet for $\mathbf{r}_\perp \neq 0$ is more intense and proportional to the last term in (4.11).

5. SPATIAL DISTRIBUTION OF THE INTENSITY IN THE PROPAGATION OF THE SIGNAL

Let us assume that the relative frequency spread in the wave packet is small and that the initial spread of the transverse wave vector is gaussian

$$\eta_2(\mathbf{k}_\perp) = r_0^2\pi^{-1}\exp(-k_\perp^2 r_0^2), \qquad (5.1)$$

where r_0 is the mean transverse dimension of the packet at the initial instant, i.e., before it enters into the substance. The spatial intensity distribution

$$I(z, \mathbf{r}_\perp) = \int d^3\mathbf{p} J(\mathbf{p}, \mathbf{r}, t)\, dt$$

relates the transverse spread of the packet with the distance z traversed in the substance. For the chosen initial wave-packet characteristics the spatial intensity distribution will take on the form

$$I(z, \mathbf{r}_\perp) = I_0 \exp(-n_0\sigma_a z)\left[1 + \frac{z^2}{4k_0^2r_0^4}\right.$$
$$\left. + n_0\sigma_s\frac{\langle\theta_0^2\rangle}{2r_0^2}\int_0^z (z-\xi)^2 d\xi\right]^{-1}$$
$$\times \exp\left\{-\left(\frac{r_\perp}{2}\right)^2\left[\alpha + \frac{r_0^2}{2} + \frac{z^2}{8k_0^2r_0^2}\right]^{-1}\right\}, \qquad (5.2)$$

where

$$\alpha = n_0\sigma_s\langle\theta_0^2\rangle\int_0^z (z-\xi)^2\, d\xi;$$

for $z \geq L$ the upper limit of integration must here and below be replaced by L, and $\langle\theta_0^2\rangle$ is given by relation (3.5).

From (5.2) one can obtain the law of attenuation of the intensity on the axis of the ray:

$$I(z, 0) = I_0\exp(-n_0\sigma_a z)\left[1 + \frac{z^2}{4k_0^2r_0^4} + n_0\sigma_s\frac{\langle\theta_0^2\rangle}{2r_0^2}\frac{z^3}{3}\right]^{-1}. \qquad (5.3)$$

It follows from here directly that in the presence of inhomogeneities the attenuation of the intensity on the axis of the ray is proportional to z^{-3}.

Expressions (5.2) and (5.3) make it possible to carry out a comparison with the results of other authors obtained by solving the kinetic transfer equation.[6-8] It follows from (5.2) that the transverse dimensions of the beam increase with increasing z like

$$\left[n_0\sigma_s\langle\theta_0^2\rangle\frac{z^3}{3} + \frac{r_0^2}{2} + \frac{z^2}{8k_0^2r_0^2}\right]^{1/2},$$

i.e., for large z like $z^{3/2}$; this is the so-called "three-halves law."[6, 8] The expression for the attenuation of the intensity on the axis of the beam (5.3) coincides with the results of the mentioned papers[6-8] if we ignore the term quadratic in z. The absence of this term in the indicated papers is due to an incorrect choice of the initial conditions which violates inequality (4.3), i.e., it is due to an incomplete account of the wave properties of the ray.

For very large z (5.3) becomes inapplicable. In the region $z \gg ar_0k_0$ one can obtain for the intensity of the beam the following expression

$$I(z, 0) = I_0 \int d^2k_\perp d^2k_\perp' r_0^4 \pi^{-2} \exp\left\{-k_\perp^2\left(r_0^2 + i\frac{z}{2k_0}\right) - k_\perp'^2\left(r_0^2 - i\frac{z}{2k_0}\right)\right\} \exp\left\{-n_0\sigma_t z + n_0\sigma_s r_0\left[\frac{\pi}{\langle\theta_0^2\rangle}\right]^{1/2}\right\},$$

or, integrating over $\mathbf{k}_\perp$ and $\mathbf{k}'_\perp$,

$$I(z, 0) = I_0\,[1 + (z/2k_0r_0^2)^2]^{-1} \exp\{-n_0\sigma_a z - n_0\sigma_s\,[z - \sqrt{\pi}r_0\langle\theta_0^2\rangle^{-1}]\}. \quad (5.4)$$

