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SELF-FOCUSING AND SELF-TRAPPING OF INTENSE LIGHT BEAMS IN A NONLINEAR MEDIUM

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A stationary theory of self-action of finite light beams in a nonlinear medium is developed in a quasi-optical approximation. The analysis has been performed both in the approximation of geometrical optics, and in the case where diffraction effects have been accounted for. The conditions required for the nonlinear medium to produce a focusing effect on the beam have been determined. It has been established that the self-focusing effect is, in general, of the aberration type, with the exception of special cases. Self-trapping conditions have been determined for two- and three-dimensional beams in a nonlinear medium. It was shown that saturation of the nonlinear part of the refraction index plays a basic role in the self-trapping phenomenon. The dimensions of the focal spot have been computed for the self-focusing beam in the nonlinear medium. The substantial influence of the nonlinearity on the structure of the focal region has been noted, particularly for the case of a cylindrical Gaussian beam. The discussion includes the self-focusing mechanisms that can be realized experimentally.

1. INTRODUCTION

THE self-action of electromagnetic waves in a nonlinear medium occupies a special place among the numerous problems in nonlinear optics that have recently come to the fore in connection with the development of high-power lasers. The self-action effect arises out of the dependence of the complex dielectric permittivity (complex refractive index) upon the intensity of the propagating wave. It is described phenomenlogically by the odd terms in the expansion of the nonlinear part of polarization of the material medium with respect to the electric field **E** of the wave. Thus, if the i-component of the nonlinear polarization vector is represented in the form of the expansions

$$P_i^{(nl)} = \hat{\chi}_{ijk}E_jE_k + \hat{\chi}_{ijkl}E_jE_kE_l + \hat{\chi}_{ijklm}E_jE_kE_lE_m + \hat{\chi}_{ijklmn}E_jE_kE_lE_mE_n + \ldots, \quad (1)$$

the self-action effects will be described by the tensors $\hat{\chi}_{ijkl}$, $\hat{\chi}_{ijklmn}$, etc.

According to (1), the index of refraction for a wave propagating in a nonlinear medium can be written as

$$n = n_0 + n_2E^2 + n_4E^4 + n_6E^6 + \ldots, \quad (2)$$

where n_2, n_4, n_6, are in general complex. Effects connected with the imaginary parts of n_2 (two-photon absorption) and n_4 (three-photon absorption) have already been frequently observed experimentally. Recent literature contains lively discussions of the effects due to the real parts of these coefficients, i.e., to the dependence of the phase velocity of the wave upon its intensity. An analysis of such effects is particularly interesting in the case of finite beams. The application of nonlinear corrections to the index of refraction causes the initially homogeneous medium to become optically inhomogeneous in the presence of a strong light field. Consequently, the beam path substantially depends upon the electromagnetic field intensity (see [1-4] and the brief review[5]).[1)]

As is shown in the papers cited, an extraordinarily interesting consequence of this inhomogeneity is the specific mode of propagation of a light beam in a nonlinear medium, the so-called self-trapping mode, whereby the light beam creates, as it were, its own optical waveguide. Steady-state profiles of self-trapping light beams were computed in [2, 3]. The first experimental observation of this effect was reported in [7]. Although the theoretically determined power P_{cr} of a self-trapping

[1)]Nonlinear corrections to phase velocity in one-dimensional problems have been treated in[6], for example, which discusses the effect of these corrections upon the generation of harmonics by plane waves and parametric amplification of traveling plane electromagnetic waves.

light beam (it weakly depends on beam diameter in the case of a three-dimensional cylindrical beam, and is inversely proportional to the linear dimension of the beam in the case of a two-dimensional beam) turns out to be comparatively low, no experimental work has as yet reported the self-trapping effect in the form predicted in [3]. The causes of this situation were discussed in [8,9], and it seems that specially selected boundary conditions are necessary to realize the self-trapping effect due solely to the n_2 term in (2).

Nevertheless, even if the self-trapping mode fails to materialize, the path of a focused (also, given sufficiently high power, of an unfocused) laser beam propagating in a material medium (including a laser oscillator or amplifier crystal), differs substantially from that computed in a linear approximation. Therefore, the form of an intense wave front, the diameter of the focal spot, and the extent of the focal region, can largely depend upon the self-action effects. The above conditions may be fairly important, particularly for the quantitative interpretation of such effects as stimulated Raman scattering,[2)] stimulated Mandel'shtam-Brillouin scattering, etc., since large power flows of $\sim 10^{10}$ W/cm^2 are required for their observation, and

$$n_0^{-1}\,\mathrm{Re}\, n_2E^2 \approx 10^{-4} - 10^{-5}.$$

The effect of the change in the beam path due to self-action will, from now on, be called the self-focusing effect.

