ON THE POSSIBILITY OF A DYNAMICAL DEFINITION OF THE THEORY IN THE WIGHTMAN AXIOMATIC SCHEME

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Submitted to JETP editor September 20, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **50**, 694–708 (March, 1966)

The concept of a system of dynamical equations of a theory is introduced in the framework of Wightman axiomatic field theory. A dynamical equation is a relation of the form $a(g) = 0$, $a(g)$ being defined in Eq. (1) below in terms of the field operator $\varphi(x)$ of the given theory and

$$g \equiv \{g_0;\ g_1(x_1);\ \ldots;\ g_n(x_1 \ldots x_n);\ 0;\ 0;\ \ldots\}$$

is an element of the test function space (on which the Wightman functional W is defined (usually the test functions $g_i(x_1, \ldots, x_i)$ are chosen to belong to the space S of infinitely differentiable rapidly decreasing functions). The relation among theories possessing identical sets of dynamical equations is analyzed. From this point of view, the choice of the space on which the functional W is defined is very important: the larger this space, the larger is the number of dynamical equations which can be written down, and consequently, the more information is contained in these equations. For a sufficiently wide class of theories (including in particular all local theories) it is shown that the functional W can be defined on a class of test functions which is wider than the space S. The fundamental result is the proof of the following statement: an irreducible Wightman functional W, defined on this extended space can be defined dynamically (i.e., is uniquely determined by the associated dynamical equations). If the functional W is reducible, but decomposes into a finite number of irreducible parts, its dynamical equation system determines the irreducible parts in a unique manner.

1. INTRODUCTION

WITHIN the axiomatic approach to field theory, which was developed successfully within the past few years, the physical theory is constructed as a representation theory of an abstract involutive algebra A by means of an algebra R(A) of operators in a Hilbert space.

From the algebraic point of view each such representation is determined by its kernel, i.e., the set of all elements of the algebra A which are taken into the zero-operator by the homomorphism $A \rightarrow R(A)$. Two representations R_1 and R_2 having the same kernels $M_1 = M_2$ are isomorphic and indistinguishable from the algebraic point of view.

This does not yet imply that the physical theories described by R_1 and R_2 are equivalent, since the physical theory is defined by the representation, and the representations R_1 and R_2, although isomorphic, can be different. It is natural to expect, however, that theories which correspond to algebraically equivalent representations will in a certain sense be close in their physical content.

Within the axiomatic school there exist two fundamental approaches: the Haag approach and the Wightman approach, differing in the choice of the fundamental algebra A. Within the Haag approach, the problem of the relation among equivalent representations has been investigated earlier.[1, 3]

Kastler and Haag[1] have shown that two algebraically equivalent representations are physically equivalent in the sense that no observer can distinguish one theory from the other by making use of the data from a finite (although arbitrarily large) number of experiments, carried out with finite (although arbitrarily small) errors.

Misra[2] has introduced another concept of physical equivalence: two representations R_1 and R_2 in the Hilbert spaces H_1 and H_2, respectively, are physically equivalent if they are algebraically isomorphic and if for each open bounded region Δ of space-time there exists an isometric mapping $U(\Delta)$ from H_1 to H_2 such that the isomorphism of the subalgebras $R_1(\Delta) \subset R_1$ and $R_2(\Delta) \subset R_2$ of quasilocal operators corresponding to the region Δ is implemented by this isometric operator

$$R_2(\Delta) = U(\Delta) R_1(\Delta) U(\Delta)^{-1}.$$

Making use of some additional assumptions, Misra has shown that algebraically isomorphic representations are physically equivalent in this sense.

Wightman[3] has obtained the same result under somewhat different assumptions.

In the present paper we shall consider the relation between algebraically equivalent representations in the framework of Wightman's axiomatics. In this scheme the initial algebra A is usually chosen to be the algebra consisting of finite sequences of functions (the elements of A will be denoted by g, h):

$$g \equiv \{g_0;\ g_1(x_1);\ldots;\ g_n(x_1\ldots x_n);0;\ 0;\ldots),$$

where $x_i \equiv (x_{i0}, \mathbf{x}_i)$, g_0 is a complex number, $g_k(x_1 \ldots x_k)$ is a function belonging to the space $S(x_1 \ldots x_k)$ of infinitely differentiable rapidly decreasing functions of the 4k variables $x_1 \ldots x_k$.

The operation of multiplication in this set of elements is defined as the "convolution:"

$$(gh)_k(x_1\ldots x_k) = \sum_{l=0}^{k} g_l(x_1\ldots x_l)\, h_{k-l}(x_{l+1}\ldots x_k)$$

and the involution $^+$ is defined by

$$(g^+)_k(x_1\ldots x_k) = \bar{g}_k(x_k\ldots x_1).$$

Each representation of such an algebra by means of an algebra of operators on a Hilbert space is defined by a (multiplicatively) positive functional W (i.e., such that $W(g^+g) \geq 0$):

$$W(g) = \sum_k \int \ldots \int dx_1 \ldots dx_k W_k(x_1\ldots x_k)\, g_k(x_1\ldots x_k),$$

where the $W_k(x_1 \ldots x_k)$ are functionals on $S(x_1 \ldots x_k)$, i.e., generalized functions of moderate growth ("moderate distributions").

The Hilbert space and the field operator $\varphi(x)$ (considered an operator-valued distribution) can be reconstructed from the functional W. One can also construct the algebra of field operators

$$a(g) = \sum_k \int \ldots \int dx_1 \ldots dx_k g_k(x_1\ldots x_k)\, \varphi(x_1)\ldots \varphi(x_k), \quad \text{(I)}$$

which realizes a representation of the algebra A.[4] By definition, the kernel M of this representation is the set of all $g \in A$ for which $a(g) = 0$. Consequently, the kernel of a representation has a simple physical interpretation in the Wightman scheme: knowing the kernel means knowing the set of dynamical equations of the theory. These equations have the form $a(g) = 0$, $g \in M$. We note that in contradistinction from the equations which are usually derived from a Lagrangian, these equations are correct, since the operator valued distribution $\varphi(x)$ "integrated" appropriately with the components $g(x_1 \ldots x_k)$ of the element $g \in A$ yields a well-defined operator in Hilbert space. As regards the equations derivable from a Lagrangian, the combination of field operators which is set equal to zero may not even be defined as a Hilbert space operator if the field exists only as an operator-valued distribution.

