ON THE THEORY OF DISPERSION RELATIONS. I

M. E. PEREL'MAN

Institute of Cybernetics, Academy of Sciences, Georgian S.S.R.

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The basic postulates of field theory, relativistic invariance and the mass spectral properties, are written down for functions of a single four-dimensional variable in the form of functional equations of constraint. The Fourier transforms of these equations lead directly to the relativistic spectral representations and dispersion relations which generalize and make more concrete the Kramers-Kronig, Källén-Lehmann, and Dyson representations and their inversions. Similar relations are derived in the x representation. All relations can be written down as "equations of motion" in the x and p spaces with an unspecified interaction. This proves the complementarity of the equations of motion method and the theory of dispersion relations.

1. INTRODUCTION

At present, the axiomatics of the local quantum field theory have been worked out in detail (see, for example, [1,2]), and there exist a number of slightly different approaches toward an axiomatic basis for the field theory: the methods of Bogolyubov,[3] Lehmann-Symanzik-Zimmermann,[4] Wightman,[5,2] Haag-Ruelle[6] (cf. also [7-8]), etc. However, none of these methods leads to a construction and a proof of general relativistic dispersion relations.

It seems to us that these methods do not make use of all the information contained in the basic postulates of the theory. We therefore attempt (in the present paper only on the example of functions of a single four-dimensional variable) to write the basic postulates of the theory (causality, the mass spectrum conditions) in the form of functional equations[1)] which completely describe these axioms mathematically (cf. Sec. 2). The Fourier transforms of these equations lead to dispersion relations (d.r.) and spectral representations (s.r.) which generalize and make more concrete the Kramers-Kronig, Källén-Lehmann,[12] and Dyson[13,14] representations (Sec. 3). Analogous integral representations, but in the x variables, follow from the conditions of constraint on the energy-momentum (Sec. 4). It is interesting that the latter are just the ones which lead to the usual equations of motion with an unspecified interaction, whereas the s.r. of the Dyson type or the usual d.r. correspond to additional equations of motion in p space (Sec. 5). All the kernels entering in the integral representations (Fourier transforms of the projectors) are expressed in terms of singular functions of the Klein-Gordon equation (cf. the Appendix).

In this paper we shall not determine more precisely the classes of functions for which the transformed equations are defined, but shall instead make the physically reasonable assumption that there exist (in the sense of the theory of generalized functions) Fourier-Laplace transforms of all functions under consideration. As usual, the number and type of subtractions determine the concrete form of the theories. This restriction, as well as the consideration of scalar fields only, is not of importance in principle.

2. CONDITIONS OF RESTRAINT

In order to take complete account, in the functions under consideration, of the general requirements of causality, finiteness of the maximal velocity of propagation of the interaction (relativity), definiteness of the sign of the energy, and existence of a mass spectrum, we must write these conditions in the form of functional equations of constraint.[10] Let us consider these constraints one by one.

1. Causality. If the function f(x) (for example, the admittance function) vanishes for $x_0 > 0$, then it can be written in the form

[1)]The mathematical meaning of the equations of restraint and a number of applications in mathematics, information theory, and physics, have been considered in a series of papers.[10,11]

$$f(x) = \theta(-x_0)\Phi(x), \quad (2.1)$$

where

$$\Phi(x) = f(x) + \theta(x_0)\Phi_1(x) \quad (2.2)$$

and $\Phi_1(x)$ is a completely arbitrary function [$\theta(\pm x_0)$ is the Heaviside function]. In particular, one may set $\Phi_1(x) = 0$. In this case we obtain the characteristic functional equation

$$\theta(x_0)f(x) = 0 \quad \text{or} \quad f(x) = \varepsilon(x_0)f(x) \quad (2.3)$$

[$\epsilon(x_0)$ is the signature function]. The last form (2.3) is sometimes used in the theory of Hilbert transformations.[15]

The representation (2.1) describes all retarded functions and is a necessary and sufficient condition for the fulfilment of the corresponding physical requirement. Equations (2.3) are only valid in the case when the function is manifestly causal.

2. Locality (relativity). If the function f(x) vanishes outside the light cone, it must be expressible in the form

$$f(x) = \theta(x^2)\Phi(x), \quad (2.4)$$

where $\Phi(x) = f(x) + \theta(-x^2)\Phi_1(x)$. This condition is necessary and sufficient for all causal (in the sense of the special theory of relativity) functions and does not require an explicit verification of the Lorentz invariance of f(x) (cf. [16]).