The subject of this paper is, first of all, a theoretical investigation of the self-focusing effect. These effects will be analyzed here both in the approximation of geometrical optics (giving the method for integrating equations of nonlinear geometrical optics for the case of a two-dimensional beam, and an example of computing rays in a nonlinear medium with $n_2 \neq 0$, $n_4 = 0$), and when diffraction effects are accounted for. It is shown that the higher-order terms in expansion (2), describing the saturation of nonlinear polarization, is particularly significant in the case of three-dimensional beams. In particular, when $n_4 < 0$, self-trapping of a three-dimensional beam takes place within a sufficiently wide class of boundary conditions. The theory is based on a quasi-optical approximation and on the method of slowly changing amplitudes, which we have previously used to analyze the generation of optical harmonics in convergent beams (see [11]).

2. BASIC EQUATIONS

The analysis of the self-action effect starts with the wave equation,

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{E} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial^2\mathbf{P}^{(l)}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial^2\mathbf{P}^{(nl)}}{\partial t^2} = 0 \quad (3)^*$$

and the material equations (harmonic waves are discussed below),

$$\mathbf{P}^{(l)} = \hat{\varkappa}\mathbf{E}, \quad (4)$$

$$\mathbf{P}^{(nl)} = \chi^{(3)}\mathbf{EEE} + \chi^{(5)}\mathbf{EEEEE}, \quad (5)$$

$$P^{(nl)}/P^{(l)} \ll 1. \quad (6)$$

The solution of Eq. (3), for a finite, weakly converging, or weakly diverging, beam, will be sought in the form,

$$\mathbf{E} = \mathbf{e}A(\mu\mathbf{r}\mathbf{z}_0;\ \sqrt{\mu}[\mathbf{r}\mathbf{z}_0])\ \exp\{i(\omega t - kz)\} \quad (7)^\dagger$$

(only the self-action effect is being considered; in the linear approximation, the medium is assumed homogeneous and incident wave harmonics are neglected. The problem is considered stationary.) Here, μ is a small parameter.

Since, according to the conditions of the problem, the beam is finite, Eq. (5) accounts for a variation of complex amplitude across as well as along the ray directed along the z axis. The variation across the ray is faster, since it involves transition into the shadow region. Substituting (7) into (3), and considering that nonlinear polarization is of the order of μ, we limit the analysis to the first-order terms with respect to μ, arriving at a simplified equation describing the self-action effect of a harmonic wave:

$$2ik\frac{\partial A}{\partial z} = \Delta_\perp A + \frac{n_2|A|^2}{n_0}k^2A + \frac{n_4|A|^4}{n_0}k^2A. \quad (8)$$

Here, $\Delta_\perp$ is the two-dimensional Laplace operator in the plane perpendicular to the ray (z axis). When $n_2 = n_4 = 0$, Eq. (8) becomes the parabolic equation used in the approximate diffraction theory. In this manner, (8) corresponds to the so-called quasi-optical approximation, and can thus describe the self-action of the normal wave in both an isotropic and an anisotropic medium. Introducing the eikonal s,

$$A = A_0 \exp(-iks), \quad (9)$$

[2)] It will be recalled that, according to recent papers, the experimentally determined gains for Stokes components materially exceed values determined theoretically[10]; at the same time, the self-action effect was not accounted for.

*rot ≡ curl.

