INVESTIGATION OF INFRARED SINGULARITIES OF THE SCATTERING CROSS SECTION BY THE FUNCTIONAL INTEGRATION METHOD

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Mutual scattering of high-energy scalar particles of mass m is studied by the functional integration method. The process is treated on basis of the scalar particle model with a Langrangian $L = g\colon \psi^2(x)\,\varphi(x)\colon$ where the field $\psi(x)$ has a mass m and $\varphi(x)$ has zero mass. The contribution of virtual quanta of field $\varphi(x)$ with $k^2 = 0$, which leads to infrared divergence of the elastic process amplitude, is taken into account exactly. A procedure for compensating infrared divergences is carried out outside the framework of perturbation theory, by taking into account in the cross section processes that involve the emission of an infinite number of real quanta from field $\varphi(x)$, and whose total energy does not exceed a certain value Δ.

1. INTRODUCTION

THE construction of amplitudes of processes by the method of functional integration entails two fundamental difficulties: first, finding the closed solutions for the Green's functions of particles with arbitrary external fields, and, second, carrying out the functional averaging of these solutions over the external field with corresponding weight. In addition to the foregoing fundamental difficulties, there is also a purely technical difficulty connected with the fact that the amplitude of the processes is expressed in terms of the corresponding Green's functions multiplied by coefficients proportional to the reciprocals of the propagation functions of the free fields, which vanish on the mass shell. Therefore, we should separate from the Green's functions the pole terms which cancel the indicated zeroes on the mass shell. In perturbation theory, this compensation is obvious, since the expression for the amplitude is made up of free propagation functions, but if the Green's function is sought by means of methods other than perturbation theory, the separation of the pole terms entails certain difficulties.

In the interesting paper by Milekhin and Fradkin,[1] devoted to the calculation of the scattering of Fermi particles with the aid of the method of functional integration, the problem described above was solved by breaking down the virtual photons into soft and hard photons, and using first-order perturbation theory with respect to the hard photons.

The present paper is an attempt to develop and improve the cited paper.[1] The proposed method is from the very outset more consistent, since it is based on an exact formula for the amplitude in terms of functional integrals. The approximations arise only when the functional quadratures are taken. The starting point is a method, proposed by one of the authors,[2] for formally solving equations for Green's functions in an external field with the aid of a functional integral. The Green's function of the Klein-Gordon-Dirac equations, obtained by this method, makes it possible to readily carry out the functional averaging over the external field, without carrying out initial quadratures, and to find in this manner the quantum Green's functions. Because of this, only one of the two mentioned fundamental difficulties remains: the determination of the functional quadratures arising during the solution of equations in the external field. The method proposed in [2] for approximately calculating the functional integrals is a good approximation to the true values in the infrared region of virtual quanta.

Using as a simple example the interaction of two scalar fields with Lagrangian

$$L_{\text{int}} = g : \psi^2(x)\,\varphi(x) : ,$$

where one field $\varphi(x)$ has zero mass, we have calculated the cross section for the interaction of the quanta of the field $\psi(x)$. We neglect here the effects of polarization of the vacuum of the field $\psi(x)$, which are insignificant in the infrared region. These calculations can be readily extended to include the case of quantum electrodynamics.

In Sec. 2 we present a method for determining the scattering amplitude in terms of the functional integral, and obtain with its aid the cross section for the scattering of the field $\psi(x)$. We use a procedure for canceling the infrared divergences by summing processes with emission of an infinite number of soft quanta of the field $\varphi(x)$. Section 3 is devoted to the transition to the mass shell of the momenta connected with external ends, and to an approximate calculation of the functional integrals.

The final results pertaining to the asymptotic values of the cross section are close to the results obtained earlier.[1, 3, 4]

2. TWO-PARTICLE GREEN'S FUNCTION AND DIFFERENTIAL SCATTERING CROSS SECTION IN THE MODE $L_{int} = g:\psi^2(x)\varphi(x)$:

The Green's function of two particles of the field $\psi(x)$ is connected with the single-particle function in an external field by the functional integral

$$G(x_1x_2\,|\,x_3x_4) = \int \delta\varphi \exp\left\{-\frac{i}{2}\int \varphi(\xi)D^{-1}(\xi\xi')\varphi(\xi')\,d\xi d\xi'\right\}$$
$$\times [G(x_1x_3|\varphi)G(x_2x_4|\varphi) + G(x_1x_4|\varphi)G(x_2x_3\,|\,\varphi)]S_0(\varphi), \qquad (1)$$

where $S_0(\varphi)$ is the average of the S matrix over the vacuum of the field $\psi(x)$. We shall henceforth disregard the contributions of the vacuum loops and put $S_0(\varphi) = 1$. This approximation is justified in the study of infrared singularities of the amplitudes (see, for example, [3]). The Green's function of a particle of field $\psi(x)$ in a classical field $\varphi(x)$ satisfies the equation

$$[i^2\partial_\mu{}^2 - m_0{}^2 + g\varphi(x)]G(x,\ y|\varphi) = -\delta(x-y). \qquad (2)$$

The solution of this equation in the form of a functional integral was obtained earlier[2] and is

$$G(x,y|\varphi) = i\int_0^\infty ds\, e^{-ism_0^2}\, C\int \delta^4\nu \exp\left\{-i\int_0^s d\xi\left[\nu_\mu{}^2(\xi)\right.\right.$$
$$\left.\left.- g\varphi\left(x - 2\int_\xi^s \nu(\eta)\,d\eta\right)\right]\right\}\delta^4\left(x - y - 2\int_0^s \nu(\eta)\,d\eta\right). (3)$$