3. Relativistic causality. Let f(x) be equal to zero everywhere outside the past light cone; then

$$f(x) = \theta(-x_0)\theta(x^2)\Phi(x) \quad \text{or}$$

$$f(x) = \theta(-t-r)\Phi(x). \quad (2.5)$$

Condition (2.5) is equivalent to the systems (2.1) and (2.4). Such types of relations can evidently also be written down for the advanced functions and their combinations with (2.5). In particular, for the singular functions of the Klein-Gordon equation we obtain

$$\Delta'_{ret}(x) = \theta(-x_0)\theta(x^2)\{\Delta'_{ret}(x) + \Phi_1(x)\}; \quad (2.6)$$

$$\Delta'(x) = \varepsilon(x_0)\theta(x^2)\{{}^1/{}_2\overline{\Delta'}(x) + \Phi_2(x)\}, \quad (2.7)$$

where Φ_1 vanishes in the past cone and Φ_2 vanishes in the entire cone.

4. Positive definiteness of the energy. The fulfilment of this condition is evidently guaranteed for the function f(p) by the equation

$$f(p) = \theta(p_0)\Phi(p). \quad (2.8)$$

5. Mass spectral properties. The absence of particles with a mass smaller than m in the interaction is guaranteed by the condition

$$f(p) = \theta(p^2 - m^2)\Phi(p). \quad (2.9)$$

The combined account of (2.8) and (2.9) yields the general equation

$$f(p) = \theta(p_0)\theta(p^2 - m^2)\Phi(p). \quad (2.10)$$

Conditions (2.1) to (2.10) can be written in the form of a single equation

$$f(\xi) = \theta(\alpha\xi_0)\theta(\xi^2 - \beta\xi^2 - \gamma)\Phi(\xi), \quad (2.11)$$

which characterizes a class of functions which vanish outside certain convex hyperboloids;[10] the physical restrictions are determined by the values of the parameters α, β, γ.

All the functional equations just written down describe a continuous spectrum of states of the functions f(p). The discrete levels can be formally included by replacing the θ functions by δ functions. Thus, in the presence of a single mass level (free particle) formula (2.10) is rewritten for states with spin 0 and $^1/_2$ as

$$\varphi(p) = \Delta^{(+)}(p, \mu^2)\varphi_0(p) + \theta(p_0)\theta(p^2 - m^2)\Phi(p); \quad (2.12)$$

$$\psi(p) = S^{(+)}(p, \mu^2)\psi_0(p) + \theta(p_0)\theta(p^2 - m^2)\Psi(p). \quad (2.13)$$

In this way, all conditions of restraint imposed in the axiomatic approach can be written in the form of functional equations. The role of f(x) and f(p) can, in particular, be played by the commutators of the interacting fields and Wightman functions.

3. SPECTRAL REPRESENTATIONS AND DISPERSION RELATIONS IN p SPACE

Let us now assume that all functions under consideration have Fourier transforms in the sense of the theory of generalized functions. (Here and below we shall not consider possible regularizations.[17]) Then the Fourier transformation of the functional equations (2.1) to (2.7) leads to integral representations of the causal functions.

The simplest relations of such type are the Kramers-Kronig relations (Hilbert transforms) of the Fourier transform of (2.3):[2)]

$$f(p) = \frac{1}{\pi i}\oint_{-\infty} d^4q \frac{f(q)}{p_0 - q_0}\delta(\mathbf{q}), \quad (3.1)$$

applicable when the known imaginary or real part of f(p) is substituted in the right-hand side. The more general relation (2.1) leads to the Sokhotskiĭ-Plemel' formulas.[18]

For the Fourier transform of a function which vanishes outside the light cone we obtain by the in-

2)The functions and their Fourier transforms are denoted by the same letter.

version of (2.4) with the help of (A.10) the known Bochner representation:[19, 8]

$$f(p)=\frac{1}{2\pi i}\int d^4q\Phi(q)\int_0^\infty d\lambda\,\nabla_1(p-q,\lambda)$$

$$=-\frac{1}{\pi^3}\int d^4q\frac{\Phi(q)}{(p-q)^4}. \qquad (3.2)$$

In virtue of the symmetry of (2.4) with respect to the inversion of the coordinates, the representation (3.2) connects the real and imaginary parts of f(p) with one another. Therefore it can be used only for a test of the self-consistency of the relativistically invariant expressions.