†$[\mathbf{r}\,\mathbf{z}_0] \equiv \mathbf{r} \times \mathbf{z}_0$.

a system of two equations for A_0 and s is obtained instead of (8); for a two-dimensional beam and a three-dimensional cylindrically-symmetrical beam, these equations are

$$2\frac{\partial s}{\partial z}+\left(\frac{\partial s}{\partial r}\right)^2=\frac{n_2A_0^2}{n_0}+\frac{n_4A_0^4}{n_0}+\frac{1}{k^2A_0}\left(\frac{\partial^2 A_0}{\partial r^2}+\frac{m}{r}\frac{\partial A_0}{\partial r}\right), \tag{10}$$

$$\frac{\partial A_0}{\partial z}+\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r}+\frac{1}{2}A_0\left(\frac{\partial^2 s}{\partial r^2}+\frac{m}{r}\frac{\partial s}{\partial r}\right)=0. \tag{11}$$

Here $m = 0$ corresponds to the two-dimensional beam and $m = 1$ to the three-dimensional beam. General solutions of (9) and (10) involve considerable difficulties. Therefore we first determine the path of the rays in a nonlinear medium in the approximation of geometrical optics ($k \to \infty$), and then consider the diffraction effects.

3. GEOMETRICAL OPTICS IN A NONLINEAR MEDIUM

Setting $k \to \infty$ and $n_4 = 0$ in (10) and (11) (the saturation effect, $n_4 \neq 0$, as will be shown below, is significant only in the discussion of phenomena near the focus, where the geometrical optics approximation loses its validity in any case), we arrive at the equations for the eikonal,

$$2\frac{\partial s}{\partial z}+\left(\frac{\partial s}{\partial r}\right)^2=\frac{n^2}{n_0}A_0^2 \tag{12}$$

and for the amplitude A_0 (after multiplying by A_0),

$$\frac{\partial A_0^2}{\partial z}+\frac{\partial s}{\partial r}\frac{\partial A_0^2}{\partial r}+A_0^2\left(\frac{\partial^2 s}{\partial r^2}+\frac{m}{r}\frac{\partial s}{\partial r}\right)=0. \tag{13}$$

It should be noted that the derivation of general solutions is difficult even for (12) and (13). Therefore we consider certain particular solutions that are of interest in the comparison of theory with experiment.

When $n_2 = 0$ (zeroth approximation: linear medium), Eqs. (12) and (13) are clearly satisfied by cylindrical and spherical waves (they are expressed here in a form corresponding to the quasi-optical approximation: see, for example, [11]). Thus when $n_2 \neq 0$ (first approximation: weakly nonlinear medium), cylindrical and spherical waves with a variable radius of curvature will constitute a natural generalization of these solutions. Here,

$$s=\frac{\beta(z)r^2}{2}+\varphi(z),\quad A_0^2=\frac{E_0^2}{f^{1+m}(z)}\left[1-\frac{r^2}{r_0^2f^2(z)}\right] \tag{14}$$

(as before, $m = 0$ corresponds to the cylindrical wave and $m = 1$ to the spherical wave). When $z = 0$

$$\beta(0)=1/R,\quad \varphi(0)=0,\quad f(0)=1; \tag{15}$$

$$A_0^2(0,r)=E_0^2(1-r^2/r_0^2) \tag{15a}$$

and consequently $R < 0$ corresponds to converging waves incident upon the boundary of the nonlinear medium, and $R > 0$ corresponds to diverging waves; within the nonlinear medium the phase front of the wave undergoes changes.

Substituting (14) into (12) and (13), we arrive at the equations for functions $\varphi(z)$, $\beta(z)$, and $f(z)$:

$$d\varphi(z)/dz=n_2E_0^2/2n_0f^{1+m}(z), \tag{16}$$

$$\beta(z)=f^{-1}(z)\,df(z)/dz, \tag{17}$$

$$d^2f(z)/dz^2=-n_2E_0^2/n_0r_0^2f^{2+m}(z). \tag{18}$$

When boundary condition (15) is accounted for, (18) has as a first integral

$$\left(\frac{df}{dz}\right)^2=\frac{2n_2E_0^2}{(1+m)n_0r_0^2f^{1+m}}+C,$$

$$C=\frac{1}{R^2}-\frac{2n_2E_0^2}{(1+m)n_0r_0^2}. \tag{19}$$

For the case of a spherical wave ($m = 1$) we have, after integration of (19),

$$f^2(z)=\left(\frac{1}{R^2}-\frac{n_2E_0^2}{n_0r_0^2}\right)z^2+\frac{2}{R}z+1. \tag{20}$$

When $f(z) = 0$, by virtue of (14), $A_0^2 \to \infty$ and the corresponding point z_f on the z axis is a focal point.

