THE SPECIFIC HEAT OF THIN SUPERCONDUCTING FILMS IN A MAGNETIC FIELD

Yu. N. OVCHINNIKOV

Moscow Physico-technical Institute

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The behavior of a thin superconducting film in a stationary magnetic field is considered. The field is assumed to be directed along the film and to satisfy the condition $(\xi_0 d)^{-1} \ll eH \ll d^{-2}$ where d is the film thickness and $\xi_0 = v/T_c$ is the correlation parameter. Under this condition the ordering parameter $\Delta(r)$ can be assumed to be the same throughout the film thickness. However, even such fields change the spectrum appreciably. It is shown that $|\Delta|$ no longer plays the role of a gap in the spectrum. The gap depends on the angles and vanishes for certain directions. The derived spectrum is used to determine the specific heat of the film. It is shown that at low temperatures $(T \ll T_c)$ the specific heat decreases according to a power law.

1. SINGLE-PARTICLE EXCITATION SPECTRUM

As usual in a superconductor problem we start from Gor'kov's temperature equations:[1]

$$\left[i\omega + \frac{1}{2m}(\nabla - ie\mathbf{A})^2 + \mu\right] G_\omega(\mathbf{r}, \mathbf{r}') + \Delta(\mathbf{r}) F_\omega^+(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$
$$\left[-i\omega + \frac{1}{2m}(\nabla + ie\mathbf{A})^2 + \mu\right] F_\omega^+(\mathbf{r}, \mathbf{r}') - \Delta^*(\mathbf{r}) G_\omega(\mathbf{r}, \mathbf{r}') = 0. \tag{1}$$

The spectrum corresponds to values $\epsilon = i\omega$, for which the corresponding homogeneous system has solutions:

$$\left[\varepsilon + \frac{1}{2m}(\nabla - ie\mathbf{A})^2 + \mu\right] g(\mathbf{r}) + \Delta(\mathbf{r}) f^+(\mathbf{r}) = 0,$$
$$\left[-\varepsilon + \frac{1}{2m}(\nabla + ie\mathbf{A})^2 + \mu\right] f^+(\mathbf{r}) - \Delta^*(\mathbf{r}) g(\mathbf{r}) = 0. \tag{2}$$

We shall consider only fields satisfying the condition $eH \ll d^{-2}$. In this case, as shown by Shapoval,[2] Δ can be considered constant throughout the entire film thickness. For $d \ll \delta$ (where δ is the penetration depth) the magnetic field H is constant inside the film. We shall restrict ourselves to the case of relatively thick films $d \gg p_0^{-1}$ (p_0 is the limiting momentum) in which the quantization in the transverse direction is not essential. In this case with specular reflection from the walls Nambu and Tuan's statement is valid:[3] the problem of a film becomes equivalent to the problem of a bulk superconductor consisting of films in which the magnetic field H is directed in opposite directions in neighboring films. Bearing in mind that the magnetic field is oriented along the film and is constant in it, the vector potential A can be written in the form

$$\mathbf{A} = (A(z), 0, 0), \tag{3}$$

where A(z) is a periodic function with a period 2d:

$$A(z) = H(-d/2 + |z|), \quad -d \leqslant z \leqslant d. \tag{4}$$

To solve the system (2), we make use of a method previously described in [4]. We seek g and f^+ in the form

$$g(\mathbf{r}) = e^{i\mathbf{p}_0\mathbf{r}}\varphi(z), \quad f^+(z) = e^{i\mathbf{p}_0\mathbf{r}}\chi(z), \tag{5}$$

where the functions $\varphi(z)$ and $\chi(z)$ change over distances of the order of d. In the quasiclassical approximation one can neglect the second derivatives with respect to φ and χ. Retaining only terms in which $\exp[i(\mathbf{p}_0 \cdot \mathbf{r})]$ is differentiated at least once, we obtain the system

$$\left[\varepsilon + \frac{ip_{0z}}{m}\frac{\partial}{\partial z} + \frac{eAp_{0x}}{m}\right]\varphi(z) + \Delta\chi(z) = 0,$$
$$\left[-\varepsilon + \frac{ip_{0z}}{m}\frac{\partial}{\partial z} - \frac{eAp_{0x}}{m}\right]\chi(z) - \Delta\varphi(z) = 0. \tag{6}$$