We substitute (3) in (1) and integrate over φ, which can readily be done since Gaussian integrals arise. We break up the field φ into two parts: external, which leaves the dependence of the quantum Green's function, and the field over which the integration is carried out. As a result we obtain

$$G(x_1x_2|x_3x_4) = \iint_0^\infty ds_1\, ds_2\, e^{-im_0^2(s_1+s_2)}$$
$$\times \iint d^4\nu_1\, d^4\nu_2 \exp\left\{-i\int_0^{s_1}\nu_1{}^2(\xi)\,d\xi - i\int_0^{s_2}\nu_2{}^2(\xi)\,d\xi\right\}$$
$$\times\,\delta\left(x_1 - x_3 - 2\int_0^{s_1}\nu_1(\eta)\,d\eta\right)\delta\left(x_2 - x_4 - 2\int_0^{s_2}\nu_2(\eta)\,d\eta\right)$$
$$\times \exp\left\{ig\int_0^{s_1} d\xi\,\varphi\left(x_1 - 2\int_\xi^{s_1}\nu_1(\eta)\,d\eta\right)\right.$$
$$\left.+\, ig\int_0^{s_2} d\xi\,\varphi\left(x_2 - 2\int_\xi^{s_2}\nu_2(\eta)\,d\eta\right)\right\}$$
$$\times \exp\left\{+\,ig^2\int_0^{s_1} d\xi\int_0^{s_2} d\zeta\, D\left(x_1 - x_2 - 2\int_\xi^{s_1}\nu_1(\eta)\,d\eta\right.\right.$$
$$\left.\left.+\,2\int_\zeta^{s_2}\nu_2(\eta)\,d\eta\right)\right\}\exp\left\{+\,i\frac{g^2}{2}\left[\int_0^{s_1}\int_0^{s_1} d\xi\, d\zeta\, D\left(2\int_\zeta^{\xi}\nu_1(\eta)\,d\eta\right)\right.\right.$$
$$\left.\left.+\int_0^{s_2}\int_0^{s_2} d\xi\, d\zeta\, D\left(2\int_\zeta^{\xi}\nu_2(\eta)\,d\eta\right)\right]\right\} \qquad (4)$$

We go over to momentum space:

$$G(p'q'|pq) = \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, e^{ip'x_1 + iq'x_2 - ipx_3 - iqx_4}G(x_1x_2|x_3x_4). \qquad (5)$$

Making a change of integration variables $y_1 = x_3$, $y_2 = x_3 - x_4$, $y_3 = x_1 - x_3$, $y_4 = x_2 - x_4$, and also a change of functional variables $\nu_1(\eta) = \omega_1(\eta) + p'$, $\nu_2(\eta) = \omega_2(\eta) + q'$, and carrying out integration with respect to y_3 and y_4, with account of the δ-functions in (4), we get

$$G(p'q'|pq) = \int dy_1\, dy_2\, e^{i(p'+q'-p-q)y_1 + i(q-q')y_2}$$
$$\times \iint_0^\infty ds_1\, ds_2\, e^{i(p_1{}^2 - m_0{}^2)s_1 + i(q_1{}^2 - m_0{}^2)s_2}$$
$$\times \iint d^4\omega_1\, d^4\omega_2 \exp\left\{-i\int_0^{s_1}\omega_1{}^2(\xi)\,d\xi - i\int_0^{s_2}\omega_2{}^2(\xi)\,d\xi\right\}$$
$$\times \exp\left\{ig\int_0^{s_1} d\xi\,\varphi\left(y_1 + 2p'\xi + 2\int_0^{\xi}\omega_1(\eta)\,d\eta\right)\right.$$
$$\left.+\, ig\int_0^{s_2} d\xi\varphi\left(y_1 - y_2 + 2q'\xi + 2\int_0^{\xi}\omega_2(\eta)\,d\eta\right)\right\}$$
$$\times \exp\left\{+\,i\frac{g^2}{2}\left[\int_0^{s_1}\int_0^{s_1} d\xi\, d\zeta\, D\left(2p'(\xi - \zeta) + 2\int_\zeta^{\xi}\omega_1(\eta)\,d\eta\right)\right.\right.$$
$$+\int_0^{s_2}\int_0^{s_2} d\xi\, d\zeta\, D\left(2q'(\xi - \zeta) + 2\int_\zeta^{\xi}\omega_2(\eta)\,d\eta\right)$$
$$+\,2\int_0^{s_1} d\xi\int_0^{s_2} d\zeta\, D\left(y_2 + 2p'\xi - 2q'\zeta + 2\int_0^{\xi}\omega_1(\eta)\,d\eta\right.$$
$$\left.\left.\left.-\,2\int_0^{\zeta}\omega_2(\eta)\,d\eta\right)\right]\right\}. \qquad (6)$$

Using this expression for the two-particle Green's function, we can obtain the scattering matrix element by means of the well-known formula

$$(2\pi)^4\delta(p'+q'-p-q)f(p'q'|pq)$$
$$= {}^1/_4 \lim_{|p^2,q^2,p'^2,q'^2|\to m^2} (p'^2-m^2)(p^2-m^2)(q'^2-m^2)$$
$$\times (q^2-m^2)G(p'q'|pq)S_0. \qquad (7)$$

However, the amplitude $f(p'q'|pq)$ will contain infrared divergences, since the field $\varphi(x)$ has zero mass. As is well known, in the cross section for the scattering process this difficulty is circumvented by taking into account the emission of a large number of soft quanta of the field $\varphi(x)$ the total energy of which does not exceed the quantity Δ, equal to the resolving power of the measuring instrument.