Thus for $z \gg ar_0k_0$ the attenuation of the intensity on the axis of the beam becomes exponential due to the diffusion of the electromagnetic radiation.[7] The pre-exponential coefficient gives an additional z^{-2} dependence connected with the presence of a beam width r_0 at the initial instant. The above expressions are confined to a region of z satisfying the inequalities

$$z < k_0^2/n_0, \qquad n_0\sigma_s\langle\theta_0^2\rangle z \ll 1,$$

in which the initial electromagnetic field equations were obtained. It should be noted that to obtain the above results by solving the kinetic equation of radiation transfer one must seek special ways of approximately solving the kinetic equation in each considered region. The use of the method proposed above makes it possible not merely to obtain the results but also to refine them. The proposed method admits readily the application of numerical methods of solution in those regions where it is difficult to obtain an analytical solution. To carry out the same in the method employing the kinetic equation seems much more difficult.

6. ANGULAR DISTRIBUTION OF THE RADIATION ON THE AXIS OF THE RAY

With the aid of the intensity distribution function $J(\mathbf{r}, \mathbf{p})$ one can obtain the experimentally observable intensity on the beam axis at various angles, the so-called function of the "brightness solid:"

$$F(z, 0, \mathbf{p}_\perp) = \int_{-\infty}^{\infty} dp_\parallel \int_S d^2\mathbf{r}_\perp J(z, \mathbf{r}_\perp; \mathbf{p}_\perp, p_\parallel), \quad (6.1)$$

where integration over $\mathbf{r}_\perp$ is carried out over the aperture of the detector S. The mean square of the angle of divergence of the radiation is then of the form

$$\langle\theta^2\rangle = k_0^{-2} \int d^2\mathbf{p}_\perp p_\perp^2 F(z, 0, \mathbf{p}_\perp). \quad (6.2)$$

Substituting (4.5) in (6.1) and (6.2), one readily obtains general formulas for the "brightness solid":

$$F(z, 0, \mathbf{p}_\perp) = \text{const} \int_S d^2\mathbf{x}_\perp [4r_0^2(4AB - C^2)]^{-1} \exp(-n_0\sigma_a z) \times \exp\left\{-\mathbf{x}_\perp^2 \frac{B}{4AB - C^2} - \mathbf{p}_\perp^2 \frac{A}{4AB - C^2} + \mathbf{p}_\perp\mathbf{x}_\perp \frac{C}{4AB - C^2}\right\}, \quad (6.3)$$

where

$$A = \frac{r_0^2}{2} + \frac{z^2}{8k_0^2r_0^2} + n_0\sigma_s\frac{\langle\theta_0^2\rangle}{12}z^3,$$

$$B = \frac{1}{8r_0^2} + n_0\sigma_s\frac{\langle\theta_0^2\rangle}{4}k_0^2 z,$$

$$C = \frac{z}{4k_0r_0^2} + n_0\sigma_s\frac{\langle\theta_0^2\rangle}{4}k_0z^2,$$

and for a mean-square angle of spread of the ray

$$\langle\theta^2\rangle = k_0^{-2}\left\{4B - \frac{R_0^2C^2}{4A^2}\exp\left(-\frac{R_0^2}{4A}\right) \times \left[1 - \exp\left(-\frac{R_0^2}{4A}\right)\right]^{-1}\right\}, \quad (6.4)$$

where R_0 is the radius of the detector.

It is interesting to analyze for the wave packet considered in the previous section the limiting cases of a "wide" ($R_0 \to \infty$) and "narrow" aperture (small R_0). In the first case it follows from (6.3) that

$$F(z, 0, \mathbf{p}_\perp)_{R_0\to\infty} = \text{const}\left(n_0\sigma_s k_0^2\frac{\langle\theta_0^2\rangle}{4}z + \frac{1}{8r_0^2}\right)^{-1} \times \exp\left\{-\frac{p_\perp^2 + n_0\sigma_a z}{n_0\sigma_s k_0^2\langle\theta_0^2\rangle z + 1/2r_0^2}\right\}, \quad (6.5)$$

where for $z \geq L$ one must substitute L for z. In this case we find from relation (6.4)

$$\langle\theta^2\rangle_{R_0\to\infty} = n_0\sigma_s\langle\theta_0^2\rangle z + (2r_0^2k_0^2)^{-1}. \quad (6.6)$$

Formulas (6.5) and (6.6) coincide with those obtained in [6, 8] by solving the transport equation if we discard the last terms containing r_0. As was pointed out above, this is due to the incorrect account in [6, 8] of the initial conditions.