Thus the problem of existence of different isomorphic representations of a given algebra and the various relations among these representations can, in our case, be reformulated as follows: do there exist different solutions of a given system of dynamical equations, and if they exist, what is the relation among these solutions? Obviously, one should not consider all possible solutions of the dynamical system, but only those which in addition satisfy the requirements of relativistic invariance and positivity of the energy-momentum spectrum. The requirement of locality (we shall consider only local theories) need not be imposed separately, since it is dynamically formulated, and consequently the corresponding equations are already included among the ones that define the kernel M.

Naturally, the most desirable solution of the problem would be the possibility to determine the theory uniquely, i.e., to determine the Wightman functional, knowing the kernel of the representation. This would mean that the theory can be dynamically defined, i.e., via a set of equations for field equations.

From the point of view of the possibility of dynamically defining a theory the choice of the starting algebra A is essential. Indeed, if one could define the Wightman functional W as a functional over some algebra A′ which contains A, the corresponding kernel M′ would contain more dynamical equations, which would make it more likely that the theory can be uniquely determined by the kernel.

On the other hand, any extension of the initial algebra leads to a restriction of the class of functionals defined on it. Therefore an algebra may be extended only to the point at which one can be sure that any functional in the class that interests us is well defined on the elements of this extended algebra.

There is a simple consideration which makes it plausible that such an extension is possible. The function $W_n(x_1, \ldots, x_n)$ is first defined as a functional over the space $S(x_1 \ldots x_n)$, i.e., may have a power-law growth as its arguments become infinitely separated. On the other hand, it is known that for a local theory these functions must converge to constant values for space-like separation of their variables. Therefore the class of local functionals is considerably narrower than the class of all possible functionals on S, and consequently the local functionals can be defined on a much wider class of test functions, having a slower decrease in space-like directions than the functions belonging to S. It turns out that one can indeed construct an extension of the algebra A, such that the functional W can be reconstructed uniquely in terms of the kernel if the functional W is irreducible. We recall that a positive functional W is irreducible if it cannot be represented as a sum $W = \rho_1 W_1 + \rho_2 W_2$, with $\rho_{1,2} > 0$ and $W_{1,2}$ positive functionals. A functional which admits such a representation is called reducible. Physically, an irreducible functional describes a theory with a nondegenerate vacuum, and a reducible functional describes a theory with degenerate vacuum.[4] The kernel of a reducible functional is the intersection of the kernels of all irreducible functionals which make it up. Therefore two reducible functionals consisting of the same irreducible components, but with different coefficients,

$$W = \sum_\alpha \rho_\alpha W_\alpha \text{ and } W' = \sum_\alpha \rho_\alpha' W_\alpha$$

have the same kernel

$$M = \bigcap_\alpha M_\alpha.$$

Consequently, a reducible functional is never uniquely determined by its kernel, since the coefficients of its irreducible components W are always arbitrary. The most one can expect here is to determine the irreducible components W_α which comprise it. It turns out that after extending the algebra this can indeed be done for a functional consisting of a finite number of irreducible functionals. For more general situations one can derive only a weaker result (theorem 3).

For simplicity we consider here the case of a single scalar field, but all results can be directly generalized to the case of an arbitrary number of fields with any spins. In fact, the class of theories to which the results derived here are applicable is wider than the class of local theories, to which we restrict our attention (cf., in this connection the remark at the end of Sec. 3).

2. CONSTRUCTION OF THE EXTENDED ALGEBRA

Consider a theory defined by a Wightman functional W. Let $g_1(x_0), \ldots, g_n(x_0)$ denote arbitrary functions of the space $S(x_0)$. Consider the expression

$$\widetilde{W}(\mathbf{x}_1 \ldots \mathbf{x}_n) \equiv \int \ldots \int dx_{10} \ldots dx_{n0} W_n(x_1 \ldots x_n) \times g_1(x_{10}) \ldots g_n(x_{n0}).$$

If one knows about $W_n(x_1 \ldots x_n)$ only that it is a functional on $S(x_1 \ldots x_n)$ one can say about $\widetilde{W}(\mathbf{x}_1 \ldots \mathbf{x}_n)$ only that it is a functional on $S(\mathbf{x}_1 \ldots \mathbf{x}_n)$. If in addition the functional W satisfies the condition that the energy is positive, then, as has been shown by Borchers,[5] $\widetilde{W}(\mathbf{x}_1 \ldots \mathbf{x}_n)$ will be a function belonging to the space $O_M(\mathbf{x}_1 \ldots \mathbf{x}_n)$ of infinitely differentiable functions of temperate growth. If in addition W satisfies the condition of locality, the function $\widetilde{W}(\mathbf{x}_1 \ldots \mathbf{x}_n)$ (and also all its derivatives) will be a bounded infinitely differentiable function. This consideration is put at the basis of the construction of the extended algebra.

In constructing the required extension it is more convenient to deal with another algebra, to be denoted by Q^S, rather than with the algebra A defined in the Introduction. In a certain sense the algebra Q^S is equivalent to the algebra A. For the construction of the algebra Q^S we select an arbitrary number n and the functions $g_1(x_0) \ldots g_n(x_0)$ belonging to $S(x_0)$. We consider the family of operators of the field operator type, but depending only on the three-dimensional vector $\mathbf{x}$:

$$A_1(\mathbf{x}), \ldots, A_n(\mathbf{x}); \quad A_i(\mathbf{x}) \equiv \int dx_0 g_i(x_0) \varphi(x).$$

Here $\varphi(x)$ is the usual field operator of the theory and $A_i(\mathbf{x})$ is at least defined as an operator valued distribution (moreover, Borchers[5] has shown that $A_i(\mathbf{x})$ is well defined as an operator on Hilbert space for any fixed $\mathbf{x}$).

Considering $A_1(\mathbf{x}), \ldots, A_n(\mathbf{x})$ as a system of n fields depending on $\mathbf{x}$ we construct the functional $\widetilde{W}$ in the same manner as the Wightman functional:

$$\widetilde{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m) \equiv \langle \Omega, A_{i_1}(\mathbf{x}_1) \ldots A_{i_m}(\mathbf{x}_m) \Omega \rangle,$$

where Ω is the vacuum vector of the theory and m takes on all the values from zero to infinity; each i_α can take on the values from 1 to n. For a fixed number m of variables there are n^m different functions $\widetilde{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$.