The amplitude of the process of scattering with emission of n quanta of the field $\varphi(x)$ can be obtained from the obtained Green's function in the external field (6), by applying to it the operator

$$\prod_{i=1}^{n}\frac{1}{\sqrt{2\omega_i}}\frac{\delta}{\delta\varphi(k_i)}$$

and using formula (7). To take into account the identity of the emitted particles, it is necessary to introduce the factor $1/\sqrt{n!}$. As a result we arrive at the formula

$$f_n(p'q'|pq) = {}^1/_4 \lim_{p'^2,p^2,q'^2,q^2\to m^2}(p'^2-m^2)(p^2-m^2)$$
$$\times (q'^2-m^2)(q^2-m^2)\int dy e^{i(q-q')y}\int_0^\infty\!\!\int ds_1\,ds_2$$
$$\times \exp\{i(p'^2-m_0^2)s_1+i(q'^2-m_0^2)s_2\}\int\int d^4\omega_1 d^4\omega_2$$
$$\times \exp\left\{-i\int_0^{s_1}\omega_1^2(\xi)d\xi - i\int_0^{s_2}\omega_2^2(\xi)d\xi\right\}$$
$$\times \exp\left\{+i\frac{g^2}{2}\left[\int_0^{s_1}\!\!\int_0^{s_1} d\xi\,d\zeta\,D\left(2p'(\xi-\zeta)+2\int_\zeta^\xi\omega_1(\eta)d\eta\right)\right.\right.$$
$$+\int_0^{s_2}\int_0^{s_2} d\xi d\zeta D\left(2q'(\xi-\zeta)+2\int_\zeta^\xi\omega_2(\eta)d\eta\right)$$
$$+2\int_0^{s_1}d\xi\int_0^{s_2}d\zeta D\left(x+2p'\xi-2q'\zeta+2\int_0^\xi\omega_1(\eta)d\eta\right.$$
$$\left.\left.\left.-2\int_0^\zeta\omega_2(\eta)d\eta\right)\right]\right\}\frac{1}{\sqrt{n!}}\prod_{i=1}^{n}\frac{ig}{(2\pi)^{3/2}\sqrt{2\omega_i}}$$
$$\times\left\{\int_0^{s_1}d\xi_i\exp\left\{2ik_i\left[p'\xi_i+\int_0^{\xi_i}\omega_1(\eta)d\eta\right]\right\}\right.$$
$$\left.+\int_0^{s_2}d\zeta_i\exp\left\{2ik_i\left[q'\left(\zeta_i-\frac{x}{2}\right)+\int_0^{\zeta_i}\omega_2(\eta)d\eta\right]\right\}\right\}. \qquad (8)$$

With the aid of (8) we can construct the cross section of the process of interest to us:

$$d\sigma = \frac{1}{(2\pi)^2(p_0q_0p_0'q_0')}\sum_{n=0}^{\infty}\theta\left(\Delta-\sum_{i=1}^{n}\omega_i\right)d^3k_1\,d^3k_2\ldots d^3k_n$$
$$\times|f_n|^2\delta^4\left(p'+q'-p-q-\sum_{i=1}^{n}k_i\right)\frac{d^3p'\,d^3q'}{J}. \qquad (9)$$

where J is the flux of the incident particles.

Summing over n, in a manner similar to that used in [1], we obtain

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_0}{d\Omega}\left[\lim_{p^2,p'^2,q^2,q'^2\to m^2}(p^2-m^2)(p'^2-m^2)(q'^2-m^2)\right.$$
$$\times(q^2-m^2)\Bigg]^2\frac{l^4}{g^4}\int_{-\infty}^{\infty}\frac{e^{i\Delta\tau}-1}{2\pi i\tau}d\tau\int dy\,dy'\,e^{il(y-y')}$$
$$\times\int_0^\infty ds_1\,ds_2\,ds_1'\,ds_2'\,e^{i(p'^2-m_0^2)(s_1-s_1')}\,e^{i(q'^2-m_0^2)(s_2-s_2')}$$
$$\times\int\int d^4\omega_1\,d^4\omega_2\,d^4\omega_1'\,d^4\omega_2'$$
$$\times\exp\left\{-i\int_0^{s_1}d\xi(\omega_1^2(\xi)-\omega_1'^2(\xi))-i\int_0^{s_2}d\xi(\omega_2^2-\omega_2'^2)\right\}$$
$$\times\exp\{F(s_1,s_2,y)+F^*(s_1',s_2',y')-2\Phi(s_1,s_2,s_1',s_2',\tau)\}. \qquad (10)$$

