MODE INTERACTION IN A GAS LASER

B. L. ZHELNOV, A. P. KAZANTSEV, and V. S. SMIRNOV

Semiconductor Physics Institute, Siberian Branch, Academy of Sciences, U.S.S.R.

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The interaction of traveling-wave modes in a gas laser with a ring resonator is considered. The interaction is based upon the nonlinearity of the medium and the mode coupling by mirror reflection. The latter mechanism can be used to explain the effects of frequency pulling and suppression of one of the traveling-wave modes.

A NUMBER of recent papers deal with the ring resonator laser capable of exciting traveling-wave modes. In solid-state lasers, a single traveling-wave mode may be stable.[1-3] In a gas laser, however, a single traveling-wave mode is unstable, the standing wave being the only stable mode.[2] It may seem strange, therefore, that the fairly weak feedback of a gas laser can be used to suppress one of the traveling-wave modes.[4] Another interesting effect, which may also be derived from the feedback between waves traveling in opposite directions, is frequency pulling in a rotating laser.[5]

The present paper deals with the interaction of opposed waves, based upon both the nonlinearity of the medium and the coupling between modes reflected from the mirrors. The latter mechanism is described phenomenologically by Eq. (4). If the mirror coupling is due to diffraction effects, the coupling coefficient is $\epsilon \lesssim 10^{-5}$. Uneven mirrors can considerably increase ϵ; nevertheless, we assume here that ϵ is fairly small. The area of interest of this paper is basically limited to the electrodynamic aspect of the problem, the active medium being defined by the polarization coefficient χ.

The first section of the paper presents equations of motion for wave amplitude and phase in a rotating system of coordinates, taking account of reflective mode coupling. The second and third sections give solutions to these equations; these solutions describe the effects of frequency pulling and suppression of one of the modes.

It should be noted that the order of magnitude of the frequency pulling region was estimated previously by Bershteĭn.[6]

1. In a slowly rotating coordinate system, Maxwell's equations in a linear approximation with respect to $\boldsymbol{\beta}$ ($\boldsymbol{\beta}(\mathbf{r})$ is the linear velocity of rotation, c = 1) are of the form,

$$\Delta \mathbf{E} - \left(\frac{\partial}{\partial t} + 2\boldsymbol{\beta}\nabla\right)\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial^2 \mathbf{P}}{\partial t^2}, \tag{1}$$

where $\mathbf{P}$ is the polarization vector of the active medium (taking account of radiation losses).

The solution to (1) shall be sought in the form of a sum of opposed traveling waves with slowly varying amplitudes:

$$E = e^{i\omega t}[E_+(x, t)e^{ikx} + E_-(x, t)e^{-ikx}] + \text{compl. conj.},$$
$$P = e^{i\omega t}[P_+(x, t)e^{ikx} + P_-(x, t)e^{-ikx}] + \text{compl. conj.} \tag{2}$$

where $\omega = |k|$ is the natural frequency of the resonator, so that $\exp(\pm ikl) = 1$, and l is the beam path length in the resonator. Considering that field amplitudes vary slowly.

$$\partial E_\pm / \partial t \ll \omega E_\pm, \quad \partial E_\pm / \partial x \ll k E_\pm,$$

Eq. (1) yields

$$\frac{\partial E_\pm}{\partial t} \pm \frac{\partial E_\pm}{\partial x} \mp i\boldsymbol{\beta}\mathbf{k}E_\pm = i\omega P_\pm. \tag{3}$$

The coupling between modes E_+ and E_-, arising on reflection from the mirror at the point x = 0, can be expressed by the equation

$$E_\pm(-0, t) - E_\pm(+0, t) = \pm i\varepsilon_\mp E_\mp. \tag{4}$$

The mode coupling coefficients with respect to amplitude, $\epsilon_\pm$, are complex in general. However, it can be readily shown that in limiting cases of $\epsilon_+ = \epsilon_-$, $\epsilon_+ \ll \epsilon_-$, which are considered below, the argument $\epsilon_\pm$ causes only a certain constant relative phase shift of the mode. Consequently, $\epsilon_\pm$ shall be considered real. The case of $\epsilon_+ \lessgtr \epsilon_-$ approximately corresponds to the case, cited in [4], in which a portion of the energy is transferred from mode $E_\mp$ to mode $E_\pm$ via an additional mirror.