The quadratic equation obtained from (20) with $f = 0$ defines in the general case two focal points z_{f1} and z_{f2}, where

$$z_{f1}=Rr_0\sqrt{\frac{n_0}{n_2E_0^2}}\left[R-r_0\sqrt{\frac{n_0}{n_2E_0^2}}\right]^{-1}. \tag{21}$$

When $n_2 > 0$, the nonlinear medium exerts a focusing action upon the light beam; it has a defocusing effect when $n_2 < 0$. The parameter characterizing the focusing effect of the nonlinear medium on a beam with an amplitude distribution described by (15) is

$$R_{nl}=r_0\sqrt{n_0/n_2E_0^2}. \tag{22}$$

The meaning of this quantity can be readily made apparent by writing (21) in the form,

$$1/z_{f_1}=1/R_{nl}-1/R. \tag{23}$$

Equation (23) shows that R_{nl} determines the distance over which the light beam with a plane phase front ($R \to \infty$) and amplitude distribution (15) is self-focused in the nonlinear medium. In the case of a converging beam ($R < 0$), the focal length decreases in a nonlinear medium with $n_2 > 0$. Moreover, an initially diverging beam ($R > 0$), becomes self-focused after entering the nonlinear medium, if R is not too small (the initial divergence is not excessive). The critical value is $R_{cr} = R_{nl}$.

The position of the second focal point is determined by the relationship,

$$1/z_{f_2} = -1/R - 1/R_{nl}. \qquad (24)$$

The second focal point exists only if $R < 0$, with $z_{f2} > z_{f1}$ (see Fig. 1). Equations (12) and (13) can also be used to determine the paths of rays in the nonlinear medium.

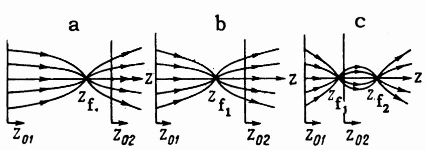

FIG. 1. Ray paths of a three-dimensional light beam with initial amplitude distribution $A^2 = E_0^2(1 - r^2/r_0^2)$. In all diagrams, section $z - z_{01}$ corresponds to the entry into the medium of a converging wave, $R < 0$. For this case, a—$|R| > R_{nl}$, weak beam convergence (a single focus appears); b—$|R| = R_{nl}$, critical convergence (a single focus remains); c—$|R| < R_{nl}$, strong convergence (two foci appear). Section $z = z_{02}$ corresponds to the entry of a diverging wave, $R > 0$. For this in a—$R < R_{nl}$ (no focus), b—$R = R_{nl}$ (no focus), C—$R > R_{nl}$ (one focus appears).

A somewhat more general analysis can be made in the case of a two-dimensional beam ($m = 0$). The geometrical-optics equations for this beam are:

$$2\frac{\partial s}{\partial z} + \left(\frac{\partial s}{\partial x}\right)^2 = \frac{n_2}{n_0} A_0^2, \qquad (25)$$

$$\frac{\partial A_0^2}{\partial z} + \frac{\partial s}{\partial x}\frac{\partial A_0^2}{\partial x} + A_0^2 \frac{\partial^2 s}{\partial x^2} = 0 \qquad (26)$$

Introducing new variables

$$u = \partial s/\partial x, \qquad \rho = A_0^2, \qquad (27)$$

the above expressions are reduced to a system of first-order equations:

$$\frac{\partial u}{\partial z} + u\frac{\partial u}{\partial x} - \gamma\frac{\partial \rho}{\partial x} = 0, \qquad \frac{\partial \rho}{\partial z} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \quad (28)$$

where $\gamma = n_2/2n_0$.

It should be noted that Eqs. (28) are of the same form as those describing a nonstationary flow of barotropic fluid (the z coordinate in (28) assumes the role of time, see [12]), and differ from the latter only in the sign of γ. This difference is admittedly quite significant, because a change of sign in γ converts the equation from hyperbolic into elliptical. System (28) can be reduced to a linear form by a hodographic transformation. This affords the opportunity to design a sufficiently general methodology of analyzing equations of nonlinear geometrical optics.