Here $l = q - q'$ is the momentum transfer, and $d\sigma_0/d\Omega$ is the cross section for scattering in first order perturbation theory, while the functions F and Φ are of the form

$$F(s_1,s_2,y)=-\frac{ig^2}{2(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}$$
$$\times\left\{\int_0^{s_1}\!\!\int^{s_1} d\xi\,d\zeta\exp\left\{2ik\left[p'(\xi-\zeta)+\int_\zeta^\xi\omega_1(\eta)d\eta\right]\right\}\right.$$
$$+\int_0^{s_2}\!\!\int^{s_2} d\xi\,d\zeta\exp\left\{2ik\left[q'(\xi-\zeta)+\int_\zeta^\xi\omega_2(\eta)d\eta\right]\right\}$$
$$+2\int_0^{s_1}d\xi\int_0^{s_2}d\zeta\exp\left\{iky+2ik\left[p'\xi-q'\zeta+\int_0^\xi\omega_1(\eta)d\eta\right.\right.$$
$$\left.\left.\left.-\int_0^\zeta\omega_2(\eta)d\eta\right]\right\}\right\},$$

$$\Phi(s_1,s_2,s_1',s_2',\tau)=-\frac{g^2}{4(2\pi)^3}\int\frac{d^3k}{\omega}e^{-i\omega\tau}\left\{\int_0^{s_1}d\xi\int_0^{s_1'}d\zeta\right.$$
$$\times\exp\quad 2ik\left[p'(\xi-\zeta)+\int_0^\xi\omega_1(\eta)d\eta-\int_0^\zeta\omega_1'(\eta)d\eta\right]\Bigg\}$$
$$+\int_0^{s_1}d\xi\int_0^{s_2'}d\zeta\exp\left\{2ik\left[p'\xi-q'\zeta+\frac{y'}{2}+\int_0^\xi\omega_1(\eta)d\eta\right.\right.$$

$$-\int_0^{\zeta}\omega_2'(\eta)\,d\eta\Big]\Big\}+\int_0^{s_2}d\xi\int_0^{s_2'}d\zeta\exp\Big\{2ik\Big[q'(\xi-\zeta)+\int_0^{\zeta}\omega_2'(\eta)\,d\eta+\int_0^{\xi}\omega_2(\eta)\,d\eta\Big]\Big\}+\int_0^{s_1'}d\xi\int_0^{s_2}d\zeta$$

$$\times\exp\Big\{2ik\Big[q'\zeta-y/2-p'\xi-\int_0^{\xi}\omega_1'(\eta)\,d\eta+\int_0^{\zeta}\omega_2(\eta)\,d\eta\Big]\Big\}\Big\}.$$

The integrals with respect to k in $F(s_1,s_2,y)$ diverge, as will be shown later. To regularize them, it is necessary to renormalize the mass, that is, to separate from $F(s_1,s_2,y)$ the terms $(s_1-s_1')\delta_1 m$ and $(s_2-s_2')\delta_2 m$, after which we go over in formula (10) to the observed masses $m_r = m_0 + \delta m$ and to the renormalized function $F_r(s_1,s_2,y)$ which has the property

$$\lim_{s_1,\,s_2\to\infty} F_r(s_1,s_2,y)<\infty.$$

The value of δm will be calculated at the end of the paper.

After performing the foregoing operations we are able to use for (10) the relation

$$\lim_{p^2\to m^2} i\int_0^{\infty}ds\exp\Big\{-is(p^2-m^2)+F_r(s)\Big\}=\lim_{p^2\to m^2}\frac{1}{m^2-p^2}e^{F_r(\infty)}, \tag{11}$$

which is valid for any finite function $F_r(s)$.[1] As a result we obtain

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_0}{d\Omega}\Big[\lim_{p^2,\,q^2\to m^2}(p^2-m^2)(q^2-m^2)\Big]^2\frac{l^4}{g^4}$$

$$\times\int d\tau\frac{e^{i\Delta\tau}-1}{2\pi i\tau}\int dy\,dy'\,e^{il(y-y')}\int d\omega_1 d\omega_2\,d\omega_1' d\omega_2'$$

$$\times\exp\Big\{-i\int_0^{\infty}d\xi\,[\omega_1^2(\xi)+\omega_2^2(\xi)-\omega_1'^2(\xi)-\omega_2'^2(\xi)]\Big\}$$

$$\times\exp\{F_r(\infty,\infty,y)+F_r^*(\infty,\infty,y')-2\Phi(\infty,\tau)\}. \tag{12}$$

After mass renormalization, some infrared divergences are left in F_r and F_r^*. The function -2Φ cancels these divergences in the real part of F_r and F_r^*. The remaining infrared divergences in the imaginary part of these functions cancel one another, something which can be written in explicit form by adding to $i\,\mathrm{Im}\,F_r$ and subtracting from $i\,\mathrm{Im}\,F_r^*$ the imaginary quantity $i\Delta = -i\,\mathrm{Im}\,F_r^*(\infty,\infty,0)$. We shall henceforth put

$$\tilde{F}_r=F_r+i\Delta,\quad \tilde{F}_r^*=F_r^*-i\Delta.$$

3. TRANSITION TO THE MASS SHELL AND APPROXIMATE CALCULATIONS OF THE FUNCTIONAL INTEGRALS

The transition to the mass shell $p^2, q^2 \to m^2$ calls for separating from formula (12) the pole terms $(p^2-m^2)^{-1}$ and $(q^2-m^2)^{-1}$, which cancel the factors (p^2-m^2) and (q^2-m^2). To this end we first carry out a transformation of formula (12), equivalent to eliminating from the amplitude the zeroth order of perturbation theory, which makes no contribution to the cross section; this is done by means of the following integral representation:

$$e^{\tilde{F}_{r_2}-\Phi_2}-1=(\tilde{F}_{r_2}-\Phi_2)\int_0^1 d\alpha\, e^{\alpha(\tilde{F}_{r_2}-\Phi_2)}, \tag{13}$$

where F_{r2} and Φ_2 are those parts of the functions F_r and Φ which depend on the variables y and y′,

$$F_r=F_{r2}(y)+F_{r1},\quad \Phi=\Phi_2(\tau,y,y')+\Phi_1(\tau).$$

Using this, we arrive at the formula

$$\frac{d\sigma}{d\Omega}\cong\frac{d\sigma_0}{d\Omega}\lim_{p^2,\,q^2\to m^2}(p^2-m^2)^2(q^2-m^2)^2\frac{l^4}{(2\pi)^8}\int_{-\infty}^{\infty}d\tau\frac{e^{i\Delta\tau}-1}{2\pi i\tau}$$

$$\times\int dy\,dy'\,e^{il(y-y')}\int d^4\omega_1 d^4\omega_2 d^4\omega_1' d^4\omega_2'$$

$$\times\exp\Big\{-i\int_0 d\eta\,[\omega_1^2(\eta)+\omega_2^2(\eta)-\omega_1'^2(\eta)-\omega_2'^2(\eta)]\Big\}$$

$$\times\int\frac{d^4k}{k^2+i\varepsilon}\int\frac{d^4k'}{k'^2-i\varepsilon}e^{iky-ik'y'}\int_0^{\infty}\int d\xi' d\zeta'$$

$$\times\exp\Big\{-2ik'\Big[p'\xi'-q'\zeta'+\int_0^{\xi'}\omega_1'(\eta)\,d\eta-\int_0^{\zeta'}\omega_2'(\eta)\,d\eta\Big]\Big\}$$

$$\times\int_0^{\infty}\int d\xi d\zeta\exp\Big\{-2ik\Big[p'\xi-q'\zeta+\int_0^{\xi}\omega_1(\eta)\,d\eta-\int_0^{\zeta}\omega_2(\eta)\,d\eta\Big]\Big\}\int_0^1\int_0^1 d\alpha d\beta\exp\{\alpha\tilde{F}_{r2}(y)-\alpha\Phi_2(\tau,y,y')$$

$$+\beta[\tilde{F}_{r2}^*(y')-\Phi_2(\tau,y,y')]\}\exp\{2\,\mathrm{Re}\,F_1-2\Phi_1(\tau)\}. \tag{14}$$

In the expression presented here we have left out terms which are of no importance in the transition to the mass shell.

We integrate with respect to the variables y, y′ and k, k′, expanding the exponential in a series in the coupling constant and using the integral representation

$$\frac{1}{k^2+i\varepsilon}=\frac{i}{\pi^2}\int\frac{d^4x}{x^2-i\delta}e^{-2ikx}. \tag{15}$$

Returning after the described operation again to the exponential, we obtain

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_0}{d\Omega}\lim_{p^2,\,q^2\to m^2}(p^2-m^2)^2(q^2-m^2)^2\frac{l^4}{\pi^4}\int\limits_{-\infty} d\tau\,\frac{e^{i\Delta\tau}-1}{2\pi i\tau}$$
$$\times\int d^4x d^4x'\,\frac{e^{-2il(x-x)}}{(x^2-i\delta)(x'^2+i\delta)}\int d^4\omega_1 d^4\omega_2 d^4\omega_1' d^4\omega_2'$$
$$\times\exp\Big\{-i\int\limits_0^\infty d\eta\,[\omega_1^2(\eta)+\omega_2^2-\omega_1'^2-\omega_2'^2]\Big\}$$
$$\times\int\int\limits_0^\infty\int\int d\xi\,d\xi'\,d\zeta\,\,d\zeta'\exp\Big\{2i(q-q')$$
$$\times\Big[\,q'\zeta+\int\limits_0^\xi\omega_2(\eta)\,d\eta-q'\zeta'-\int\limits_0^{\zeta'}\omega_2'd\eta\Big]\Big\}$$
$$\times\exp\Big\{2i(p-p')\Big[\,p'\xi+\int\limits_0^\xi\omega_1 d\eta-p'\xi'-\int\limits_0^{\xi'}\omega_1'(\eta)\,d\eta\Big]\Big\}$$
$$\times\int\limits_0^1\int\limits_0^1 d\alpha\,d\beta\exp\{\alpha[\bar F_{r2}'(x)-\Phi_2'(\tau)]\}$$
$$\times\exp\{\beta[\bar F_{r2}'^*(x')-\Phi_2'(\tau)]\}\exp\{2[\mathrm{Re}\,F_1'-\Phi_1'(\tau)]\},\qquad(16)$$

where

$$F_1'=-i\frac{g^2}{2(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}\int\int\limits_0^\infty d\xi\,d\zeta\Big[\,\exp\Big\{i2k\Big[\,p'(\xi-\zeta)$$
$$+\int\limits_\zeta^\xi\omega_1(\eta)\,d\eta\Big]\Big\}+\exp\Big\{i2k\Big[\,q'(\xi-\zeta)+\int\limits_\zeta\omega_2(\eta)\,d\eta\Big]\Big\}\Big]$$
$$+\,i\delta_1 m^2 s_1+i\delta_2 m^2 s_2,$$