We now construct the algebra $Q^S(n, g_1 \ldots g_n)$ on which the functional W is defined in the same manner as one constructs the algebra A for a system of n fields. The only distinction consists in the fact that now the elements of the algebra will be functions depending on the three-dimensional x only. Thus the elements of the algebra $Q^S(n, g_1 \ldots g_n)$ consist of terminating sequences:

$$g \equiv \{g_0;\ g_{i_1}(\mathbf{x}_1);\ g_{i_1 i_2}(\mathbf{x}_1\mathbf{x}_2);\ \ldots;\ g_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m);\ \ldots\},$$

where each index i_α takes values from 1 to n, g_0 is a complex number and the $g_{i_1\ldots i_k}(\mathbf{x}_1\ldots\mathbf{x}_k)$ are functions belonging to the space $S(\mathbf{x}_1\ldots\mathbf{x}_k)$. The functional $\widetilde{W}$ is defined on g in the following manner:

$$\widetilde{W}(g) = \sum_m \sum_{i_1\ldots i_m=1}^{n} \int\ldots\int d\mathbf{x}_1\ldots d\mathbf{x}_m \widetilde{W}_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m) \times g_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m).$$

The usual multiplication and involution laws hold in $Q^s(n, g_1\ldots g_n)$:

$$(gh)_{i_1\ldots i_k}(\mathbf{x}_1\ldots\mathbf{x}_k) = \sum_{l=0}^{k} g_{i_1\ldots i_l}(\mathbf{x}_1\ldots\mathbf{x}_l) h_{i_{l+1}\ldots i_k}(\mathbf{x}_{l+1}\ldots\mathbf{x}_k),$$

$$(g^+)_{i_1\ldots i_k}(\mathbf{x}_1\ldots\mathbf{x}_k) = \overline{g}_{i_k\ldots i_1}(\mathbf{x}_k\ldots\mathbf{x}_1)$$

and the space S has the usual topology.

The algebra Q^s is constructed as the set-theoretic union of all $Q^s(n, g_1\ldots g_n)$:

$$Q^s \equiv \bigcup_{n,\, g_1\ldots g_n} Q^s(n, g_1\ldots g_n).$$

W is defined as a functional over Q^s in the following manner: from a given $g \in Q^s$ one determines the algebra $Q^s(n, g_1\ldots g_n)$ containing that g; then one constructs the corresponding $\widetilde{W}$ and one takes $\widetilde{W}(g)$. The algebra Q^s is equivalent to A in the sense that each W defined on A is uniquely defined on Q^s and vice versa. In constructing the algebra Q^s only the fact has been taken into account that $W_n(x_1\ldots x_n)$ is a functional on $S(x_1\ldots x_n)$, and consequently that $\widetilde{W}_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m)$ is a functional on $S(x_1\ldots x_m)$. Making use of the result of Borchers, according to which all $W_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m)$ are infinitely differentiable bounded functions, it is not hard to extend the algebra Q^s. For the construction of such an extension it is sufficient to select some topology on the set of infinitely differentiable bounded functions and to consider the set of functionals which are continuous in this topology. As will become clear in the sequel, for the proof of the assertion made in the Introduction, it is sufficient to select the topology of the space $C(\mathbf{x}_1, \ldots \mathbf{x}_m)$ of continuous bounded functions; $C(\mathbf{x}_1\ldots\mathbf{x}_m)$ is a Banach space (i.e., a complete normed space) with the norm

$$\|f\| = \sup_{\mathbf{x}_1\ldots\mathbf{x}_m} |f(\mathbf{x}_1\ldots\mathbf{x}_m)|.$$

The general form of a functional on $C(\mathbf{x}_1\ldots\mathbf{x}_m)$ is given by the Radon integral (cf. Appendix):

$$\Phi(f) = \int\ldots\int f(\mathbf{x}_1\ldots\mathbf{x}_m)\, d\Phi(\mathbf{x}_1\ldots\mathbf{x}_m),$$

where $\Phi(x_1\ldots x_n)$ is a finitely additive set function of bounded variation, defined for all subsets of the 3m-dimensional space of $\mathbf{x}_1\ldots\mathbf{x}_m$.

Since it is more convenient to consider together the set of all functionals for all $i_1\ldots i_m$, rather than each individual $\widetilde{W}_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m)$, it will be expedient to introduce the Banach space $C_n(\mathbf{x}_1\ldots\mathbf{x}_m)$, having as elements the ensembles of of n^m functions

$$f_n \equiv \{f_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m)\},\quad i_\alpha = 1, 2, \ldots, n$$

and with the topology defined, for instance, by the norm

$$\|f_n\| = \max_{i_1\ldots i_m} \sup_{\mathbf{x}_1\ldots\mathbf{x}_m} |f_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m)|.$$

The general form of a linear functional on this space is

$$\Phi_n(f_n) = \sum_{i_1\ldots i_m=1}^{n} \int\ldots\int f_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m)\, d\Phi_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m).$$

Therefore we can take as the extended algebra $Q(n, g_1\ldots g_n)$ the set of terminating sequences

$$\Phi \equiv \{\Phi_0;\ \Phi_{i_1};\ \Phi_{i_1 i_2};\ \ldots;\ \Phi_{i_1\ldots i_m};\ \ldots\},$$

with the usual laws of multiplication and involution.

Clearly $Q^s(n, g_1\ldots g_n) \subset Q(n, g_1\ldots g_n)$ and to the elements of the space $Q^s(n, g_1\ldots g_n)$ are associated measures admitting the representation

$$d\Phi_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m) = g_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m)\, d\mathbf{x}_1\ldots d\mathbf{x}_m,$$

with

$$g_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m) \in S(\mathbf{x}_1\ldots\mathbf{x}_m).$$

The extended algebra is defined as the union

$$Q \equiv \bigcup_{n,\, g_1\ldots g_n} Q(n, g_1\ldots g_n).$$

It is easy to see that for given $n, g_1\ldots g_n$, Φ the expression

$$a(\Phi) = \sum_m \sum_{i_1\ldots i_m=1}^{n} \int\ldots\int A_{i_1}(\mathbf{x}_1)\ldots A_{i_m}(\mathbf{x}_m) \times d\Phi_{i_1\ldots i_m}(\mathbf{x}_1\ldots\mathbf{x}_m)$$

is a well defined unbounded operator in Hilbert space. Its domain contains at least the vectors of the form $a(\Phi')\Omega$, where Ω is the vacuum vector of the theory, Φ' is an arbitrary element of Q (here and in the following the symbol Φ denotes not only the functional Φ but also indicates to which algebra $Q(n, g_1\ldots g_n)$ it belongs).

