RELATION BETWEEN ASYMPTOTIC VALUES OF THE TOTAL CROSS SECTION AND THE RATIOS OF THE IMAGINARY TO REAL PARTS OF THE AMPLITUDE FOR ELASTIC SCATTERING THROUGH ZERO ANGLE

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Submitted to JETP editor August 9, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) 50, 672–679 (March, 1966)

A number of upper and lower bounds of the asymptotic values of the total cross section are obtained as functions of the asymptotic value of the ratio of the imaginary and real parts of the elastic scattering amplitude. It is shown that if the amplitude is asymptotically not purely imaginary, the cross section increases according to a power law if the real part is positive and decreases according to a power law if the real part is negative. If the elastic-scattering amplitude is asymptotically purely imaginary, the cross section may increase or decrease only at a slower rate than that at which the energy in any degree may vary.

KHURI and Kinoshita[1] have shown, using Meĭman's theorem,[2] that a close connection exists between the asymptotic behaviors (for large E, where E is the energy in the laboratory system) of the symmetrical amplitudes $f(E) = f_+(E) + f_-(E)$ and the ratio of its imaginary and real parts:

$$\xi(E) = \operatorname{Im} f(E) / \operatorname{Re} f(E). \qquad (1)$$

(Here $f_+(E)$ and $f_-(E)$ are respectively the zero-scattering amplitudes of the particle and antiparticle.) In this paper this connection is investigated in greater detail.

Khuri and Kinoshita have shown, in particular, that the Greenberg-Low inequality[3] [1)]

$$|f(E)| \leqslant CE^2(\ln E)^2 \qquad (2)$$

(C is a constant), which is based only on the most general principles of the theory, can be strengthened to[2)]

$$|f(E)| \leqslant CE^{2-\alpha/2}(\ln E)^2, \qquad (2a)$$

if

$$|\xi| \geqslant \operatorname{tg} \pi\alpha, \quad 0 < \alpha \leqslant 1/2. \qquad (3)^*$$

We shall show that this result can be greatly improved, and that the limitations on the scattering amplitude are obtained for certain very natural physical assumptions, not only from above but also from below. In particular, it can be stated that, subject to the usual assumptions concerning the analytic properties of f(E) and satisfaction of inequality (3), if we exclude the case of strong decrease of the cross section, the following double inequality holds

$$CE^{2\alpha-\varepsilon} \leqslant |f(E)| \leqslant CE^{2-2\alpha+\varepsilon}(\ln E)^2, \qquad (4)$$

where C is a constant and $\epsilon > 0$ is arbitrarily small.[3)]

To obtain the left side of inequality (4) we assume that if the total cross section does decrease with increasing energy, it does so more slowly than the reciprocal of the first power of the energy

$$\sigma(E) \geqslant C / E^{1+2\alpha-\varepsilon},$$

where $\epsilon > 0$ and is infinitesimally small. This assumption seems to be quite likely in light of the available experimental data. Making the assumption (3) concerning ξ(E) more concrete, we obtain more definite limitations on f(E) and by the same token on the asymptotic behavior of the total cross section:

a) if Re f(E) has in the asymptotic expression a definite sign and the inequality (29) (see below) is satisfied, then formulas (30) and (33) are valid respectively for Re f(E) > 0 and Re f(E) < 0.

1)Here and throughout, the equalities and inequalities are satisfied if E is sufficiently large or for arbitrary E.

2)To simplify the notation, we denote all the constants by a single letter C.

*tg ≡ tan.

3)As graciously reported to the author by Professor Meĭman, he independently proved the right side of the double inequality (4).

b) if $\xi(E)$ tends to a definite limit (see (34) below), then formulas (35a)–(39a) hold.

Thus, if we exclude the possibility of a strong decrease in the cross section (formulas (38) and (39)), then in the case of the pure imaginary amplitudes, $\alpha = {}^1/_2$, we get

$$CE^{1-\varepsilon} \leqslant |f(E)| \leqslant CE^{1+\varepsilon}(\ln E)^2. \quad (4a)$$

In addition, we can prove the inverse theorem (41) and (42).