$$\bar F_2'(x)=-i\frac{g^2}{(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}\int\int\limits_0^\infty d\xi'\,d\zeta'$$
$$\times\exp\Big\{i2k\Big[-x+p'(\xi'-\xi)-q'(\zeta'-\zeta)$$
$$+\int\limits_\xi^{\xi'}\omega_1 d\eta-\int\limits_\zeta^{\zeta'}\omega_2 d\eta\Big]\Big\},$$

$$\Phi_1'(\tau)=-\frac{g^2}{4(2\pi)^3}\int\frac{d^3k}{\omega}e^{-i\tau\omega}\int\int\limits_0^\infty d\xi''\,d\zeta''$$
$$\times\Big[\,\exp\Big\{i2k\Big[\,p'(\xi''-\xi)-p'(\zeta''-\xi')+\int\limits_\xi^{\xi''}\omega_1 d\eta$$
$$-\int\limits_{\xi'}^{\zeta''}\omega_1'\,d\eta\,\Big]\Big\}+\exp\Big\{i2k\Big[\,q'(\xi''-\zeta)-q'(\zeta''-\zeta')$$
$$+\int\limits_\zeta^{\xi''}\omega_2\,d\eta-\int\limits_{\zeta'}^{\zeta''}\omega_2'\,d\eta\,\Big]\Big\}\Big],$$

$$\Phi_2'(\tau)=-\frac{g^2}{4(2\pi)^3}\int\frac{d^3k}{\omega}e^{-i\tau\omega}\int\int\limits_0^\infty d\xi''\,d\zeta''$$
$$\times\Big[\,\exp\Big\{i2k\Big[\,p'(\xi'-\xi)-q'(\zeta''-\zeta')+\int\limits_\xi^{\xi''}\omega_1\,d\eta$$
$$-\int\limits_{\zeta'}^{\zeta''}\omega_2'\,d\eta\,\Big]\Big\}+\exp\Big\{i2k\Big[-p'(\xi''-\xi')+q'(\zeta''-\zeta)$$
$$-\int\limits_{\xi'}^{\xi''}\omega_1'\,d\eta+\int\limits_\zeta^{\zeta''}\omega_2\,d\eta\Big]\Big\}\Big].$$

Making in (16) a transformation of the functional variables ($l'=p'-p$)

$$\omega_1(\eta)=\lambda_1(\eta)-l'\theta(\xi-\eta),\;\;\omega_2(\eta)=\lambda_2(\eta)+l\theta(\zeta-\eta),$$
$$\omega_1'(\eta)=\lambda_3(\eta)-l'\theta(\xi'-\eta),\;\;\omega_2'(\eta)=\lambda_4(\eta)+l\theta(\zeta'-\eta),\qquad(17)$$

we arrive at integrals of the type

$$\int\limits_0^\infty d\xi\exp\{i(p^2-p'^2)\xi+\psi(\xi)\},\qquad(18)$$

for which formula (11) is applicable when $p^2\to p'^2$, since $\psi(\xi)$ is a finite function. This procedure separates the remaining pole terms containing the factors $(p^2-m^2)^2$ and $(q^2-m^2)^2$. We ultimately obtain the cross section in the form

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_0}{d\Omega}\frac{l^4}{\pi^4}\int\limits_{-\infty}^\infty\frac{e^{i\Delta\tau}-1}{2\pi i\tau}d\tau\int dy_1\,dy_2\,\frac{e^{-i2l(y_1-y_2)}}{(y_1^2-i\varepsilon)(y_2^2+i\varepsilon)}$$
$$\times\int d^4\lambda_1\,d^4\lambda_2\,d^4\lambda_3\,d^4\lambda_4\exp\Big\{-i\int\limits_0^\infty d\eta\,[\lambda_1^2+\lambda_2^2-\lambda_3^2-\lambda_4^2]\Big\}$$
$$\times\int\limits_0^1\int\limits_0^1 d\alpha\,d\beta\exp\{2(\mathrm{Re}\,F_{r1}''-\Phi_1'')\}$$
$$\times\exp\{\alpha(F_{r2}''(y_1)-\Phi_2'')+\beta(F_{r2}''^*(y_2)-\Phi_2'')\},\qquad(19)$$

where ($A,s_1,s_2\to\infty$)

$$F_{r1}''=-i\frac{g^2}{2(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}\int\int\limits_{-\infty}^\infty d\xi\,d\zeta$$
$$\times\Big[\,\exp\Big\{i2k\Big([p'\theta(\xi)+p\theta(-\xi)]\xi-[p'\theta(\zeta)+p\theta(-\zeta)]\zeta$$
$$+\int\limits_{\zeta+A}^{\xi+A}\lambda_1(\eta)\,d\eta\Big)\Big\}+\exp\Big\{i2k\Big([q'\theta(\xi)+q\theta(-\xi)]\,\xi$$
$$-[q'\theta(\zeta)+q\theta(-\zeta)]\,\zeta+\int\limits_{\zeta+A}^{\xi+A}\lambda_2(\eta)\,d\eta\Big)\Big\}\Big]$$
$$+\,i\delta_1 m^2 s_1+i\delta_2 m^2 s_2,$$