The kernel M of the algebra Q is the set of those Φ for which $a(\Phi) = 0$. Obviously

$$M(n, g_1 \ldots g_n) = M \cap Q(n, g_1 \ldots g_n),$$

$$M = \bigcup_{n, g_1 \ldots g_n} M(n, g_1 \ldots g_n).$$

If for two theories W and W′ the kernels M and M′ coincide, then obviously

$$M(n, g_1 \ldots g_n) = M'(n, g_1 \ldots g_n)$$

for any $n, g_1 \ldots g_n$. This means that the isomorphism of the algebras

$$R(Q) \equiv \{a(\Phi);\ \Phi \in Q\} \text{ and } R'(Q) \equiv \{a'(\Phi);\ \Phi \in Q\}$$

is equivalent to the isomorphism of

$$R(Q(n, g_1 \ldots g_n)) \text{ and } R'(Q(n, g_1 \ldots g_n))$$

for any $n, g_1 \ldots g_n$.

3. PROOF OF THE FUNDAMENTAL RESULTS

This section contains only a purely formal proof of the assertions made in the introduction, the principal among which was the following: the kernel of an irreducible representation of the extended algebra determines this representation uniquely.

We first prove several lemmas.

Let Δ_{x_0} be an open bounded set on the time axis, and $\boldsymbol{\Delta}_{\mathbf{x}}$ an open bounded set in the three-space of four-dimensional space-time. We denote by Q_Δ the set of all $\Phi \in Q$ for which the functions $g_1(x_0) \ldots g_n(x_0)$ are nonvanishing only in Δ_{x_0}, and all $d\Phi_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$ are nonvanishing only for $\mathbf{x}_1 \in \boldsymbol{\Delta}_{\mathbf{x}} \ldots \mathbf{x}_m \in \boldsymbol{\Delta}_{\mathbf{x}}$.

Lemma 1. Assume that for some $\Phi \in Q$

$$\langle a(\Phi')\Omega,\ a(\Phi)a(\Phi')\Omega\rangle = 0$$

for all $\Phi' \in Q^S_\Delta$ (Δ fixed, arbitrary). Then $a(\Phi) = 0$, i.e., $\Phi \in M$.

This is a direct consequence of the Reeh-Schlieder theorem.[6] Indeed, taking

$$\Phi' = \lambda_1\Phi_1' + \lambda_2\Phi_2', \quad \Phi_{1,2}' \in Q_\Delta{}^s,$$

we have $\Phi'_{1,2} \in Q^S_\Delta$ and consequently

$$\langle a(\Phi_1')\Omega,\ a(\Phi)a(\Phi_2')\Omega\rangle = 0$$

for any $\Phi'_{1,2} \in Q^S$. According to the Reeh-Schlieder theorem, the set of elements of the form $a(\Phi'_1)$, $\Phi'_1 \in Q^S_\Delta$, is dense in H. Consequently, for any $G \in Q$, and any $\Phi'_2 \in Q^S_\Delta$

$$\langle a(G)\Omega,\ a(\Phi)a(\Phi_2')\Omega\rangle = 0.$$

Letting the operator $a(\Phi)$ act on the left vector and making use again of the Reeh-Schlieder theorem, we obtain

$$\langle a(\Phi^+)a(G)\Omega,\ a(G')\Omega\rangle = \langle a(G)\Omega,\ a(\Phi)a(G')\Omega\rangle = 0$$

for all $G, G' \in Q$, which proves the lemma.

Lemma 2. Let the functions $g_1(x_0) \ldots g_n(x_0)$ belonging to $S(x_0)$ be nonvanishing only for $x_0 \in \Delta'_{x_0}$, where Δ'_{x_0} is an arbitrary but fixed bounded set on the time axis. We consider the function

$$\langle a(\Phi)\Omega,\ A_1(\mathbf{x}_1)\ldots A_n(\mathbf{x}_n)a(\Phi)\Omega\rangle,$$

$$A_i(\mathbf{x}) \equiv \int dx_0 g_i(x_0)\varphi(x),$$

where $\Phi \in Q^S_\Delta$ (here $Q^S_\Delta \equiv Q^S \cap Q_\Delta$; Δ is also an arbitrary fixed domain of space-time). Then for an irreducible functional

$$\lim_{|\mathbf{a}|\to\infty} \langle a(\Phi)\Omega, A_1(\mathbf{x}_1+\mathbf{a})\ldots A_n(\mathbf{x}_n+\mathbf{a})a(\Phi)\Omega\rangle$$
$$= \|a(\Phi)\Omega\|^2\langle\Omega, A_1(\mathbf{x}_1)\ldots A_n(\mathbf{x}_n)\Omega\rangle$$

for any fixed $\mathbf{x}_1 \ldots \mathbf{x}_n$.

Proof. It is clear that for any fixed $\mathbf{x}_1 \ldots \mathbf{x}_n$, Δ, Δ'_{x_0} there exists a number $r > 0$, such that for all $\mathbf{a}$ ($|\mathbf{a}| > r$) any point from any of the segments $(\mathbf{x}_{i0}, \mathbf{x}_i + \mathbf{a})$, $x_{i0} \in \Delta'_{x_0}$, in four-dimensional space-time will be separated from any point of the domain Δ by a space-like interval. Due to local commutativity for such $\mathbf{a}$ we have

$$\langle a(\Phi)\Omega, A_1(\mathbf{x}_1+\mathbf{a})\ldots A_n(\mathbf{x}_n+\mathbf{a})a(\Phi)\Omega\rangle$$
$$= \langle a(\Phi)\Omega, a(\Phi)A_1(\mathbf{x}_1+\mathbf{a})\ldots A_n(\mathbf{x}_n+\mathbf{a})\Omega\rangle$$
$$= \langle a(\Phi^+\Phi)\Omega, U(\mathbf{a})A_1(\mathbf{x}_1)\ldots A_n(\mathbf{x}_n)\Omega\rangle,$$

where $U(\mathbf{a})$ is the unitary operator representing the space-like translation by $\mathbf{a}$. For an irreducible theory, we have for arbitrary Φ and Φ'

$$\lim_{|\mathbf{a}|\to\infty}\langle a(\Phi)\Omega, U(\mathbf{a})a(\Phi')\Omega\rangle = \langle a(\Phi)\Omega, \Omega\rangle\langle\Omega, a(\Phi')\Omega\rangle,$$

which completes the proof of the lemma.