The solutions to Eq. (3) must also satisfy the periodicity condition:

$$E_\pm(x, t) = E_\pm(x - l, t). \qquad (5)$$

The dependence of $E_\pm$ upon x can be found by perturbation theory, as in the paper of Kuznetsova and Rautian.[7] When the reflection coefficient of the mirrors is close to unity, only the first two terms of the expansion can be considered:

$$E_\pm(x, t) = E_\pm^{(0)}(t) - \frac{x}{l} E_\pm^{(1)}(t),$$

$$\pm \partial E_\pm^{(0)}/\partial t + i\beta k E_\pm^{(0)} \mp i\omega P_\pm^{(0)} = E_\pm^{(1)}/l, \qquad (6)$$

where $P_\pm^{(0)}$ depends only upon $E_\pm^{(0)}$.

Substituting (6) into (5), and taking (4) into account,

$$l\frac{\partial E_\pm}{\partial t} \mp i\Delta E_\pm - \chi_\pm E_\pm + i\epsilon_\mp E_\mp = 0, \quad \frac{\Delta}{l} = \frac{\omega}{l}\oint \beta\, dl; \qquad (7)$$

Δ/l is the Doppler frequency shift due to rotation, and $\chi_\pm$ is the coefficient of polarization of the active medium with allowance for resonator losses.

Lamb[8] computed $\chi_\pm$ for standing-wave modes. Similar computations for the case of traveling-wave modes of any amplitude lead to the following expression:

$$\chi_\pm = \xi r\left(1 - |E_\pm|^2 - \frac{|E_\mp|^2}{1 + i\delta}\right). \qquad (8)$$

Here r is the mirror transmittance ($r \ll 1$), $\xi = I/I_0 - 1$ is a so-called generation parameter defining the excess of pump energy I over the threshold value I_0, $\delta = (\omega - \omega_0)\tau$ is a dimensionless detuning of field frequency ω relative to the transition frequency ω_0, and τ is the characteristic lifetime of an excited atom. The electric field amplitudes $E_\pm$ are dimensionless (a system of units is used where $\hbar/2\tau d = 1$, d being the dipole moment of transition).

The criterion of validity of the formula for $\chi_\pm$ is based on the assumption that $\tau dE_\pm/dt \ll E_\pm$. In the case of field $E_\pm$, constrained by the reflective wave coupling, this condition is practically met for $\epsilon_\pm \lesssim r$.

Let us note that when $\epsilon_\pm = \Delta = 0$, Eq. (7) reduces to the usual laser equations; the wave energies are then determined by the condition $\mathrm{Re}\,\chi_\pm = 0$ and turn out to be equal, while $\mathrm{Im}\,\chi_\pm$ determines the frequency pulling effect. In (7) it is convenient to separate the amplitudes and phases. Assuming that $E_\pm = \rho_\pm \exp(i\varphi_\pm)$ and substituting $\alpha_\pm = \epsilon_\pm/\xi r$, $\Omega = 2\Delta/\xi r$ and $t \to tl/\xi r$, we obtain

$$\dot{\rho}_\pm = \rho_\pm(1 - \rho_+^2 - \rho_-^2) + \delta^2\rho_\pm\rho_\mp^2 \mp \alpha_\mp\rho_\mp \sin\varphi, \qquad (9)$$

$$\dot{\varphi} = \Omega + \delta(\rho_+^2 - \rho_-^2) + \frac{\alpha_+\rho_+^2 - \alpha_-\rho_-^2}{\rho_+\rho_-}\cos\varphi, \qquad (10)$$

$$\dot{\varphi}_\pm = \pm\frac{1}{2}\Omega - \delta\rho_\mp^2 - \alpha_\mp\frac{\rho_\mp}{\rho_\pm}\cos\varphi, \quad \varphi = \varphi_+ - \varphi_-. \qquad (11)$$

Equations (9)–(11) contain an expansion in terms of δ, since the effects discussed below basically occur only when the detuning is small.