The system (6) is equivalent to one second-order equation

$$\left\{-\left(\frac{p_{0z}}{m}\right)^2\frac{d^2}{dz^2} + \left[\Delta^2 - \left(\varepsilon + \frac{eAp_{0x}}{m}\right)^2 + \frac{iep_{0x}p_{0z}}{m}\frac{dA}{dz}\right]\right\}\varphi(z) = 0, \tag{7}$$

where A(z) is defined by Eq. (4).

The coefficients of Eq. (7) are periodic functions of z, and therefore its solution should be of the form

$$\varphi(z) = e^{igz}\psi(z), \qquad \psi(z+2d) = \psi(z). \tag{8}$$

In each strip Eq. (7) is brought to the form

$$\left[\frac{d^2}{dz^2} + a\left(z + \frac{b}{2a}\right)^2 + 2i\sqrt{a}\left(\nu - \frac{1}{2}\right)\right]\psi(z) = 0. \tag{9}$$

Its solutions are the parabolic cylinder functions.[5] Two linearly independent solutions can be chosen in the form

$$D_{-\nu}[(1-i)a^{1/4}(z+b/2a)], \; D_{-1+\nu}[(1+i)a^{1/4}(z+b/2a)],$$

where

$$\nu = i\Delta^2 m^2 / 2eHp_{0x}p_{0z}. \tag{10}$$

Matching the solutions on each boundary and taking into account (8), we find that exp (2igd) is the eigenvalue of the transition matrix B:

$$B = -i\begin{pmatrix} D_{-\nu}(-iz) & D'_{-\nu}(-iz) \\ D_{-1+\nu}(z) & iD'_{-1+\nu}(z) \end{pmatrix} \times \begin{pmatrix} D'_{-1-\nu}(-iz) & iD'_{\nu}(-z) \\ D_{-1-\nu}(-iz) & -D_{-\nu}(-z) \end{pmatrix} \times \begin{pmatrix} D_{\nu}(-\tilde{z}) & iD'_{\nu}(-\tilde{z}) \\ D_{-1-\nu}(-i\tilde{z}) & -D_{-1-\nu}(-i\tilde{z}) \end{pmatrix} \times \begin{pmatrix} iD'_{-1+\nu}(\tilde{z}) & -D'_{-\nu}(-i\tilde{z}) \\ -D_{-1+\nu}(\tilde{z}) & +D_{-\nu}(-i\tilde{z}) \end{pmatrix}, \tag{11}$$

where

$$z = (1+i)\sqrt{\frac{eHp_{0x}}{p_{0z}}}\left(\frac{d}{2} + \frac{\varepsilon m}{eHp_{0x}}\right),$$

$$\tilde{z} = (1+i)\sqrt{\frac{eHp_{0x}}{p_{0z}}}\left(-\frac{d}{2} + \frac{\varepsilon m}{eHp_{0x}}\right). \tag{12}$$

In sufficiently strong fields, when $\nu \ll 1$, the transition matrix simplifies considerably, and we obtain for the beginning of the band ($gd \ll 1$) the spectrum

$$\varepsilon^2 = \xi^2 + \Delta^2 f(|\rho \operatorname{tg}\varphi|), \quad \xi = V(p - p_0), \tag{13*}$$

$$|\operatorname{tg}\varphi| = |p_{0x}/p_{0z}|,$$

$$f(\lambda) = {}^1/_4\left[e^{i\lambda}\int_0^1 e^{-i\lambda x^2}dx + e^{-i\lambda}\int_0^1 e^{i\lambda x^2}dx\right]^2. \tag{14}$$

The function $f(\lambda)$ can also be expressed in terms of Fresnel integrals:

$$f(\lambda) = \frac{\pi}{2\lambda}[C(\sqrt{\lambda})\cos\lambda + S(\sqrt{\lambda})\sin\lambda]^2. \tag{15}$$

For $|\rho \tan\varphi| \ll 1$ one can expand in (14) in powers of λ, and we obtain for the spectrum the expression

$$\varepsilon^2 = \xi^2 + \Delta^2[1 - 0.13\rho^2(p_{0x}/p_{0z})^2], \quad \rho = eHd^2/4. \tag{16}$$

In other words, for particle motion at not too small angles to the film surface the anisotropy of the spectrum is small and the gap is close to Δ.