We derive first one auxiliary relation (see formula (9)). Let

$$|f(E)| \leqslant CE^{2-\lambda}(\ln E)^2. \quad (5)$$

f(E) has the usual analytic properties and the crossing-symmetry properties (for arbitrary E):

$$f(-E-i0) = f(E+i0), \quad f^*(E-i0) = f(E+i0). \quad (6)$$

Consequently, we also have

$$f^*(-E+i0) = f(E+i0). \quad (7)$$

Generalizing the method of Khuri and Kinoshita[1] we construct an auxiliary function

$$\omega(E) = \left(1 - i\,\mathrm{tg}\frac{\pi\lambda}{2}\right)E^{-2+\lambda}\left(\ln E - \frac{i\pi}{2}\right)^{-\gamma} f(E), \quad \gamma > 2. \quad (8)$$

$\omega(E)$ is analytic in the upper half-plane (everywhere except a semicircle of finite radius, which is immaterial). It is obvious that in accordance with (5) (since $\gamma > 2$) we have $\omega(E) \to 0$, if $E \to \pm\infty$. Then, however, in accordance with the Phragmen-Lindelof theorem[2] we have $\omega(E) \to 0$ if $|E| \to \infty$ everywhere in the upper half-plane. In the dispersion relations[1] we can prove that f(E) has no zeroes in the upper half-plane when $|E|$ is sufficiently large. Therefore also $\omega(E) \neq 0$ for sufficiently large $|E|$. It can be readily seen, taking (7) into account, that

$$\omega^*(-E+i0) = \omega(E+i0). \quad (9)$$

Relation (9) shows that $\omega(E)$ has the same properties as f(E) when E + i0 is replaced by −E + i0. This circumstance will be consistently used in what follows.

From (8) and (9) we readily obtain

$$\left|\frac{\mathrm{Im}\,\omega(E)}{\mathrm{Re}\,\omega(E)}\right| = \left|\frac{\mathrm{Im}\,\omega(-E)}{\mathrm{Re}\,\omega(-E)}\right| = \left|\frac{\xi(E) - \mathrm{tg}(\pi\lambda/2)}{1+\xi(E)\,\mathrm{tg}(\pi\lambda/2)}\right|. \quad (10)$$

The latter expression is written out accurate to terms $1/\ln E \to 0$ as $E \to \infty$ (by E and −E are meant E + i0 and −E + i0).

We now proceed to prove formula (4), assuming relation (3) to be satisfied. We first prove that

$$|f(E)| \leqslant CE^{2-2\alpha+\varepsilon}(\ln E)^2,$$

where $\epsilon > 0$ is arbitrarily small. Let

$$|f(E)| \leqslant CE^{2-\lambda}(\ln E)^2 \equiv CE^{2-2\alpha+2\alpha-\lambda}(\ln E)^2, \quad (11)$$

$$2\alpha > \lambda \geqslant \alpha/2. \quad (12)$$

From (10) and (3), taking (12) into account, we readily obtain

$$\left|\frac{\mathrm{Im}\,\omega(E)}{\mathrm{Re}\,\omega(E)}\right| = \left|\frac{\xi(E) - \mathrm{tg}(\pi\lambda/2)}{1+\xi(E)\,\mathrm{tg}(\pi\lambda/2)}\right| \geqslant \frac{\mathrm{tg}\,\pi\alpha - \mathrm{tg}(\pi\lambda/2)}{1+\mathrm{tg}\,\pi\alpha\,\mathrm{tg}(\pi\lambda/2)} = \mathrm{tg}\,\pi\left(\alpha - \frac{\lambda}{2}\right). \quad (13)$$

We use further Meĭman's theorem,[2] which states that if a function $\varphi(E)$ which is analytic in the upper half-plane, satisfies the conditions

$$\varphi^*(-E+i0) = \varphi(E+i0), \quad |\xi(E)| \geqslant \mathrm{tg}\,\pi\varkappa,$$

$$0 < \varkappa \leqslant {}^1/_2,$$

and if $\varphi(E) \to 0$ when $|E| \to \infty$, then

$$|\varphi(E)| \leqslant CE^{-\varkappa/2} \quad (14)$$

(C is a constant). In addition, $\varphi(E)$ must have no zeroes in the upper half-plane when $|E|$ is sufficiently large.