Thus, according to the optical theorem, the total cross section of a two-particle process is

$$\sigma(p,k)=\frac{2\pi}{|\mathbf{k}|}\int d^4xe^{ikx}\left\langle p\left|\left[j\left(\frac{x}{2}\right),\ j\left(-\frac{x}{2}\right)\right]\right|p\right\rangle. \qquad (3.3)$$

Hence the Fourier transform of $|\mathbf{k}|\sigma(p, k)$ vanishes outside the light cone and

$$\sigma(p,k)=-\frac{1}{\pi^3|\mathbf{k}|}\int d^4q\frac{|\mathbf{q}|\sigma(p,k)}{(k-q)^4}. \qquad (3.4)$$

From (3.4) one can, for example, determine approximately the asymptotic form of the total cross section [it should be recalled that in (3.4), in contrast to (3.2), one must substitute a completely definite function in the right-hand side].

It is clear that the Fourier transforms of (2.5) and (2.6), which take into account the definite symmetry of the functions with respect to inversion of the time [cf. (A.9) and (A.11)], must contain a large amount of information:

$$f(p)=\frac{1}{2\pi i}\int d^4q\Phi(q)\int_0^\infty d\lambda\,\nabla^{(+)}(q-p,\lambda)$$

$$=-\frac{1}{2\pi^3}\int d^4q\Phi(q-p)(q^2-i\varepsilon q_0)^{-2}; \qquad (3.5)$$

$$f(p)=\frac{1}{2\pi i}\int d^4q\Phi(q)\int_0^\infty d\lambda\,\nabla(p-q,\lambda)$$

$$=-\frac{1}{2\pi^3}\int d^4q\Phi(q-p)\frac{\partial}{\partial q^2}\Delta(q,0). \qquad (3.6)$$

Let us show that (3.6) is a more precise version of the known Dyson representation. Indeed, transforming the ∇ function in (3.6),

$$\nabla(u,\lambda)={}^1/_2\,\varepsilon(u_0)\overline{\nabla}(u^2,\lambda)={}^1/_2\,\varepsilon(u_0)\int_0^\infty d\nu\,\delta(u^2-\nu)\overline{\nabla}(\nu,\lambda)$$

and introducing the new spectral function

$$\Phi_2(q,\nu)=\frac{1}{4\pi i}\Phi(q)\int_0^\infty d\lambda\overline{\nabla}(\nu,\lambda)=\frac{1}{4\pi^2 i}\Phi(q)\delta'(\nu),$$

we obtain the Dyson representation

$$f(p)=\int d^4q\int_0^\infty d\nu\,\Phi_2(q,\nu)\varepsilon(p_0-q_0)\delta[(p-q)^2-\nu]. \qquad (3.7)$$

which has been derived by more complicated methods.[13, 14, 20, 7, 19] The function

$$\Phi_2(q,\nu)=\frac{1}{4\pi^2 i}\delta'(\nu)\,[f(q)+\Phi_1(q)] \qquad (3.8)$$

[$\Phi_1(q)$ is the Fourier transform of a function which vanishes inside the cone] can be regarded as a solution of the integral equation (3.7).

The restrictions from the existence of a mass spectrum can in all the above representations be taken into account by substituting the corresponding formulas of Sec. 2 in the right-hand side or by integrating over more complicated regions as is done in the derivation of the Dyson representation. The spectral restrictions are, in virtue of (3.8), essentially identical for the left and right hand sides of (3.7) (cf. [20]). It is also worth noting that (3.6) is the only representation in which the interaction vanishes automatically for $(p-2)^2 < 0$ (a similar effect in x space occurs in the usual Lehmann representation).

The starting formula (3.5) is a more exact version of the s.r. used by Lehmann for the proof of the analyticity of the scattering amplitude in the momentum transfer inside the Lehmann ellipse.[21] It follows from a comparison of (3.5) with (3.7) and (3.8) that it is impossible to improve these estimates of Lehmann. It is important, however, that in our method of establishing the d.r. (we do not discuss the possibilities of solving the equations) we do not need the principle of analyticity.

Indeed, since $\Phi_1(p)$ in (3.8) is arbitrary, we can substitute the function f(p) itself in the right-hand side of (3.6) and (3.5), i.e., go from the s.r. to the d.r. This relation does not involve any additional variables and solves the basic problem of the present paper: the construction of relativistic d.r. without knowledge of the analyticity.