Definition. Let B be a Banach space (i.e., a complete normed space) and B_1, B_2 two of its subspaces. We call such subspaces separated if $s_1 \equiv B_1 \cap s$ and $s_2 \equiv B_2 \cap s$, where s is the unit sphere of B, are separated by a finite interval, i.e., if

$$\inf\|x_1 - x_2\| = d > 0 \text{ for } x_1 \in s_1,\ x_2 \in s_2.$$

Lemma 3. If two subspaces B_1 and B_2 of a Banach space B are separated, the set

$$B_1 + B_2 \equiv \{x_1 + x_2\colon x_1 \in B_1,\ x_2 \in B_2\}$$

is closed.

Proof. Let $x \in \overline{B_1 + B_2}$ ($\bar{L}$ denotes the closure of the set L in the norm of B). Consequently there exists a sequence $x_n \in B_1 + B_2$ such that $x_n \to x$. By definition of $B_1 + B_2$ each x_n can be represented in the form

$$x_n = x_n^{(1)} + x_n^{(2)}, \text{ where } x_n^{(1)} \in B_1,\ x_n^{(2)} \in B_2.$$

We prove that the set $\|x_n^{(1)}\|$ is bounded from above for all n. Assuming the contrary, one could select from the sequence $x_n^{(1)}$ a subsequence $x_{n_i}^{(1)}$ such that $\|x_{n_i}^{(1)}\| \to \infty$. Dividing by $\|x_{n_i}^{(1)}\|$ we have

$$\frac{x_{n_i}^{(1)}}{\|x_{n_i}^{(1)}\|} + \frac{x_{n_i}^{(2)}}{\|x_{n_i}^{(1)}\|} = \frac{x_{n_i}}{\|x_{n_i}^{(1)}\|} \to 0,$$

since $x_{n_i} \to x$ by definition. Thus $\|x_{n_i}^{(2)}\|/\|x_{n_i}^{(1)}\| \to 1$. Indeed:

$$\left|1 - \frac{\|x_{n_i}^{(2)}\|}{\|x_{n_i}^{(1)}\|}\right| = \left|\left\|\frac{x_{n_i}^{(1)}}{\|x_{n_i}^{(1)}\|}\right\| - \left\|\frac{x_{n_i}^{(2)}}{\|x_{n_i}^{(1)}\|}\right\|\right| \leqslant \left\|\frac{x_{n_i}^{(1)}}{\|x_{n_i}^{(1)}\|} + \frac{x_{n_i}^{(2)}}{\|x_{n_i}^{(1)}\|}\right\| \to 0.$$

But then

$$\left\|\frac{x_{n_i}^{(1)}}{\|x_{n_i}^{(1)}\|} + \frac{x_{n_i}^{(2)}}{\|x_{n_i}^{(2)}\|}\right\| = \left\|\frac{x_{n_i}^{(1)}}{\|x_{n_i}^{(1)}\|} + \frac{x_{n_i}^{(2)}}{\|x_{n_i}^{(1)}\|} + x_{n_i}^{(2)}\left(\frac{1}{\|x_{n_i}^{(2)}\|} - \frac{1}{\|x_{n_i}^{(1)}\|}\right)\right\| \leqslant \left\|\frac{x_{n_i}^{(1)}}{\|x_{n_i}^{(1)}\|} + \frac{x_{n_i}^{(2)}}{\|x_{n_i}^{(1)}\|}\right\| + \left|1 - \frac{\|x_{n_i}^{(2)}\|}{\|x_{n_i}^{(1)}\|}\right|.$$

It has been shown that the right hand side of the inequality converges to zero. Since $x_{n_i}^{(1)}/\|x_{n_i}^{(1)}\| \in s_1$ and $-x_{n_i}^{(2)}/\|x_{n_i}^{(2)}\| \in s_2$, it follows that

$$\inf \|x_1 - x_2\| = 0 \quad (x_1 \in s_1, \quad x_2 \in s_2),$$

in contradiction with the assumed separability of the subspaces B_1, B_2. Consequently the set $\|x_n^{(1)}\|$ is bounded. Obviously, the same is true for the set $\|x_n^{(2)}\|$.

Since a bounded set in a Banach space is weakly compact, one can select from $x_n^{(1)}$ a weakly convergent subsequence. Without loss of generality we can assume that the sequence $x_n^{(1)}$ itself converges weakly to some $x^{(1)}$ and the sequence $x_n^{(2)}$ to some $x^{(2)}$. Since a subspace of a Banach space contains the limits of weakly convergent sequences of elements of that subspace, it follows that $x^{(1)} \in B_1$ and $x^{(2)} \in B_2$.

By assumption, the sequence $x_n = x_n^{(1)} + x_n^{(2)}$ converges strongly to the element x, and, as proved, converges weakly to the element $x^{(1)} + x^{(2)}$. Consequently $x = x^{(1)} + x^{(2)} \in B_1 + B_2$, Q.E.D.

Let us consider two subspaces of the Banach space $C(x_1, \ldots x_m)$ of bounded continuous functions: the subspace $C^0(\mathbf{x}_1 \ldots \mathbf{x}_m)$ consisting by definition of those functions $f(\mathbf{x}_1 \ldots \mathbf{x}_m)$ for which

$$\lim_{|\mathbf{a}|\to\infty} f(\mathbf{x}_1 + \mathbf{a}, \ldots, \mathbf{x}_m + \mathbf{a}) = 0$$

for any fixed $\mathbf{x}_1 \ldots \mathbf{x}_m$, and the subspace $C^t(\mathbf{x}_1 \ldots \mathbf{x}_m)$ consisting of translationally invariant functions, i.e., functions for which $f(\mathbf{x}_1 + \mathbf{a}, \ldots, \mathbf{x}_m + \mathbf{a})$ does not depend on $\mathbf{a}$.