It is easily seen that $\omega(E)$ satisfies all the conditions of Meĭman's theorem, then, applying formula (14) to $\omega(E)$ and using (13), we obtain[4)]

$$|\omega(E)| \leqslant CE^{-(\alpha-\lambda/2)/2}. \quad (15)$$

Consequently, if it is known with respect to f(E) that (11) is satisfied, then, using (15), we find that the upper bound of the asymptotic expression for f(E) drops to

$$|f(E)| \leqslant CE^{2-\lambda_1}(\ln E)^2, \quad \lambda_1 = 2\alpha - {}^3/_4(2\alpha - \lambda). \quad (16)$$

The upper bound of f(E) can be lowered even more by systematically applying the same procedure. We construct for this purpose a new function:

$$\omega_1 = f(E)\left(1 - i\,\mathrm{tg}\frac{\pi\lambda_1}{2}\right)E^{-2+\lambda_1}\left(\ln E - \frac{i\pi}{2}\right)^{-\gamma}, \quad \gamma > 2, \quad (8a)$$

which also satisfies the conditions of the Meĭman theorem. Applying (14) again, we obtain

4)As noted in[1], it is necessary to impose on $\varphi(E)$ one more limitation: $\varphi(E)$ does not satisfy the Meĭman theorem if $|\varphi(E)|$ as $E \to \infty$ can oscillate an infinite number of times with amplitude, the ratio of which to the minimum of $|\varphi(E)|$ in the corresponding interval of E becomes arbitrarily large. We assume that no such oscillations occur in $\omega(E)$.

$$|f(E)| \leqslant CE^{2-2\alpha+(3/4)^2(2\alpha-\lambda)}(\ln E)^2. \quad (16a)$$

Continuing this process n times, we obtain

$$|f(E)| \leqslant CE^{2-2\alpha+(3/4)^n(2\alpha-\lambda)}(\ln E)^2 \rightarrow CE^{2-2\alpha+\varepsilon}(\ln E)^2, \quad (17)$$

where for sufficiently large n the value of ϵ can be taken arbitrarily small. (We cannot put $\epsilon = 0$, since terms of order $\sim 1/\ln E$ are not taken into account in (10).)

In the particular case when $\alpha = 1/2$, that is, $|\xi| \rightarrow \infty$, we obtain

$$|f(E)| \leqslant CE^{1+\varepsilon}(\ln E)^2. \quad (17a)$$

This is the so-called Froissart threshold,[4] with the only difference that in place of E we have $E^{1+\epsilon}$, where ϵ is arbitrarily small.

We note that it is not clear how to improve the threshold (17a). However, if in place of (17a) we take a limitation that is apparently equivalent to it physically

$$|f(E)| \leqslant CE(\ln E)^k, \quad (18)$$

where $k > 0$ is arbitrary, then this limitation can already be made stronger. Repeating for arbitrary k the proofs given in [1] for $k = 2$, we find, as in [1]:

a) $|f(E)| \leq CE(\ln E)^{-k'}$ for any $k' > 0$, if

$$|\xi(E)| \leqslant (\ln E)^{+\alpha}, \quad 0 < \alpha < 1.$$

If f(E) has asymptotically the form

$$f(E) = C(\ln E)^k(\ln\ln E)^\delta(\ln\ln\ln E)^\lambda\ldots,$$

where δ and λ are arbitrary, then

b) $|f(E)| \leq CE(\ln E)^{2\alpha/\pi}$, if

$$\xi(E) \sim a^{-1}\ln E, \quad 0 < a \leqslant 1/2\pi k;$$

c) $|f(E)| \leq CE(\ln E)^{\epsilon}$, where ϵ is arbitrarily small, if

$$|\xi(E)\ /\ln E| \rightarrow \infty.$$

We note that if in the latter case we assume that

$$f(E) = C(\ln E)^{-k}(\ln\ln E)^\delta(\ln\ln\ln E)^\lambda\ldots,$$

where $k > 0$ (arbitrary) and δ and λ are arbitrary, then, following the method of [1] (theorem III), we obtain $k = \epsilon$, where ϵ is arbitrarily small. In other words, if $|\xi(E)/\ln E| \rightarrow \infty$ and if in addition (18) holds true, then

$$CE(\ln E)^{-\varepsilon} \leqslant |f(E)| \leqslant C(\ln E)^\varepsilon E. \quad (19)$$