However, if we take, for example, the real part of (3.5), then the right-hand side will contain integrals over the imaginary as well as the real parts of f(p), i.e., in substituting Im f(p) in (3.5) it turns into a singular integral equation for Re f(p). This equation can be solved in general form, but the answer is rather complicated. At least in two cases of practical importance one can obtain a simpler relation: 1) if it is known beforehand that f(p) vanishes outside the light cone, then it satisfies the d.r. (3.2), and, as is easily seen, (3.5) goes over into (3.6); 2) if the function can be expanded into a symmetric and an antisymmetric part with respect to time reversal, i.e., into an absorptive and a dispersive part, then we obtain again (3.6) owing to a condition of the type (2.7). This formula yields the simplest relativistic generalization of the Kramers-Kronig relation:[3)]

$$f_D(\omega, \mathbf{k}) = \frac{1}{(2\pi)^2 i} \int d\mathbf{n} \oint_{-\infty}^{\infty} d\eta \left(\frac{1}{\eta} + \frac{\partial}{\partial\eta}\right) \times f_A(\omega - \eta, \mathbf{k} - \mathbf{n}|\eta|), \qquad (3.9)$$

where $\mathbf{n}$ is a unit vector. In using the shortened form of the d.r. (3.9), the final results must in the first case, of course, be investigated as to their self-consistency with the help of (3.2).

Formulas (3.5), (3.6) or (3.9) can be applied, for example, to the amplitude for elastic two-particle scattering:

$$R(p_1k_1; p_2k_2) = i \int d^4x e^{ikx} \theta(x_0) \langle p_1 | [j_1(x/2), j_2(-x/2)] | p_2 \rangle, \qquad (3.10)$$

where $k = (k_1 + k_2)/2$, with arbitrary fixed momenta p_1 and p_2. The relation (3.9) is most simply used by dividing the amplitude into a dispersive and an absorptive part. It is more interesting, however, to consider the dependence on all arguments, as will be done in a subsequent paper with the help of the full reduction formula, taking commutators of higher order into account.

The representations just discussed are very complicated. They simplify appreciably only for functions which depend only on the square of the interval. Thus we obtain for the Fourier transform of (2.6), instead of (3.6),

$$\Delta'(p) = \frac{1}{2\pi i} \oint_{-\infty}^{\infty} d\varkappa^2 \left(\frac{1}{p^2 - \varkappa^2} - \frac{d}{d\varkappa^2}\right) \{\bar{\Delta}'(\varkappa^2) + \Phi_1(\varkappa^2)\},$$

or

$$\Delta'(p) = \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \frac{d\varkappa^2}{p^2 - \varkappa^2} (\bar{\Delta}'(\varkappa^2) + \Phi_1(\varkappa^2)) + \sum_n \operatorname{Res} \bar{\Delta}'(\mu_n^2 + i\Gamma_n^2) \qquad (3.11)$$

the Lehmann representation up to quasilocal operators.[12, 3] It is interesting that (3.11) is, in contrast to the usual derivations, obtained from the causality condition, whereas the mass spectrum conditions are necessary only to restrict the limits of integration in (3.11). This shows (at least for the two-particle functions) that the causality condition and the mass spectrum condition are not independent.

In concluding this section we note that the first attempt to construct d.r. of the type (3.5) without the use of the analyticity condition was apparently made by Leontovich for functions of a single space variable.[22] He obtained, however, a relation of the type (3.6) (since he used the signature function instead of the unit step function), so that only functions of a definite class can be substituted in the right-hand side; he did not notice the necessity of two terms on the right-hand side, corresponding to the two bisectors of the (tx) plane. Dispersion relations of the type (3.5) were established in the work of Silin and Rukhadze;[23] their result is necessary but, it seems to us, not sufficient, since the cone is actually defined by the condition $x_0^2 - \alpha\mathbf{x}^2 \geq 0$, where $\alpha \leq 1$, which requires a verification of the resulting functions for all possible α.

4. SPECTRAL REPRESENTATIONS IN x SPACE

In x space, integral representations of similar form can be obtained by using the condition of the definiteness of the sign of the energy and the restrictions on the mass spectrum.