In the same way one can introduce the subspaces $C_n^0(\mathbf{x}_1 \ldots \mathbf{x}_m)$ and $C_n^t(\mathbf{x}_1 \ldots \mathbf{x}_m)$, consisting of elements for which each component $f_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$ belongs respectively to $C^0(\mathbf{x}_1 \ldots \mathbf{x}_m)$ or $C^t(\mathbf{x}_1 \ldots \mathbf{x}_m)$.

Lemma 4. The subspaces $C^0(\mathbf{x}_1 \ldots \mathbf{x}_m)$ and $C^t(\mathbf{x}_1 \ldots \mathbf{x}_m)$ of the space $C(\mathbf{x}_1 \ldots \mathbf{x}_m)$ are separated.

Let

$$f_0 \in C^0(\mathbf{x}_1 \ldots \mathbf{x}_m); \quad f_t \in C^t(\mathbf{x}_1 \ldots \mathbf{x}_m).$$

We select arbitrary, fixed $\mathbf{x}_1 \ldots \mathbf{x}_m$. Then

$$\|f_0 - f_t\| \geqslant \sup_{\mathbf{a}} |f_0(\mathbf{x}_1 + \mathbf{a}, \ldots, \mathbf{x}_m + \mathbf{a}) - f_t(\mathbf{x}_1 + \mathbf{a}, \ldots, \mathbf{x}_m + \mathbf{a})| \geqslant \lim_{|\mathbf{a}|\to\infty} |f_0(\mathbf{x}_1 + \mathbf{a}, \ldots, \mathbf{x}_m + \mathbf{a}) - f_t(\mathbf{x}_1 \ldots \mathbf{x}_m)| = |f_t(\mathbf{x}_1 \ldots \mathbf{x}_m)|.$$

Since the $\mathbf{x}_1 \ldots \mathbf{x}_m$ are arbitrary,

$$\|f_0 - f_t\| \geqslant \sup_{\mathbf{x}_1 \ldots \mathbf{x}_m} |f_t(\mathbf{x}_1 \ldots \mathbf{x}_m)| = \|f_t\|.$$

If $\|f_0\| = \|f_t\| = 1$, $\|f_0 - f_t\| \geq 1$, which completes the proof. In the same manner one can prove the separateness of the subspaces $C_n^0(\mathbf{x}_1 \ldots \mathbf{x}_m)$ and $C_n^t(\mathbf{x}_1 \ldots \mathbf{x}_m)$ of the space $C_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$.

We are now ready to prove the fundamental theorems.

Theorem 1. Consider two functionals W and W′, satisfying the requirements of relativistic invariance, positive energy and locality, as functionals over the extended algebra Q. If the theory described by W is irreducible and if $M' \supseteq M$, then $W = W'$ and consequently $M' = M$.

We take some Δ'_{x_0}, an arbitrary number n and the functions $g_1(x_0), \ldots, g_n(x_0)$ which are nonvanishing only for $x_0 \in \Delta'_{x_0}$. We consider the algebra $Q(n, g_1 \ldots g_n)$. The assumptions of the theorem imply:

$$M'(n, g_1 \ldots g_n) \supseteq M(n, g_1 \ldots g_n).$$

We select arbitrary m, $\Delta \equiv (\Delta_{x_0} \boldsymbol{\Delta}_{\mathbf{x}})$ and consider the set of elements of the Banach space $C_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$ of the form

$$\omega_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m) \equiv \langle a(\Phi)\Omega, A_{i_1}(\mathbf{x}_1) \ldots A_{i_m}(\mathbf{x}_m) a(\Phi)\Omega \rangle,$$

where $\Phi \in Q_\Delta^S$ and each i_α takes on values from 1 to n.

We prove first that under the assumptions of the theorem the elements $\tilde{W}'_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$ of the Banach space $C_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$ belong to the closure of the linear hull of the elements $\omega_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m)$.

A known[7] necessary and sufficient condition for this is the following criterion: any functional which vanishes on all elements $\omega_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m)$ must vanish also on the element $\widetilde{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$.

Assume now that some functional Φ' vanishes on all ω. This means that

$$\langle a(\Phi)\Omega, a(\Phi')a(\Phi)\Omega\rangle = 0$$

for all $\Phi \in Q^S_\Delta$. It follows (lemma 1) that $\Phi' \in M(n, g_1 \ldots g_n)$ and since by assumption $M' \subseteq M$, $\Phi' \in M'$. Consequently

$$\mathcal{W}'(\Phi') = \sum_{i_1 \ldots i_m = 1}^{n} \int \ldots \int \mathcal{W}'_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m) \times d\Phi'_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m) = 0$$

(the functional W always vanishes on the elements of its kernel). It follows that there exists a sequence of linear combinations of elements $\omega_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m)$ which is norm-convergent to the element $\widetilde{W}'_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$.

Consider now the function

$$\omega^t_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m) \equiv \lim_{|\mathbf{a}| \to \infty} \omega_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 + \mathbf{a}, \ldots, \mathbf{x}_m + \mathbf{a}).$$

According to lemma 2,

$$\omega^t_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m) = \|a(\Phi)\Omega\|^2 \mathcal{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m).$$

Consequently an arbitrary element ω admits the representation

$$\omega_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m) = \omega^0_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m) + \|a(\Phi)\Omega\|^2 \mathcal{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m),$$

with $\omega^0_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m) \in C^0_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$ and the element $\widetilde{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$ forms a one-dimensional subspace $C^W_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$ contained in $C^t_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$ and consequently separated from $C^0_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$.