$$F_{r2}''(y_1)=-i\frac{g^2}{(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}\int\int\limits_{-\infty}^\infty d\xi d\zeta$$
$$\times\exp\Big\{\,i2k\Big(-y_1+[p'\theta(\xi)+p\theta(-\xi)]\xi$$

$$-[q'\theta(\zeta)+q\theta(-\zeta)]\zeta+\int_A^{\xi+A}\lambda_1(\eta)\,d\eta-\int_A^{\zeta+A}\lambda_2(\eta)\,d\eta\Big)\Big\},$$

$$\Phi_1''(\tau)=-\frac{g^2}{4(2\pi)^3}\int\frac{d^3k}{\omega}e^{-i\tau\omega}\int_{-\infty}^{\infty}\int d\xi\,d\zeta\Big[\exp\Big\{i2k\Big([p'\theta(\xi)$$

$$+p\theta(-\xi)]\xi-[p'\theta(\zeta)+p\theta(-\zeta)]\zeta+\int_A^{\xi+A}\lambda_1\,d\eta+\int_A^{\zeta+A}\lambda_3\,d\eta\Big)\Big\}$$

$$+\exp\Big\{i2k\Big([q'\theta(\xi)+q\theta(-\xi)]\xi-[q'\theta(\zeta)+q\theta(-\zeta)]\zeta$$

$$+\int_A^{\xi+A}\lambda_2\,d\eta+\int_A^{\zeta+A}\lambda_4\,d\eta\Big)\Big\}\Big],$$

$$\Phi_2''(\tau)=-\frac{g^2}{4(2\pi)^3}\int\frac{d^3k}{\omega}e^{-i\tau\omega}\int_{-\infty}^{\infty}\int d\xi\,d\zeta\Big[\exp\Big\{i2k\Big([p'\theta(\xi)$$

$$+p\theta(-\xi)]\xi-[q'\theta(\zeta)+q\theta(-\zeta)]\zeta+\int_A^{\xi+A}\lambda_1\,d\eta+\int_A^{\zeta+A}\lambda_4\,d\eta\Big)\Big\}$$

$$+\exp\Big\{i2k\Big([q'\theta(\xi)+q\theta(-\xi)]\xi-[p'\theta(\zeta)+p\theta(-\zeta)]\zeta$$

$$+\int_A^{\xi+A}\lambda_2\,d\eta+\int_A^{\zeta+A}\lambda_3\,d\eta\Big)\Big\}\Big]$$

The next step is the calculation of the functional integrals with respect to $\lambda_1(\eta)$, $\lambda_2(\eta)$, $\lambda_3(\eta)$ and $\lambda_4(\eta)$, but an exact calculation is impossible, so that we obtain an approximate expression. In [2] there was developed a method for approximately calculating similar integrals. It is based on the formula

$$C\int\delta^4\nu\exp\Big\{-i\int_0^s\nu^2(\eta)\,d\eta\Big\}e^{F(s,\nu)}=e^{\overline{F(s)}}C\int\delta^4\nu$$
$$\times\exp\Big\{-i\int_0^s\nu^2(\eta)\,d\eta\Big\}\sum_{n=0}^{\infty}\frac{(F-\overline{F})^n}{n!},\tag{20}$$

where

$$\overline{F}(s)=C\int\delta^4\nu\exp\Big\{-i\int_0^s\nu^2(\eta)\,d\eta\Big\}F(s,\nu).$$

Confining ourselves to the first term in the expression (20), we get

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_0}{d\Omega}\frac{l^4}{\pi^4}\int_{-\infty}^{\infty}d\tau\frac{e^{i\Delta\tau}-1}{2\pi i\tau}\int\int dy_1\,dy_2\frac{e^{-i2l(y_1-y_2)}}{(y_1^2-i\delta)(y_2^2+i\delta)}$$
$$\times\int_0^1\int_0^1 d\alpha\,d\beta\exp\Big\{\alpha[\overline{F}_2(y_1)-\overline{\Phi}_2]+\beta[\overline{F}_2^*(y_2)-\overline{\Phi}_2]$$
$$+2(\mathrm{Re}\,\overline{F}_1-\overline{\Phi}_1)\Big\},\tag{21}$$

where

$$\overline{F}_1=-i\frac{g^2}{2(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}$$
$$\times\Big\{\Big[\frac{1}{k^2+2kp+i\varepsilon}-\frac{1}{k^2+2kp'+i\varepsilon}\Big]^2$$
$$+\Big[\frac{1}{k^2+2kq+i\varepsilon}-\frac{1}{k^2+2kq'+i\varepsilon}\Big]^2\Big\},$$

$$\overline{F}_2(y_1)=i\frac{g^2}{(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}e^{-i2ky_1}\Big[\frac{1}{k^2-2kp+i\varepsilon}$$
$$+\frac{1}{k^2+2kp'+i\varepsilon}\Big]\Big[\frac{1}{k^2+2kq+i\varepsilon}+\frac{1}{k^2-2kq'+i\varepsilon}\Big],$$

$$\overline{\Phi}_1(\tau)=-\frac{g^2}{2(4\pi)^3}\int\frac{d^2k}{\omega}e^{-i\tau\omega}\Big\{\Big(\frac{1}{kp'}-\frac{1}{kp}\Big)^2$$
$$+\Big(\frac{1}{kq'}-\frac{1}{kq}\Big)^2\Big\},$$

$$\overline{\Phi}_2(\tau)=-\frac{g^2}{(4\pi)^3}\int\frac{d^3k}{\omega}e^{-i\tau\omega}\Big(\frac{1}{kp'}-\frac{1}{kp}\Big)\Big(\frac{1}{kq'}-\frac{1}{kq}\Big).$$