Thus the condition that the energy be nonnegative (2.8) leads to the d.r. for the decay amplitude derived in [24]. From (2.9) and (A.8) we obtain the analog of (3.2):

$$f(x) = \frac{1}{2\pi i} \int d^4y \Phi(y) \int_{m^2}^{\infty} d\lambda \Delta_1(x - y, \lambda) = \frac{2}{\pi i} \int d^4y \Phi(y - x) \frac{\partial}{\partial y^2} \Delta_1(y, m^2), \qquad (4.1)$$

and (2.10) and (A.9) lead to a representation of the Fourier transform of a function which vanishes outside the upper convex hyperboloid:

$$f(x) = \frac{1}{2\pi i} \int d^4y \Phi(y) \int_{m^2}^{\infty} d\lambda \Delta^{(+)}(x - y, \lambda) = \frac{2}{\pi i} \int d^4y \Phi(y - x) \frac{\partial}{\partial y^2} \Delta^{(+)}(y, m^2). \qquad (4.2)$$

Analogously, we obtain for a function which is antisymmetric in the energy and vanishes for $p^2 < m^2$

$$f(x) = \frac{1}{2\pi i} \int d^4y \Phi(y) \int_{m^2}^{\infty} d\lambda \Delta(x - y, \lambda). \qquad (4.3)$$

If in (2.10) or similar equations, f(p) depends only on the square of the four-momentum and the sign of the energy, as is the case, e.g., for the two-particle Wightman functions, then it follows

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3)All integrals are to be taken in the sense of regularized values. Equation (3.9) may also contain quasilocal operators (ref. [17], ch. III, sec. 2).

from the obvious (up to quasilocal operators) equality

$$\int d^4y\Phi(y)\Delta(x-y, m^2) = \rho(m^2)\Delta(x, m^2) \qquad (4.4)$$

that (4.3) goes over into the Källén-Lehmann representation. (Here one must perform the necessary subtractions.)

It is interesting to note that (4.3) is the exact analog of the Dyson representation in x space: the transformation corresponds formally to the transformation of (3.6) into (3.7). We also note that this and only this representation [containing more information than (4.1)] satisfies automatically the locality requirements on the interaction.

One can also obtain more general integral relations. Let us consider, for example, the commutator

$$f_{PQ}(x) = \langle P|[A(x/2), B(-x/2)]|Q\rangle = f_+(x) - f_-(x) \qquad (4.5)$$

with the obvious spectral properties [a = (P + Q)/2]

$$f_\pm(p) = 0 \text{ for } (p_0 \pm a_0) < 0 \quad \text{or} \quad (p+a)^2 \leqslant m_\pm^2. \qquad (4.6)$$

The condition (4.6) corresponds to the equation of constraint

$$f_\pm(p) = \theta(p_0 \pm a_0)\theta[(p+a)^2 - m_\pm^2]\Phi_\pm(p), \qquad (4.7)$$

which, by Fourier transformation, goes over into the spectral representation for (4.5):

$$f_{PQ}(x) = \frac{1}{2\pi i}\int d^4y\theta(y^2)\int_0^\infty d\lambda\,\{\Phi_+(y) \times \Delta^{(+)}(x-y, m_+^2+\lambda)e^{-iay} + \Phi_-(y)\Delta^{(-)}(x-y, m_-^2+\lambda)e^{iay}\}. \qquad (4.8)$$

In the same way one can take into account the mass spectrum properties of any pair of fields in the higher Wightman functions. In all these examples the support of the function (the region of the argument where the function is different from zero) is determined from the causality condition.

The discrete mass states are easily included in the spectral representations: going over to the Fourier transform of (2.12) we obtain in analogy to (4.2)

$$f(x) = \frac{1}{2\pi i}\int d^4y\Phi(y) \times \int_0^\infty d\lambda\Delta^{(+)}(x-y, \lambda)\left\{\theta(\lambda - m^2) + \sum_n \delta(\lambda - \mu_n^2)\right\}. \qquad (4.9)$$

5. CONNECTION BETWEEN THE DISPERSION RELATIONS AND THE EQUATIONS OF MOTION

It is easy to show that the Fourier transforms of the characteristic functions (A.7) to (A.9) are singular functions of the six-dimensional Klein-Gordon operator:

$$\Box_x^{(6)} - m^2 \equiv \frac{\partial^2}{\partial x_0^2} - \sum_{i=1}^5 \frac{\partial^2}{\partial x_i^2} - m^2. \qquad (5.1)$$

Applying (5.1) to (4.1) to (4.3), we therefore obtain formal equations of motion for the corresponding f(x), which follow from the mass spectrum conditions.