In our case, however, we shall use a simpler approach: a substitution of variables,

$$x - uz = \xi, \qquad z\rho = \eta, \qquad (29)$$

converts system (28) to the system,

$$\frac{\partial u}{\partial \eta} - \frac{\gamma}{\rho}\frac{\partial \rho}{\partial \xi} = 0, \qquad \frac{\partial \rho}{\partial \eta} + \frac{\partial u}{\partial \xi} = 0. \qquad (30)$$

System (30) has a partial solution in the form,

$$u = -\frac{2\gamma z\rho}{a}\,\mathrm{th}\left(\frac{x - uz}{a}\right),$$

$$\rho = \left[\rho_0 + \frac{\gamma(z\rho)^2}{a^2}\right]\mathrm{ch}^{-2}\left(\frac{x - uz}{a}\right). \qquad (31)^*$$

In the approximation of geometrical optics, Eqs. (31) describe a wave propagating in a nonlinear medium and having in the $z = 0$ section a plane phase front ($u = \partial s/\partial x = 0$) and an intensity distribution

$$\rho(x, 0) = \rho_0\,\mathrm{ch}^{-2}(x/a). \qquad (32)$$

Using (31), one can determine the paths of rays and the beam profile in various sections of the nonlinear medium; it should be remembered here that

$$\tan \widehat{S_0 z_0} \approx \widehat{S_0 z_0} \approx \partial s/\partial x = u \qquad (33)$$

i.e., the angle between a ray vector $\mathbf{S}_0$ and the z axis (unit vector $\mathbf{z}_0$) approximately equals u for near-axial rays.

Figure 2 shows paths of rays plotted by the isocline method and described by (31). Figure 3 gives the corresponding graphs of intensity distribution in various sections of the beam. As follows from Fig. 2, the rays at first approach the beam axis and self-focusing occurs; however, in contrast to the above case of a self-focused spherical wave with initial profile (15a), in the present case the self-focused beam is subject to fairly strong aberrations. The peripheral rays cross the z axis later than the near-axial rays.

Two regions can be distinguished here: in the first region, the rays converge (intensity is in-

*ch ≡ cosh, th ≡ tanh.

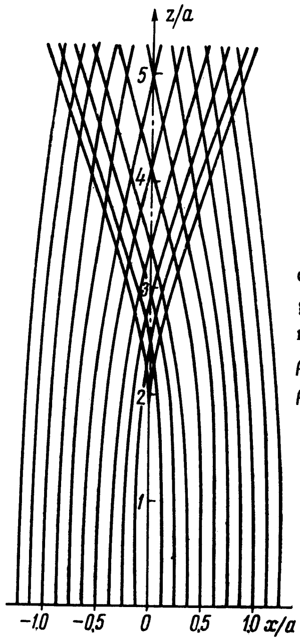

FIG. 2. Ray paths in a two-dimensional light beam propagating in a nonlinear medium with $n_2 > 0$. When $z = 0$, $u = \partial s/\partial x = 0$, $\rho = \rho_0 \cosh^{-2}(x/a)$, parameter $\rho_0\gamma = 1/16$.

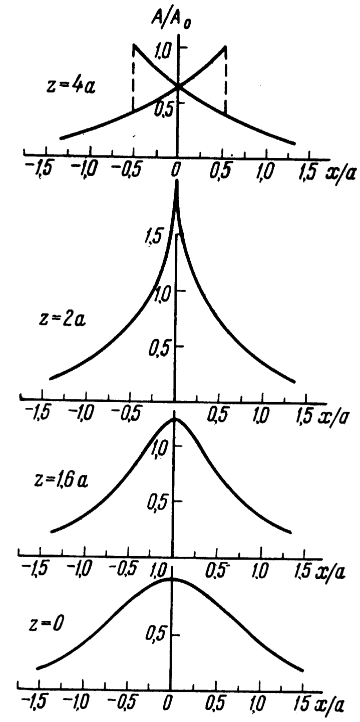

FIG. 3. Intensity distribution over the cross section of a two-dimensional light beam propagating in a nonlinear medium, for various values of the parameter z/a. Beam parameters are the same as in Fig. 2.

creasing), and in the second, the rays diverge (intensity is decreasing). The light intensity reaches maximum in a plane situated at a distance z_f from the boundary of the nonlinear medium:

$$z_f = a / 2\sqrt{\rho_0\gamma} = a\sqrt{2n_0 / n_2E_0^2}. \qquad (34)$$

It should be noted that the integration of Eq. (19), which corresponds to the absence of nonlinear aberration, leads to

$$z_f = \frac{\pi a}{2\sqrt{2}}\sqrt{\frac{n_0}{n_2E_0^2}}$$

(see also R_{nl} in (22)).