Here we have already separated from $\overline{F}_1$ in explicit fashion the terms responsible for the mass renormalization. They are equal to

$$\delta_1m^2=i\frac{g^2}{2(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}\Big[\frac{1}{k^2+2kp+i\varepsilon}$$
$$+\frac{1}{k^2-2kp+i\varepsilon}+\frac{1}{k^2+2kp'+i\varepsilon}+\frac{1}{k^2-2kp'+i\varepsilon}\Big],$$

$$\delta_2m^2=i\frac{g^2}{2(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}\Big[\frac{1}{k^2+2kq+i\varepsilon}+\frac{1}{k^2-2kq+i\varepsilon}$$
$$+\frac{1}{k^2+2kq'+i\varepsilon}+\frac{1}{k^2-2kq'+i\varepsilon}\Big]\tag{22}$$

The next approximation, which is permissible in the investigation of the asymptotic behavior of the cross section for high energies of the scattering particles, consists in neglecting the dependence of F on the variable y. In this approximation, after integrating with respect to k and τ, the cross section takes the form

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_0}{d\Omega}\int_0^1\int_0^1 d\alpha\,d\beta$$
$$\times\frac{\exp\{(\alpha+\beta)[A_1\ln(m/\Delta)+A_2]+A_1'\ln(m/\Delta)+A_2'\}}{\Gamma[1-(\alpha+\beta)A_1-A_1']},\tag{23}$$

where

$$A_1=\frac{g^2}{(2\pi)^2}\Big[\frac{1}{\sqrt{(pq')^2-m^4}}\ln\frac{\sqrt{pq'+m^2}+\sqrt{pq'-m^2}}{m\sqrt{2}}$$
$$-\frac{1}{\sqrt{(pq)^2-m^4}}\ln\frac{\sqrt{pq+m^2}+\sqrt{pq-m^2}}{m\sqrt{2}}\Big],$$

$$A_1' = \frac{g^2}{(2\pi)^2}\left[-\frac{1}{2m^2} + \frac{1}{\sqrt{(pp')^2 - m^4}} \times \ln\frac{\sqrt{pp'+m^2}+\sqrt{pp'-m^2}}{m\sqrt{2}}\right],$$

$$A_2 = \frac{g^2}{(4\pi)^2}\left\{\frac{1}{\sqrt{(pq')^2-m^4}}\left[\ln\frac{2(pq'+m^2)}{m^2} \times \ln\frac{\sqrt{pq'+m^2}+\sqrt{pq'-m^2}}{m\sqrt{2}} + \Phi\left(\frac{1}{2}\left[1-\left(\frac{pq'-m^2}{pq'+m^2}\right)^{1/2}\right]; \frac{1}{2}\left[1+\left(\frac{pq'-m^2}{pq'+m^2}\right)^{1/2}\right]\right)\right] - \frac{1}{\sqrt{(pq)^2-m^4}} \times\left[\ln\frac{2(pq-m^2)}{m^2}\ln\frac{\sqrt{pq+m^2}+\sqrt{pq-m^2}}{m\sqrt{2}} + \Phi\left(\frac{1}{2}\left[1-\left(\frac{pq+m^2}{pq-m^2}\right)^{1/2}\right]; \frac{1}{2}\left[1+\left(\frac{pq+m^2}{pq-m^2}\right)^{1/2}\right]\right)\right]\right\},$$

$$A_2' = \frac{g^2}{(4\pi)^2}\frac{1}{\sqrt{(pp')^2-m^4}}\left[\ln\frac{2(pp'+m^2)}{m^2} \times \ln\frac{\sqrt{pp'+m^2}+\sqrt{pp'-m^2}}{m\sqrt{2}} + \Phi\left(\frac{1}{2}\left[1-\left(\frac{pp'-m^2}{pp'+m^2}\right)^{1/2}\right]; \frac{1}{2}\left[1+\left(\frac{pp'-m^2}{pp'+m^2}\right)^{1/2}\right]\right)\right],$$

$$\Phi(x, y) = \int_x^y \frac{dt}{t}\ln|t-1|.$$

The final expression for the cross section (23) differs little from the result of Milekhin and Fradkin.[1] Indeed, we apply the mean-value theorem to the integral with respect to α and β, which leads to a good approximation, since the integrand varies slowly in the integration region (A_1 and A_2 are small when $p^2 \gg m^2$). We can readily see with its aid that these two results practically coincide.

CONCLUSION

The method proposed in this paper makes it possible to find the exact differential cross section of processes in which particles with zero mass participate. The cross section is free of infrared divergences and is expressed in the form of functional integrals. In the calculation of the latter we used approximations that differ from the usual perturbation theory, making it possible to obtain, even during the first stage, the correct asymptotic behavior of the cross sections at large energies of the scattering particles. The corrections of higher orders, as shown earlier,[2] make no essential contributions to the asymptotic behavior.

For further progress in the region of interest to us, it is essential to develop a method for approximately calculating functional integrals with allowance for the vacuum polarization. The method, which we have demonstrated with a simple model of scalar particles, can be readily extended to the real case of electrodynamics.

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