Acting with the usual Klein-Gordon operator on these spectral representations [for definiteness, on (4.2)], we can obtain an interesting equation (we assume that it is possible to differentiate under the sign of the singular integral):

$$(\Box_x - m^2)f(x) = \frac{1}{2\pi i}\int d^4y\Phi(y)\int_0^\infty d\lambda\cdot\lambda\Delta^{(+)}(x-y, m^2+\lambda) \qquad (5.2)$$

or

$$(\Box_x - \hat{M}^2)f(x) = 0, \qquad (5.3)$$

where we have introduced the mass operator

$$\hat{M}^2 f(x) = \frac{1}{2\pi i}\int d^4y\Phi(y)\int_0^\infty d\lambda(m^2+\lambda)\Delta^{(+)}(x-y, m^2+\lambda). \qquad (5.4)$$

If in particular, f(x) is a complete propagation function of any type (i.e., a combination of two-particle Wightman functions), then (5.3) corresponds to the Schwinger equation,[25] where however, the method of solution of (5.4) still remains an open problem. A solution of equations of this type may yield the mass spectrum of the particles.[26]

If f(x) satisfies the equation

$$(\Box_x - m_i^2)f(x) = J(x), \qquad (5.5)$$

where J(x) is the total "current," then (4.9) leads to the following integral representation for the "current":

$$J(x) = \frac{1}{2\pi i}\int d^4yf(y)\int_0^\infty d\lambda(\lambda - m_i^2)\Delta^{(+)}(x-y, \lambda) \times \left\{\theta(\lambda - \mu^2) + \sum \delta(\lambda - m_n)\right\}. \qquad (5.6)$$

Thus we may assert that in a certain sense the equations of motion are equivalent to the mass spectrum requirements. This assertion is not too unexpected, for if the Klein-Gordon equation for the free particle is the operator form of the identity $p^2 = m^2$, then any interaction moves the particle away from this discrete level into the hyperboloid $p^2 \geq m^2$, as it were.

It follows from this that the mass spectrum condition, and hence also the equations (4.1) to (4.3), must correspond to the Klein-Gordon equation with an unspecified interaction, and for example, (5.6) includes all conceivable currents consistent with the basic postulates of the theory.

The representations of Sec. 4, from which we obtained the "equations of motion" written down above, are analogous in form to the s.r. and d.r. of Sec. 3. It is therefore natural to apply the corresponding differential operators also to these formulas.

Thus it is known that (A.7) to (A.9) are the eigenfunctions of $\Box_p^{(6)}$, the D'Alembert operator in six-dimensional p space,[7, 19] 4) but it is more interesting to use operators which can be given a physical meaning.

Since ∇ functions occur in the formulas of Sec. 3, such an operator is evidently the D'Alembert operator in p space [cf. (A.13)]:

$$\Box_p \equiv \frac{\partial^2}{\partial p_0{}^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial p_i{}^2}. \tag{5.7}$$

Acting with the operator (5.7) on (3.7), we obtain

$$\Box_p f(p) = \frac{1}{2\pi i} \int d^4 q \Phi(q) \int_0^\infty d\lambda \cdot \lambda \nabla(p-q, \lambda) \tag{5.8}$$

or

$$(\Box_p - \hat{S}^2) f(p) = 0, \tag{5.9}$$

where we have introduced the integral operator

$$\hat{S}^2 f(p) = \frac{1}{2\pi i} \int d^4 q \Phi(q) \int_0^\infty d\lambda \cdot \lambda \nabla(p-q, \lambda)$$

$$= \frac{8}{\pi i} \int d^4 q \, \Phi(q-p) \frac{\partial^2 \Delta(q, 0)}{\partial (q^2)^2}, \tag{5.10}$$

which is logically called the interval operator.

Indeed, in going over to the Fourier transform, (5.9) corresponds to the equation

$$(x^2 - \hat{S}^2) f(x) = 0, \tag{5.11}$$

which is complementary to (5.3). The most important difference lies in the fact that the Klein-Gordon equation [or (5.3) for $\hat{M}^2 f(p) = m^2 f(p)$] can be written for an arbitrary, in general stable, free particle, whereas (5.11) for $\hat{S}^2 f(x) = 0$ corresponds only to a noninteracting particle with vanishing mass. It is clear, in analogy to the treatment of (5.3), that any interaction, even one which leads to a lowering of the mass, moves particles away from the surface of the light cone, i.e., increases the value of the interval in (5.11). We note that the spectrum of the operator $\hat{S}^2$ is continuous and has, in contrast to the spectrum of $\hat{M}^2$, apparently only one—the ground—discrete state.

Thus there exists a complete symmetry between the mass spectrum requirements [d.r. of Sec. 4 and equations of motion of the type (5.3)] and relativistic causality [d.r. of Sec. 3 and equations of motion of the type (5.9)]. Only to the extent that we basically use the energy description is it more convenient to use the equations of motion of the type (5.3) and the d.r. of Sec. 3. It is evidently also important that for the solution of the equations one can, in the relations of Sec. 3 and in (5.10), use the exact or the approximate unitarity condition, whose role in x space is evidently played by the nonnegative condition of Wightman,[5, 7] which is the nonlinear part of the x theory.