Since ϵ cannot be set equal to zero, we cannot state that the obtained limitation is equivalent to the Froissart limitation $|f(E)| \leq CE(\ln E)^2$. Physically, the limitation (17a), as noted above, is apparently equivalent to (18). As is well known, the Froissart threshold was obtained using much stronger assumptions than employed in the present work, namely, Froissart started from double dispersion relations (see Martin's discussion[5]).

We proceed to prove the left side of the double inequality (4)

$$|f(E)| \geqslant CE^{2\alpha-\varepsilon}. \quad (20)$$

We assume that the inverse inequality holds

$$|f(E)| \leqslant CE^{2\alpha-\varepsilon}, \quad (21)$$

and prove that it follows from (21) that

$$|f(E)| \leqslant CE^{-2\alpha+\varepsilon}. \quad (22)$$

Indeed, let

$$|f(E)| \leqslant CE^{2\alpha'}, \quad -2\alpha+\varepsilon \leqslant 2\alpha' \leqslant 2\alpha-\varepsilon. \quad (23)$$

Formula (21) corresponds to $2\alpha' = 2\alpha - \epsilon$, while formula (5) corresponds to $\lambda = 2 - 2\alpha'$. We can again form a function $\omega(E)$ in accordance with (8) (but now assuming that $\gamma > 0$ and not $\gamma > 2$, since the factor $(\ln E)^2$ in (23) is missing). Since $|\xi(E)| \geq \tan \pi\alpha$, we can now readily prove by taking (23) into account that

$$\left|\frac{\xi(E)+\operatorname{tg}\pi\alpha'}{1+\xi(E)\operatorname{tg}\pi\alpha'}\right| \geqslant \operatorname{tg}\pi(\alpha-\tilde{\alpha}') \geqslant \operatorname{tg}\frac{\pi\varepsilon}{2},$$

$$\tilde{\alpha}' \equiv |\alpha'|. \quad (24)$$

According to (14), it follows from (24) that

$$|\omega(E)| \leqslant CE^{-\varepsilon/4}, \quad (25)$$

that is, we obtain in lieu of (23)

$$|f(E)| \leqslant CE^{2\alpha'-\varepsilon/4}. \quad (26)$$

This result (26) is valid for arbitrary $\alpha' > -\alpha + \epsilon/2$ regardless of the value of α'. Since ϵ is a definite number (although we can make it as small as desired), we obtain the required inequality (22) by constructing the functions $\omega(E)$ a sufficient number of times.

However, if $|f(E)| \leq CE^{-2\alpha/\epsilon}$, then the total cross section $\sigma(E)$ decreases rapidly with energy like $\sim C/E^{1+2\alpha-\epsilon}$. The total cross section is apparently in sharp contrast to experiment. Assuming that it is not realized, we find that inequality (21) is impossible. Consequently, either f(E) oscillates, becoming alternately larger and smaller than $CE^{2\alpha-\epsilon}$, or else we have at all times $|f(E)| \leq CE^{2\alpha-\epsilon}$. Now, using the factual information on the cross section, we shall show that the first possibility is in practice forbidden.

Indeed, the experimental data on the cross section apparently do not admit of a more rapid decrease of the cross section (faster than 1/E).

Consequently, we can assume that

$$|f(E)| \geqslant CE^{-2\alpha+\varepsilon}. \tag{27}$$

Then $|f(E)|^{-1} \leq C'E^{2\alpha-\epsilon}$, with 1/f(E) having the same analytic properties and crossing-symmetry properties as f(E). Therefore, repeating for 1/f(E) the reasoning which led to (22) for f(E), we obtain

$$|f(E)|^{-1} \leqslant C'E^{-2\alpha+\varepsilon}, \qquad |f(E)| \geqslant CE^{2\alpha-\varepsilon}. \tag{28}$$

Thus, from the limitation imposed by the experiment there follows again the left-side inequality in (4). This completes the proof of (4).