It can be shown that there exist λ_k such that $f(\lambda_k) = 0$. This means that there exist directions in which there is no gap in the spectrum. The first zero of the equation $f(\lambda) = 0$ is for $\lambda = 2.6$, so that

$$|\operatorname{tg}\varphi| = 2.6/\rho. \tag{17}$$

2. SPECIFIC HEAT

To calculate the specific heat per unit volume we use its usual expression in terms of the elementary excitation spectrum:[1]

$$C = \frac{\partial(E/V)}{\partial T} = 2\sum_{\mathbf{k}}\varepsilon_{\mathbf{k}}\frac{\partial}{\partial T}[1 + \exp(\varepsilon_{\mathbf{k}}/T)]^{-1}. \tag{18}$$

The main contribution to the specific heat is due to excitations with small quasimomenta ($g \ll d^{-1}$). Therefore one can use for ϵ_k Eq. (13). From Eqs. (18) and (13) we obtain

$$C = \frac{8mp_0}{(2\pi)^3}\int_0^{2\pi}\int_0^{\infty}d\xi\sqrt{\xi^2 + \Delta^2 f}\frac{\partial}{\partial T}\left[1 + \exp\left(\frac{\sqrt{\xi^2 + \Delta^2 f}}{T}\right)\right]^{-1}. \tag{19}$$

At temperatures $T \sim T_c(H)$ account of the angular dependence of f in (18) leads to a small correction to the specific heat (a correction of the order of ρ). However, at low temperatures [$T \ll T_c(H)$] the specific heat is determined by excitations with a small gap. In this case the main contribution to the integral (18) is due to narrow regions in the neighborhood of the zeroes of the function f in which

$$f(\lambda) = \alpha_n(\lambda - \lambda_n)^2, \quad \alpha_n = \frac{\pi}{2\sqrt{\lambda_n}}\left[\frac{1}{\sqrt{2\pi\lambda_n}} - \frac{C(\sqrt{\lambda_n})}{\sin\lambda_n}\right]^2. \tag{20}$$

Here λ_n is the n-th root of the equation $f(\lambda) = 0$.

Substituting expansion (20) in (18) and going over from integration over φ to a sum of integrals over $(\lambda - \lambda_n)$, we obtain

$$C = \frac{12mp_0}{(2\pi)^3}\frac{\rho T^2}{\Delta}\sum_1^{\infty}\frac{1}{\lambda_n^2\sqrt{\alpha_n}}\int_0^{\infty}\frac{x^2dx}{1+e^x}. \tag{21}$$

Calculating the sum and the integral, we find the specific heat per unit volume to be

$$C = 1.2C_n(T/\Delta)eHd^2, \tag{22}$$

where C_n is the electronic part of the specific heat in the normal phase and Δ is taken at $T = 0$

*tg ≡ tan.

but is a function of the field. It follows from (22) that in a rather wide range of temperatures $[T \ll T_c(H)/|\ln \rho|]$ the specific heat decreases according to a power law. Equation (22) for the specific heat is valid when the field H is not too small. The lower bound on the field is obtained from the condition $\nu \ll 1$ [ν is obtained from (10)]. Taking into account the fact that the important angles are those of the order of ρ, we find that

$$eH \gg 1/\xi_0 d. \tag{23}$$

If the film is sufficiently thin, the field is bounded by the condition

$$eH \gg (p_0 d^3)^{-1}, \tag{24}$$

due to the violation of the quasiclassical nature of the motion. Condition (24) is violated before condition (23) only when $d^2 < \xi_0/p_0$.

It should be noted that all the equations are valid only for pure films and in specular reflection from the walls.

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