Thus one may say that the relativistic d.r. are the integral form of the equations of motion which, however, arise not from the requirement $p^2 \geq m^2$, but from the condition $x^2 \geq 0$. The fundamental solutions to (5.3) which follow from the energy requirements will be the functions $\Delta^{(\pm)'}(x, m)$; causality and the condition $x^2 \geq 0$ are introduced into the resulting relations from outside, and the fundamental solutions of (5.9) are the functions

$$\nabla'_{\substack{ret \\ adv}}(p, S) = \Delta^{(\pm)'}(x \to p, m \to S), \tag{5.12}$$

in which the mass spectrum requirements and the condition of a definite sign of the energy are not observed.

6. CONCLUSION

Let us summarize our results.

1. The functional equations of restraint are equivalent to the considered postulates of the theory and lead to an algorithm for the construction of the corresponding s.r.

2. In the framework of the axiomatic method, single-particle relativistic s.r. and d.r. are established. The number and type of subtractions defining the theory are supplied from outside.

3. These d.r. are the relativistic generalization of the Kramers-Kronig relations and take account of all phenomena connected with spatial dispersion.

4. The same type of s.r. and d.r. are obtained

4)This circumstance probably inspired the idea of going to six-dimensional space in the usual derivation of the Dyson representation (3.7).

in x space as a consequence of the mass spectrum conditions.

5. The s.r. obtained contain as special cases the representations of Källén-Lehmann and Jost-Lehmann-Dyson, and also their inversions.

6. The estimates of Lehmann for the region of analyticity can apparently not be improved. The construction of s.r. and d.r. is possible without studying the regions of analyticity.

7. The s.r. following from the restrictions on the mass spectrum can be written in the form of Schwinger equations with a certain mass operator.

8. The s.r. and d.r. following from the condition of relativistic causality can be written in the form of equations of motion in the momentum space. The equations of this type are the analogs to the usual equations of motion in space-time.

9. Thus the dispersion approach is complementary to the usual Hamiltonian or Lagrangian formalism.

More generally one may conclude that if the equations of motion are in some sense the operator form of equations of the type of conservation laws or of $p^2 = m^2$, then the s.r. and d.r. are the operator form of the inequalities $p^2 \geq m^2$ or $x^2 \geq 0$, and the form of their kernels is further made more precise by inequalities of the type $p_0 > 0$ or $x_0 < 0$.

In subsequent papers we shall apply this method to the description of more complicated cases and actual interactions of fields and particles. We shall also consider some consequences and special cases of the relations obtained.

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APPENDIX

FOURIER TRANSFORMS AND REPRESENTATIONS OF CHARACTERISTIC FUNCTIONS AND THEIR CONNECTION WITH THE GREEN'S FUNCTIONS

Let us introduce the Fourier transformation operator

$$F_p[f(x)] \equiv f(p) = (2\pi)^{-4} \int d^4x f(x) e^{ipx}. \quad (A.1)$$

The convolution of the Fourier transforms of the functions $\Phi(x)$ of interest to us with the Fourier transforms of the generalized functions $Q(x)$ to be calculated below follows the usual rules:

$$f(x) = Q(x+a)\Phi(x) \to f(p) = \int dq Q(q) e^{-iqa} \Phi(p-q). \quad (A.2)$$

The Fourier transforms of the characteristic functions can be calculated directly. For example, (cf. [19, 27]),

$$F_p[\varepsilon(x_0)\theta(x^2)] = \frac{i}{2\pi^2}\frac{1}{|\mathbf{p}|}\frac{\partial}{\partial|\mathbf{p}|}\varepsilon(p_0)\delta(p^2)$$

$$= \frac{1}{\pi^2 i}\varepsilon(p_0)\delta'(p^2). \quad (A.3)$$

(The integration goes first over the spatial and then over the time variables.) It is simpler, however, to use the integral representation of the θ function:

$$\theta(a) = \int_0^\infty db\,\delta(a-b). \quad (A.4)$$

In the calculations we encounter integrals of the form

$$\int_{m^2}^\infty d\lambda\,\Delta(x,\lambda) = 4\frac{\partial}{\partial x^2}\Delta(x, m^2); \quad (A.5)$$

$$\int_0^\infty d\lambda\cdot\lambda^n\Delta(x, m^2+\lambda) = n!\left(4\frac{\partial}{\partial x^2}\right)^{n+1}\Delta(x, m^2)$$