In order to obtain stronger limitations for $|f(E)|$ than given in (4), it is necessary to make more concrete assumptions concerning the behavior of $\xi(E)$, namely, we must assume in addition to the lower bound also the existence of an upper bound for $|\xi(E)|$. We therefore assume that

$$\operatorname{tg}\pi\nu \geqslant |\xi(E)| \geqslant \operatorname{tg}\pi\alpha, \quad 0 < \alpha \leqslant 1/2, \quad 0 < \nu \leqslant 1/2. \tag{29}$$

It is necessary here to distinguish between the two possible cases Re f(E) > 0 and Re f(E) < 0. We cannot strengthen the inequality (4) if Re f(E) can reverse sign an infinite number of times when $E \to \infty$.

1. Re f(E) > 0. We can prove here that

$$CE^{2-2\nu-\varepsilon} \leqslant |f(E)| \leqslant CE^{2-2\alpha+\varepsilon}(\ln E)^2. \tag{30}$$

The right-side inequality was already proved (see (4)). As to the left-side inequality, we assume the opposite inequality and prove that we should then obtain

$$|f(E)| \leqslant CE^{-2\alpha+\varepsilon}. \tag{31}$$

The latter would mean a rapid decrease of the cross section, which we have agreed to regard as impossible.

Indeed, by constructing in the usual manner the auxiliary function $\omega(E)$ (8) and using (29), we readily obtain from (10) for the function f(E) satisfying the condition $|f(E)| \leq CE^{2-\lambda}$, $\lambda \geq 2\nu + \epsilon$,

$$\left|\frac{\operatorname{Im}\omega(E)}{\operatorname{Re}\omega(E)}\right| \geqslant \operatorname{tg}\pi\frac{\varepsilon}{2}, \quad \text{if} \quad \frac{\lambda}{2} \leqslant 1+\alpha-\frac{\varepsilon}{2}.$$

Using formula (14) we get

$$|f(E)| \leqslant CE^{2-\lambda-\varepsilon/4}. \tag{32}$$

This result (32) is valid for arbitrary λ, provided $2 + 2\alpha - \epsilon \geq \lambda \geq 2\nu + \epsilon$. Inasmuch as ϵ is a defined although arbitrarily small number, repeating for a sufficient number of times the construction of the function $\omega(E)$ (and lowering each time the degree of $|f(E)|$ by $\epsilon/4$), we obtain (31) as a result. Consequently, the inequality $|f(E)| \leq CE^{2-2\nu-\epsilon}$ is also inadmissible, and by excluding the possibility of strong oscillations, as in the derivation of (20), we arrive at the required formula (30).

2. Re f(E) < 0. In this case we can greatly strengthen the right-side of inequality (4). Namely, we prove that

$$CE^{2\alpha-\varepsilon} \leqslant |f(E)| \leqslant CE^{2\nu+\varepsilon}(\ln E)^2. \tag{33}$$

For the proof we construct $\omega(E)$ in accordance with (8):

$$\left|\frac{\operatorname{Im}\omega(E)}{\operatorname{Re}\omega(E)}\right| = \left|\frac{\tilde{\xi}(E)+\operatorname{tg}(\pi\lambda/2)}{1-\tilde{\xi}(E)\operatorname{tg}(\pi\lambda/2)}\right| \geqslant \operatorname{tg}\frac{\pi\varepsilon}{2},$$

so long as $\lambda/2 < 1 - \nu - \epsilon/2$ regardless of the value of λ; ϵ is arbitrarily small and $\tilde{\xi}(E) = -\xi(E) > 0$. Using the customary reasoning, we obtain $|f(E)| \leq CE^{2\nu+\epsilon}$, that is, formula (33).