For the case of a narrow aperture (small R_0) one readily obtains

$$F(z, 0, \mathbf{p}_\perp) = \text{const}\cdot\exp(-n_0\sigma_a z)\,[4r_0^2(4AB - C^2)]^{-1} \times \exp[-p_\perp^2 A/(4AB - C^2)] \quad (6.7)$$

and

$$\langle\theta^2\rangle = k_0^{-2}\left[\frac{4AB - C^2}{A} + \frac{C^2}{8A^2}R_0^2\right]. \quad (6.8)$$

Comparison of (6.6) and (6.8) shows that depending on the properties of the aperture, the mean square of the angle changes by a factor of four (for sufficiently large z), the dependence on $zn_0\sigma_S\langle\theta_0^2\rangle$ being entirely analogous. If we plot the z dependence of $\langle\theta^2\rangle$, then depending on the area of the aperture the $\langle\theta^2\rangle = \Phi_S(z)$ curves will fall between the straight line (6.6) and the curve (6.8).

7. LIMITS OF APPLICABILITY OF THE RESULTS

The region of applicability of the obtained results is bounded, first, by the condition of applicability of the initial equations (1.3) and (1.3′), i.e., by the inequality $\omega a \gg 1$. This inequality gives rise to the anisotropy of the scattering by an individual inhomogeneity.

Secondly, the region of applicability of the results is bounded by the applicability condition of Eq. (1.7), i.e., by the condition that the wave is scattered independently by each inhomogeneity. For this it is necessary that the scattering amplitude of the wave at zero angle be small compared with the distance between the inhomogeneities.

Finally, in obtaining the distribution function (4.5)–(4.10) use was made of the assumption that the mean-square of the scattering angle in the considered layer of the substance is small compared with unity. This limits the lengths considered by the condition

$$z \ll [n_0\sigma_s\langle\theta_0^2\rangle]^{-1},$$

from which it follows in particular that in the scattering by an individual inhomogeneity the incident wave differs little from a plane wave. The same conditions are usually introduced in solving the transfer equation. For example, from the transport equation

$$\frac{\partial J(\vartheta, z)}{\partial z} = -\sigma_t J(\vartheta, z) + \int \sigma(\chi) J(\vartheta - \chi, z)\, d\Omega_\chi \qquad (7.1)$$

one can obtain with the aid of the Fokker-Planck method a gaussian approximation for the distribution function in the substance.[8] For this it is sufficient to expand the integrand in the angle χ up to second-order terms. The fact that this expansion is possible is connected with the anisotropy of the angular distribution of single-center scattering. The solution of the obtained diffusion-type equation

$$\frac{\partial J(\vartheta, z)}{\partial z} = \frac{1}{4}\langle\chi^2\rangle_z \left\{\frac{\partial^2 J(\vartheta, z)}{\partial\vartheta^2} + \frac{1}{\vartheta}\frac{\partial J(\vartheta, z)}{\partial\vartheta}\right\}, \qquad (7.2)$$

where

$$\langle\chi^2\rangle_z = 2\pi\int_0^\infty \chi^3\, d\chi\, \sigma(\chi),$$

is of the form

$$J(\vartheta, z) = [\pi\langle\chi^2\rangle_z]^{-1} \exp\{-\vartheta^2/\langle\chi^2\rangle_z\}. \qquad (7.3)$$

Comparing the derivation of (7.3) with the derivation of (4.5), one can readily verify that identical assumptions are made, in particular that there is a complete analogy in the expansion in deriving (7.3) with the expansion in the derivation of (4.5).

It is seen from the above considerations that the proposed method yields results [for example, (5.2), (5.3), (6.5), and (6.6)] which coincide with the results obtained from the kinetic equation. Moreover, the proposed method allows one to find the intensity distribution function which depends on three angles and three coordinates, a problem which is much more difficult with the kinetic equation.

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