$$(n = 0, 1, 2, \ldots). \quad (A.6)$$

The proof of (A.5) and (A.6) is easily carried out with the help of the known integral representation

$$\Delta(x, m^2) = \frac{\varepsilon(x_0)}{4\pi^2}\int_{-\infty}^\infty d\alpha\exp\left(ix^2\alpha + i\frac{m^2}{4\alpha}\right)$$

and the equality

$$\int_{-\infty}^\infty dx\cdot x^{-2}\,\delta(x)\exp\left(i\frac{a^2}{x}\right) = 0,$$

which follows from the rules for the multiplication of singular functions[28] (the point $a = 0$ must be considered separately).

With the help of (A.4) and (A.5) we obtain

$$F_x[\varepsilon(p_0)\theta(p^2-m^2)] = \frac{1}{2\pi i}\int_{m^2}^\infty d\lambda\Delta(x,\lambda) = \frac{2}{\pi i}\frac{\partial}{\partial x^2}\Delta(x, m^2); \quad (A.7)$$

$$F_x[\theta(p^2-m^2)] = \frac{1}{2\pi i}\int_{m^2} d\lambda\Delta_1(x,\lambda) = \frac{2}{\pi i}\frac{\partial}{\partial x^2}\Delta_1(x, m^2); \quad (A.8)$$

$$F_x[\pm\theta(\pm p_0)\theta(p^2-m^2)] = \frac{1}{2\pi i}\int_{m^2}^\infty d\lambda\Delta^{(\pm)}(x,\lambda)$$

$$= \frac{2}{\pi i}\frac{\partial}{\partial x^2}\Delta^{(\pm)}(x, m^2). \quad (A.9)$$

Thus the characteristic functions arising from the mass spectrum conditions (and the definiteness of the sign of the energy) are the Fourier transforms of the derivatives with respect to the interval of the singular functions of the Klein-Gordon equation.

For the Fourier transforms of the characteristic functions entering in the causality equations we obtain in the same way a number of expressions:

$$F_p[\varepsilon(x_0)\theta(x^2)] = \frac{1}{2\pi i}\int_0^\infty d\lambda\, \nabla(p,\lambda) = \frac{2}{\pi i}\frac{\partial}{\partial p^2}\nabla(p,0)$$

$$= -\frac{1}{2\pi^3}\frac{\partial}{\partial p^2}\Delta(p,0) = \frac{1}{\pi^3}\bar{\Delta}(p,0)\Delta(p,0); \qquad (A.10)$$

$$F_p[\theta(x^2)] = \frac{1}{2\pi i}\int_0^\infty d\lambda\, \nabla_1(p,\lambda) = \frac{2}{\pi i}\frac{\partial}{\partial p^2}\nabla_1(p,0)$$

$$= \frac{1}{\pi^3}\frac{\partial}{\partial p^2}\bar{\Delta}(p,0); \qquad (A.11)$$

$$F_p[\pm\theta(\pm x_0)\theta(x^2)] = \frac{1}{2\pi i}\int_0^\infty d\lambda\, \nabla^{(\pm)}(p,\lambda)$$

$$= -\frac{1}{2\pi^3}\frac{\partial}{\partial p^2}\Delta_{\substack{ret\\adv}}(p,0). \qquad (A.12)$$

Equations (A.10) to (A.12) contain ∇ functions defined by

$$\{\nabla_1(p); \nabla(p); \nabla^{(\pm)}(p)\} = \frac{i}{(2\pi)^3}\int d^4x e^{ipx}\delta(x^2-\lambda)\{1, \varepsilon(x_0),$$

$$\pm\theta(\pm x_0)\}. \qquad (A.13)$$

These are evidently the singular functions of a quadratic equation in p space, the symmetric Klein-Gordon equation:

$$(\Box_p - \lambda)\nabla(p,\lambda) \equiv (x^2-\lambda)\nabla(p,\lambda) = 0 \qquad (A.14)$$

etc. Formulas (A.7) to (A.11) can be regarded as an application of a transformation of the type of the Radon transformation,[29] since the Δ or ∇ functions, etc., are the areas of the hyperplanes in volumes determined by the characteristic functions.

The characteristic functions themselves are the fundamental solutions of the homogeneous equations $(\Box_p - m^2)^2\varphi(p) = 0$ [function $\theta(p^2 - m^2)$, etc.] and $\Box_x^2\varphi(x) = 0$ [function $\theta(x^2)$, etc.], respectively.

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