It has been shown that any $\omega_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m)$ is an element of

$$C^0_n(\mathbf{x}_1 \ldots \mathbf{x}_m) + C^w_n(\mathbf{x}_1 \ldots \mathbf{x}_m).$$

Applying now lemma 3 to the separated subspaces $C^0_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$ and $C^W_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$, we conclude that any limit of a sequence of linear combinations of elements belongs to

$$C^0_n(\mathbf{x}_1 \ldots \mathbf{x}_m) + C^w_n(\mathbf{x}_1 \ldots \mathbf{x}_m),$$

i.e., admits the representation

$$f^0_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m) + C\mathcal{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_n),$$

where C is a constant and

$$f^0_{i_1 \ldots i_m}(\mathbf{x}_1 + \mathbf{a}, \ldots, \mathbf{x}_m + \mathbf{a}) \to 0 \quad \text{as} \quad |\mathbf{a}| \to \infty.$$

If this limit belongs to $C^t_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$ (here this is a consequence of translation invariance of W'), then $f^0_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$ vanishes identically. Consequently

$$\mathcal{W}'_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m) = C\mathcal{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m).$$

The reasoning was carried out for a fixed algebra $Q(n, g_1 \ldots g_n)$. We show that the value of the constant C is independent of the choice of $n, g_1 \ldots g_n$. Let us assume that for two different algebras $Q(n, g_1 \ldots g_n)$ and $Q(l, g'_1 \ldots g'_l)$ the constant C takes on different values. We then take the algebra

$$Q(n + l, g_1 \ldots g_n, g_{n+1} \ldots g_{n+l}), \; g_{n+k} \equiv g_k'.$$

Repeating the reasoning given above for this case we find

$$\mathcal{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m) = C\mathcal{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m),$$

where each i_α ranges over the values 1 to $n + l$. This system of equalities (their number equals the number of sets $i_1 \ldots i_m$) contains as subsystems the same kind of relations as obtained directly for the algebras $Q(n, g_1 \ldots g_n)$ and $Q(l, g'_1 \ldots g'_l)$. Since the constant C does not depend on the choice of the indices $i_1 \ldots i_m$, the constants for $Q(n, g_1 \ldots g_n)$ and $Q(l, g_1 \ldots g_l)$ cannot differ from each other.

It remains to be shown that the value of the constant C is independent of m also, i.e., does not depend on the number characterizing the given Wightman function.

We select arbitrarily the numbers m and m' and construct the Banach space of pairs $(f_m, f_{m'})$

$$f_m \in C_n(\mathbf{x}_1 \ldots \mathbf{x}_m), \quad f_{m'} \in C_n(\mathbf{x}_1' \ldots \mathbf{x}_{m'}').$$

The norm in this space is defined by

$$\|(f_m, f_{m'})\| = \max\{\|f_m\|, \|f_{m'}\|\},$$

where $\|f_m\|$ and $\|f_{m'}\|$ denote the norms of the elements f_m and $f_{m'}$ considered respectively as elements of the subspaces $C_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$ and $C_n(\mathbf{x}'_1 \ldots \mathbf{x}'_m)$. For the space of pairs $(C_n(\mathbf{x}_1 \ldots \mathbf{x}_m), C_n(\mathbf{x}'_1 \ldots \mathbf{x}'_{m'}))$ the previous arguments remain unchanged, and one obtains

$$(\mathcal{W}'_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m), \mathcal{W}'_{i_1' \ldots i'_{m'}}(\mathbf{x}_1' \ldots \mathbf{x}_{m'}')) = C(\mathcal{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m), \mathcal{W}_{i_1' \ldots i'_{m'}}(\mathbf{x}_1' \ldots \mathbf{x}_{m'}')).$$

It is clear that the values of the constants C are the same for any two m, m'. If the two functionals W and W' satisfy the usual normalization condition $W(1) = W'(1) = 1$, it follows that $C = 1$. Consequently

$$\tilde{W}'_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m) = \tilde{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m).$$

Since from the very beginning the functions $g_1(x_0) \ldots g_n(x_0)$ which generate the algebra $Q(n, g_1 \ldots g_n)$ have been selected so that they are nonvanishing only in a given region Δ'_{x_0}, and are now otherwise completely arbitrary, the preceding relation means that the Wightman functions of the theories W and W′ must coincide when the arguments $x_{10}, \ldots, x_{m0}$ belong to the given region Δ'_{x_0} and the $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are arbitrary.

It should be noted that since the Wightman functions can be considered as boundary values of analytic functions of several complex variables,[4] their equality in a given region implies their equality for all $x_{10}, \ldots, x_{m0}, \mathbf{x}_1, \ldots, \mathbf{x}_m$, thus concluding the proof of the theorem.

Theorem 2. Let W be a functional consisting of a finite number of irreducible ones:

$$W = \sum_{\alpha=1}^{N} \rho_\alpha W_\alpha, \quad \rho_\alpha > 0, \quad \sum_{\alpha=1}^{N} \rho_\alpha = 1.$$

Let W′ be a functional such that $M' \supseteq M$. Then W′ is constructed from the same irreducible functionals as W, i.e.,

$$W' = \sum_{\alpha=1}^{N} \rho_\alpha' W_\alpha, \quad \rho_\alpha' \geqslant 0, \quad \sum_{\alpha=1}^{N} \rho_\alpha' = 1.$$

The numbers ρ'_α are all nonvanishing, only if $M' = M$.

The proof of this theorem is carried out in the same manner as that of theorem 1. The difference consists only in the fact that instead of the system of functions

$$\omega_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m) \equiv \langle a(\Phi)\Omega, A_{i_1}(\mathbf{x}_1) \ldots A_{i_m}(\mathbf{x}_m) a(\Phi)\Omega \rangle$$

one must now take the larger system

$$\omega^{(\alpha)}_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m) \equiv \langle a(\Phi)\Omega_\alpha, A_{i_1}(\mathbf{x}_1) \ldots A_{i_m}(\mathbf{x}_m) a(\Phi)\Omega_\alpha \rangle,$$

where $\alpha = 1, 2, \ldots, N$ and Ω_α is the vacuum vector of the functional W_α, i.e.,

$$\langle \Omega_\alpha, a(\Phi)\Omega_\alpha \rangle = W_\alpha(\Phi).$$

The subspace $C_n^W(\mathbf{x}_1 \ldots \mathbf{x}_m)$ will now be a finite-dimensional subspace spanned by the N elements $\tilde{W}^{(\alpha)}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$, $\alpha = 1, 2, \ldots, N$, and not a one-dimensional space, as before. In proving that

$$\tilde{W}'_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m) \in \overline{C_n{}^0(\mathbf{x}_1 \ldots \mathbf{x}_m) + C_n{}^w(\mathbf{x}_1 \ldots \mathbf{x}_m)},$$

one must take into account the fact that the vanishing of the functional Φ' for all $\omega^{(\alpha)}_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m)$ for fixed α implies (cf. the proof of theorem 1) that $\Phi' \in M_\alpha$ (M_α is the kernel of the functional W_α) and the vanishing of this functional on all $\omega^{(\alpha)}_{i_1 \ldots i_m}(\Phi, \mathbf{x}_1 \ldots \mathbf{x}_m)$ for any α implies that

$$\Phi' \in \bigcap_\alpha M_\alpha = M.$$

All numbers ρ'_α must be nonnegative, due to the positiveness of the functional W′ and their sum is one, due to the normalization condition of the functional.