Finally, let us consider also a more definite form of $\xi(E)$. Namely, let $\xi(E)$ have the limit

$$\lim_{E\to\infty} \xi(E) = \operatorname{tg}\pi\alpha, \qquad |\alpha| \leqslant 1/2. \tag{34}$$

In particular, if $\alpha > 0$ then we get from (30) ($\alpha = \nu$)

$$CE^{2-2\alpha-\varepsilon} \leqslant |f(E)| \leqslant CE^{2-2\alpha+\varepsilon}(\ln E)^2. \tag{35}$$

On the other hand, if $\alpha < 0$, then from (33) ($\alpha = \nu$) it follows that

$$CE^{2\alpha-\varepsilon} \leqslant |f(E)| \leqslant CE^{2\alpha+\varepsilon}(\ln E)^2. \tag{36}$$

Relations (35) and (36) can also be obtained directly from the formulas written out at the end of Meĭman's paper.[2]

The case of existence of a definite limit for $\xi(E)$ (34) will be analyzed in greater detail also from the point of view of relation[5]

$$|f(E)| \geqslant C/E^2, \tag{37}$$

which supplements the limitation (2). Applying our analysis not only to f(E), but also to 1/f(E), taking into account only the analytic properties, we obtain that the following relations could hold true in lieu of (32) and (33):

$$CE^{-2\alpha-\varepsilon} \leqslant |f(E)| \leqslant CE^{-2\alpha+\varepsilon}, \quad \alpha > 0; \tag{38}$$

$$CE^{-2\alpha-2-\varepsilon} \leqslant |f(E)| \leqslant CE^{-2\alpha-2+\varepsilon}, \quad \alpha < 0. \tag{39}$$

(In addition, the function f(E) can oscillate in such a manner that for certain large energies it satisfies formulas (35) and (36), and for other energies formulas (38) and (39). We shall assume henceforth that there are no such oscillations in f(E).) Relations (38) and (39) denote such a decrease in the cross section with increasing energy, which we refute on the basis of the existing experimental data. Were it to turn out, nonetheless, that a rapid decrease in the cross section is possible, then (38) and (39) would impose a limit on the asymptotic value of such a cross section. (In more general cases, that is, when only (3) is satisfied, or (29) is satisfied, we can also obtain several new relations that are analogs of formulas (4), (30), and (33). For example, if (3) is satisfied, then $f(E) > C/E^{2-2\alpha+\epsilon}$.)

Thus, in accordance with (35), (36), (38), and (39), if (37) is satisfied and $\xi(E) \to \tan \pi\alpha$ when $E \to \infty$, $|\alpha| \leq 1/2$, then for large energies with $\alpha > 0$ we have either

$$f(E) = \varphi_1(E) E^{2-2\alpha}, \tag{35a}$$

or

$$f(E) = \varphi_2(E) E^{-2\alpha}; \tag{38a}$$

when $\alpha < 0$ we have either

$$f(E) = \varphi_3(E) E^{-2\alpha}, \tag{36a}$$

or

$$f(E) = \varphi_4(E) E^{-2\alpha-2}. \tag{39a}$$

Here $\varphi_i(E)$ is a slowly varying function ($\epsilon > 0$ is arbitrarily small):

$$CE^{-\varepsilon} < \varphi_i(E) < CE^{\varepsilon}. \tag{40}$$

(We prove in the Appendix that $\lim [\varphi_i(-E)/\varphi_i(E)] = 1$, $E \to \infty$.)

Let us now prove the inverse theorem: if

$$\lim_{E\to\infty} \frac{f(E)}{E^{\beta}\varphi(E)} = C \neq 0 \tag{41}$$

exists (where $\varphi(E)$ satisfies formula (40)) and if $\lim [\varphi(-E)/\varphi(E)] = 1$, $E \to \infty$, then there exists also

$$\lim_{E\to\infty} \frac{\operatorname{Im} f(E)}{\operatorname{Re} f(E)} = -\operatorname{tg}\frac{\pi\beta}{2}. \tag{42}$$

Indeed, according to the Phragmen-Lindelof theorem

$$\lim_{E\to\infty} \frac{f(-E+i0)}{(-E+i0)^{\beta}\varphi(-E+i0)} = \lim_{E\to\infty} \frac{f(E+i0)}{(E+i0)^{\beta}\varphi(E+i0)}.$$

(In the opposite case, f(E) would increase more rapidly than exponentially in the upper half-plane and we assume, as is customary, that $|f(E)| < e^{\epsilon E}$, $\epsilon > 0$ is arbitrarily small.) Taking (7) into account and the fact that $\lim [\varphi(-E)/\varphi(E)] = 1$ when $E \to \infty$, we obtain (42) by direct calculation.