2. When the radiation energies are not excessively low, one can assume that $\alpha_\pm \ll 1$. Since the frequency pulling effect occurs when $\Omega \lesssim \sqrt{\alpha_\pm}$, we have also $\Omega \ll 1$.

In terms of (9) and (10), frequency pulling means that φ = const and $\rho_\pm$ = const. Let us consider the conditions under which such a mode can be realized. The problem basically depends upon the magnitude δ of detuning. Let us first note the important property of the wave gain $\mathrm{Re}\,\chi_\pm$: when $\delta \neq 0$ it follows from (8) that $\mathrm{Re}\,\chi_+ > \mathrm{Re}\,\chi_-$ if $|E_-|^2 > |E_+|^2$, and vice versa. This means that the active medium tends to equalize energy among the modes. The larger δ, the larger is this effect. On the other hand, the rotation and various coefficients of mode coupling $\alpha_\pm$ cause a non-uniform distribution of energy among the modes. It can be readily shown that both effects are of the same order when $\delta^2 \sim \alpha_\pm$. Therefore, when $\delta^2 \gg \alpha_\pm$, we have $\rho_+^2 \approx \rho_-^2 \approx 1/2$. The value of $\rho_+^2 - \rho_-^2$ can be found from (9) neglecting $\rho_\pm \sim \alpha^2$. As a result, we have

$$\dot{\varphi} = \Omega - \delta^{-1}(\alpha_+ + \alpha_-)\sin\varphi + (\alpha_+ - \alpha_-)\cos\varphi. \qquad (12)$$

This equation will obviously have the solution φ = const only when $|\Omega| < \Omega_c$, where

$$\Omega_c = [(\alpha_+ + \alpha_-)^2/\delta^2 + (\alpha_+ - \alpha_-)^2]^{1/2}. \qquad (13)$$

When $|\Omega| > \Omega_c$, oscillations appear which, in the limit of $|\Omega| \gg \Omega_c$, describe the usual Doppler frequency shift.

In the second limiting case $\delta^2 \ll \alpha_\pm$, the wave amplitudes $\rho_\pm$ differ in general. One can merely assert that $\rho_+^2 + \rho_-^2 = 1$. Assuming that $\rho_+ = \cos(\psi/2)$ and $\rho_- = \sin(\psi/2)$, the following equations will be obtained for ψ and φ:

$$\dot{\psi} = [(\alpha_+ + \alpha_-) + (\alpha_+ - \alpha_-)\cos\psi]\sin\varphi + \delta^2\sin\psi\cos\psi, \qquad (14)$$

$$\dot{\varphi} = \Omega + \delta\cos\psi + \frac{\cos\varphi}{\sin\psi}[(\alpha_+ - \alpha_-) + (\alpha_+ + \alpha_-)\cos\psi]. \qquad (15)$$

If the small term with δ^2 is dropped from (14), several oscillating solutions are possible for (14) and (15); the term with δ^2 determines which of these solutions are stable.

Let us consider the stability of stationary states defined by the equations $\dot{\psi} = \dot{\varphi} = 0$, for $\alpha_+ = \alpha_-$

$= \alpha/2$. Linearizing (14) and (15) near the equilibrium point, we obtain the following dispersion relation for perturbations of the exp (Γt) type:

$$\Gamma^2 + \Gamma\delta^2(1 - 3\cos^2\psi) + \left(\delta + \frac{\Omega}{\cos\psi}\right)\left(\frac{\Omega}{\cos\psi} + \delta\cos^2\psi\right) = 0, \quad (16)$$

$$\sin\varphi = 0, \quad \Omega + \delta\cos\psi \pm \alpha\cot\psi = 0. \quad (17)$$

When $\Omega = 0$, there are two equilibrium positions: $\cos\psi = 0$ $(\rho_+ = \rho_-)$, and $\sin\psi = |\alpha/\delta|$, if $|\delta| > \alpha$. The first solution is stable, while the second is not. As a consequence of rotation, the first equilibrium position changes, the amplitudes are no longer equal, and the equilibrium becomes unstable.