Let us prove the last assertion of the theorem. If there are no zeros among the numbers then obviously $M' = \bigcap_\alpha M_\alpha = M$. Conversely, let $M' = M$. Assume that some $\rho'_{\alpha_0} = 0$. Interchanging the roles of the functionals W and W′ (this can be done since $M = M'$) we would reach the conclusion that W is composed of the same irreducible functionals as W′, i.e., W does not contain the functional W_{α_0} either. This contradiction proves the assertion.

For theories with infinitely degenerate vacuum similar reasoning leads only to a weaker result.

Theorem 3. Let W be a functional with infinitely degenerate vacuum, i.e., a functional which admits the representation

$$W = \int W_{(\alpha)} d\mu(\alpha).$$

If for some W′ we have

$$M' \supseteq M = \bigcap_\alpha M_{(\alpha)},$$

where $M_{(\alpha)}$ is the kernel of the irreducible component $W_{(\alpha)}$, then for any choice of $n, g_1(x_0), \ldots, g_n(x_0)$, the element $\tilde{W}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$ of the space $C_n(\mathbf{x}_1 \ldots \mathbf{x}_m)$ can be obtained as a strong limit of some sequence of linear combinations of elements $\tilde{W}^{(\alpha)}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$.

A complete analog of theorem 2 in this case would be the assertion that W′ admits the representation

$$W' = \int W_{(\alpha)} d\mu'(\alpha)$$

with another positive measure $\mu'(\alpha)$. Unfortunately, in distinction from the preceding case, we have not succeeded in proving that any limit of a sequence of linear combinations of elements $\tilde{W}^{(\alpha)}_{i_1 \ldots i_m}(\mathbf{x}_1 \ldots \mathbf{x}_m)$ admits such a representation.

Remark. It is clear from the proof that the requirement of locality is necessary only insofar as locality automatically guarantees the fulfillment of the following two statements:

1. $\tilde{W}_{i_1\dots i_m}(\mathbf{x}_1\dots\mathbf{x}_m)$

$$= \int\dots\int dx_{10}\dots dx_{m0}\, g_{i_1}(x_{10})\dots g_{i_m}(x_{m0})\, W_m(x_1\dots x_m)$$

is a bounded function for any $g_1(x_0), \dots, g_n(x_0) \in S(x_0)$.

2. $\lim\limits_{|\mathbf{a}|\to\infty} \tilde{W}_{i_1\dots i_m}(\mathbf{x}_1,\dots,\mathbf{x}_k,\ \mathbf{x}_{k+1}+\mathbf{a},\dots,\mathbf{x}_{k+l}+\mathbf{a},$

$\mathbf{x}_{k+l+1},\dots,\mathbf{x}_m) = \tilde{W}_{i_1\dots i_k i_{k+l+1}\dots i_m}(\mathbf{x}_1\dots\mathbf{x}_k\mathbf{x}_{k+l+1}\dots\mathbf{x}_m)$

$\times \tilde{W}_{i_{k+1}\dots i_{k+l}}(\mathbf{x}_{k+1}\dots\mathbf{x}_{k+l})$

holds for any k, l, m if the theory is irreducible. Therefore the class of theories to which the results derived here are applicable is wider than the class of local theories and includes all those for which the Wightman functions possess these two properties.

I make use of the occasion to thank L. V. Prokhorov for useful discussions and a reading of the manuscript.

APPENDIX

SOME PROPERTIES OF ADDITIVE FUNCTIONS OF BOUNDED VARIATION AND OF THE RANDOM INTEGRAL

For simplicity we restrict our attention to the consideration of real additive functions of bounded variation, defined on the set T of all subsets of the one-dimensional Euclidean space R_1. Detailed information about additive functions of bounded variation and the Radon integral can be found in the book by Kantorovich and Akilov[7] (Chapter 6, § 4).

Let T denote the set of all subsets of the one-dimensional Euclidean space R_1. If we associate to each subset $e \in T$ a real number $\Phi(e)$ we define a set function Φ on T. The function Φ is said to be additive if for any two disjoint $e_{1,2} \in T$ we have $\Phi(e_1 \cup e_2) = \Phi(e_1) + \Phi(e_2)$.

The function Φ is said to be of bounded variation if

$$\bar{\Phi}(R_1) = \sup \sum_{k=1}^{n} |\Phi(e_k)| < \infty,$$

where the upper bound is taken over all partitions of R_1 into parts $e_1 \dots e_n$. If the function is of bounded variation, obviously $\sup\limits_{e\in T} |\Phi(a)| < \infty$. One can show that this condition is also sufficient.

Let f(x) be some real function defined on R_1. We consider the partition of the real line

$$\dots < l_{-m} < \dots < l_{-1} < l_0 < l_1 < \dots < l_m < \dots,$$

such that

$$l_m \underset{m\to\infty}{\to} \infty; \quad l_{-m} \underset{m\to\infty}{\to} -\infty; \quad \lambda \equiv \sup_k |l_{k+1} - l_k| < \infty.$$

Define the sets

$$e_k = \{x \in R_1 : l_k \leqslant f(x) < l_{k+1}\}.$$

Let Φ be an additive function of bounded variation. The Radon integral

$$\int_{R_1} f(x)\, d\Phi(e)$$

is defined by

$$\lim_{\lambda\to 0} \sum_{k=-\infty}^{\infty} l_k \Phi(e_k),$$

if the limit exists and does not depend on the choice of the successive partitions $\{l_k\}$.

If, in particular, f(x) is a bounded function, the Radon integral exists and the following estimate holds:

$$\left|\int_{R_1} f(x)\, d\Phi(e)\right| \leqslant \sup_x |f(x)| \cdot \bar{\Phi}(R_1).$$

It follows that

$$\int_{R_1} f(x)\, d\Phi(e)$$

is a continuous linear functional on the space M_{R_1} of all bounded functions on R_1 (the norm in M_{R_1} is defined as $\|f\| = \sup\limits_x |f(x)|$). It can be shown that any bounded linear functional on M_{R_1} is of this form.

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Translated by Meinhard E. Mayer
88