It is easy to verify that the formulas (41) and (42) correspond to (34), (35a), (38a), (36a), and (39a).

Thus, a unique relation exists between the energy exponent, which determines the rate of growth of f(E) as $E \to \infty$ (or the exponent which gives the rate of decrease when f(E) decreases) and the limit $\lim \xi(E)$ as $E \to \infty$.

In conclusion we note that if ($\epsilon > 0$ is arbitrarily small)

$$CE^{\beta-\varepsilon} \leqslant |f(E)| \leqslant CE^{\beta+\varepsilon}(\ln E)^2,$$

then, using the Phragmen-Lindelof theorem, we obtain the following connection between the amplitudes f_+ and f_-:

$$-\operatorname{Im} f_-(E) = \operatorname{Im} f_+(E)\cos\pi\beta + \operatorname{Re} f_+(E)\sin\pi\beta,$$
$$\operatorname{Re} f_-(E) = \operatorname{Re} f_+(E)\cos\pi\beta - \operatorname{Im} f_+(E)\sin\pi\beta. \tag{43}$$

(By $f_+(E)$ and $f_-(E)$ we mean respectively $\lim f_+(E)$ and $\lim f_-(E)$ as $E \to \infty$.) From this it follows that

$$\operatorname{Im} f_+(E) - \operatorname{Im} f_-(E) = (\operatorname{Re} f_+(E) - \operatorname{Re} f_-(E)) \operatorname{ctg}(\pi\beta/2),$$
$$\beta \neq 2, 1, 0.$$

Thus, if $\beta \neq 2, 1, 0$, then from the inequality $\operatorname{Im} f_-(E) = \operatorname{Im} f_+(E)$ there follows also $\operatorname{Re} f_+(E) = \operatorname{Re} f_-(E)$. For these inequalities it is necessary to have

$$\frac{\operatorname{Im} f_-(E)}{\operatorname{Re} f_-(E)} = \frac{\operatorname{Im} f_+(E)}{\operatorname{Re} f_+(E)} = -\operatorname{tg}\frac{\pi\beta}{2}.$$

If β is equal to 2 or to 0, then f(E) is pure real and $\operatorname{Re} f_+(E) = \operatorname{Re} f_-(E)$. If $\beta = 1$, then f(E) is pure imaginary and $\operatorname{Im} f_-(E) = \operatorname{Im} f_+(E)$—the Pomeranchuk theorem.[6]

In conclusion I am deeply grateful to E. L. Feĭnberg for continuous interest in the work, N. N. Meĭman for a very useful discussion, and to D. S. Chernavskiĭ, I. M. Dremin, and I. I. Roĭzen for valuable discussions.

APPENDIX

Let us prove, for example

$$\lim_{E\to\infty} \frac{\varphi_1(-E)}{\varphi_1(E)} = 1.$$

We construct

$$\tilde{\omega}(E) = f(E) / E^{2-2\alpha}\varphi_1(E).$$

According to the Phragmen-Lindel theorem (see the derivation of (42)) we have

$$\lim_{E\to\infty} \tilde{\omega}(E+i0) = \lim_{E\to\infty} \tilde{\omega}(-E+i0). \qquad (I)$$

On the other hand, from (34) we find that

$$\lim_{E\to\infty} \frac{f(-E+i0)}{(-E+i0)^{2-2\alpha}} = \lim_{E\to\infty} \frac{f(E+i0)}{(E+i0)^{2-2\alpha}}. \qquad (II)$$

The proved statement follows directly from formulas (I) and (II).

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Translated by J. G. Adashko
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