We determine the critical value Ω_c at which instability sets in from the condition Re $\Gamma(\Omega_c) = 0$, so that the frequency pulling region will, as before, be defined by the condition $|\Omega| < \Omega_c$. From (16) and (17) it follows that

$$\Omega_c = \alpha/\sqrt{2} + |\delta|/\sqrt{3}. \quad (18)$$

It should be noted that within the frequency region $\delta^2 \ll \alpha$ under consideration, $\cos^2\psi = 1/3$ at the stability limit, and the wave energies differ by an amount of the order of unity.

The dependence of Ω_c upon detuning is qualitatively shown in Fig. 1. This type of relationship can be explained as follows. In the stationary state, frequency shift due to rotation is compensated by unequal frequency pulling for various modes, insofar as the latter effect is determined by the $\delta\rho_\pm^2$ term in Eq. (11). The approximate equilibrium condition is

$$\Omega + \delta(\rho_+^2 - \rho_-^2) \approx 0. \quad (19)$$

When $|\delta| \ll \sqrt{\alpha}$, the mode energies may markedly differ from one another, so that $\rho_+^2 - \rho_-^2 \lesssim 1$, and the frequency pulling effect will be possible only when $\Omega \lesssim \delta$. When $|\delta| \gg \sqrt{\alpha}$, the mode energies are almost the same: $\rho_+^2 - \rho_-^2 \approx \alpha\delta^{-2}\sin\varphi$. In this case $\Omega_c = |\alpha/\delta|$. It should be noted for large and small δ, the critical velocity of rotation does not depend upon the radiation energy. Indeed, when expressed in dimensional units, $\Delta_c \sim \epsilon$ for $|\delta| \ll \epsilon/\xi r$ and $\Delta_c \sim \epsilon/\delta$ for $|\delta| \gg \sqrt{\epsilon/\xi r}$. However, the peak value of Δ_c is proportional to the root of radiation energy: $\Delta_c \sim \sqrt{\epsilon\xi r}$ for $|\delta| \approx \sqrt{\epsilon/\xi r}$.

According to [5], frequency pulling occurs within a region of several hundred cps, so that $(\Delta\omega/\omega)_c \sim 10^{-12}$. The same order of magnitude can be obtained from the above formulas, taking $\epsilon \sim 10^{-5}$, $kl \sim 5 \times 10^6$ and $\delta \sim 1$.

3. Let us now consider the effect of suppressing one of the modes. To be more explicit, let us consider that the minus mode ($\psi \ll 1$) is being suppressed. This can occur only if $\alpha_- \gg \alpha_+$ and $\delta \ll \alpha_-$. Considering that $\sqrt{\alpha_+/\alpha_-} \ll \psi \ll 1$, Eqs. (14) and (15) can be simplified:

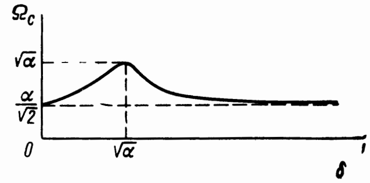

FIG. 1

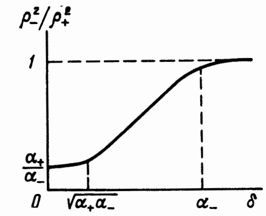

FIG. 2

$$\dot\psi = \tfrac{1}{2}\alpha_-\psi^2\sin\varphi + \delta^2\psi, \quad (20)$$

$$\dot\varphi = \delta - \tfrac{1}{2}\alpha_-\psi\cos\varphi. \quad (21)$$

The stationary solution of these equations is represented by $\psi = 2|\delta/\alpha_-|$ and $\varphi \approx -\delta$. The energy ratio of traveling waves in dimensional units is

$$\rho_-^2/\rho_+^2 = (2\delta\xi r/\varepsilon_-)^2. \quad (22)$$

Linearizing (20) and (21) in terms of small perturbations, we can readily show that solution (22) is stable. The small-oscillation damping increment, Γ, will in this case be

$$\Gamma = -\delta^2 \pm i\delta. \quad (23)$$

When $\delta \lesssim \sqrt{\alpha_+\alpha_-}$, expression (22) becomes invalid and should be replaced by the relation

$$\rho_-^2/\rho_+^2 = \psi^2 \sim \alpha_+/\alpha_-. \quad (24)$$

The variation in energy ratio of two modes as a function of δ is shown schematically in Fig. 2.

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