When $z > z_f$, the rays begin to intersect one another. Along the boundary rays in this region,

$$u = \pm 2\sqrt{\rho_0\gamma}\sqrt{1 - z_f / z}.$$

The presence of discontinuities in the boundary rays is, of course, due to the use of the geometrical-optics approximation. On the other hand, the effects on the beam boundary, and a correct interpretation of the phenomena near the focal point (such an investigation is particularly needed in the case of the beam described by (14) and (15)), require the consideration of diffraction effects and, as shown by the analysis, of higher-order nonlinear effects. These are the problems we consider next.

4. PHENOMENA NEAR THE FOCAL POINT IN A NONLINEAR MEDIUM

It is well known that allowance for diffraction effects in the linear theory eliminates discontinuities and infinities which appear in equations derived in the geometrical-optics approximation. It is interesting to note that, in general, the situation becomes substantially more complicated in nonlinear problems.

Indeed, the presence of a positive coefficient n_2 in (2) may lead to a drastic "pinching" of the beam in the focal point (as illustrated in Fig. 1, the angles between the rays and the z axis are close to 90° in the vicinity of the focal point). In some cases, the expanding (defocusing) "forces" due to diffraction[3)] are unable to balance the self-action of the beam (this was pointed out in [8]; see also [9]), particularly in the case of a cylindrical beam described by (14) and (15). Consequently, a complete analysis of near-focal phenomena in the nonlinear theory should, in general, include not only diffraction effects, but also other nonlinear effects not accounted for by the real part of coefficient n_2. Minor effects of this type are those associated with the imaginary part of n_2 (two-photon absorption) and effects associated with coefficient n_4 in expansion (2).

If $n_2 = n_2' + in_2''$ and $n_4 = 0$ are used in lieu of (10) and (11), we have the system

$$2\frac{\partial s}{\partial z} + \left(\frac{\partial s}{\partial r}\right)^2 = \frac{n_2'}{n_0}A_0^2 + \frac{1}{k^2A_0}\left(\frac{\partial^2 A_0}{\partial r^2} + \frac{m}{r}\frac{\partial A_0}{\partial r}\right), \qquad (35)$$

3)The corresponding terms in the equations do not, of course, have the dimensions of force, and the expression "force" should not be taken literally.

$$\frac{\partial A_0}{\partial r}+\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r}+\frac{1}{2}A_0\left(\frac{\partial^2 s}{\partial r^2}+\frac{m}{r}\frac{\partial s}{\partial r}\right)+\delta A_0^3=0, \quad (36)$$

where δ is a nonlinear absorption coefficient determined by n_2''. An analysis of (35) and (36), carried out for the near-axial region of the beam, shows that in this approximation nonlinear absorption has no significant effect on the path of rays in the near-focal region and, in particular, does not prevent pinching (turning the field to infinity) of the three-dimensional beam. Therefore, further analysis is based on Eqs. (10) and (14) which, along with a nonlinear second-order field term, include a fourth-order field term. According to the above considerations, the most interesting case here is that of $\mathrm{Im}\, n_2 = \mathrm{Im}\, n_4 = 0$, $\mathrm{Re}\, n_2 > 0$, $\mathrm{Re}\, n_4 < 0$, since the n_4 term thus defines the nonlinear-polarization saturation effect.

As in Sec. 2, the eikonal is represented in the form,

$$s=\beta(z)r^2/2+\varphi(z), \quad (37a)$$

and a Gaussian distribution is assumed for the amplitude

$$A_0^2(r,z)=\frac{E_0^2}{f^{1+m}(z)}\exp\left\{-\frac{r^2}{r_0^2 f^2(z)}\right\}; \quad (37b)$$

it should be noted that the general solution of (11) is of the form,

$$A_0^2=\frac{1}{f^{1+m}}F\left(\frac{r}{f}\right), \qquad \frac{1}{f}\frac{df}{dz}=\beta.$$

Boundary conditions (15) will be used, as before. Substituting (37) into (10) and (11), and limiting the region under consideration to the vicinity of the beam axis (for this purpose, the nonlinear terms are expanded in powers of r up to terms containing r^2), we arrive at the equations,

$$2\frac{d\varphi}{dz}=\frac{n_2E_0^2}{n_0 f^{1+m}}-\frac{n_4E_0^4}{n_0 f^{2+2m}}-\frac{(m+1)}{k^2r_0^2f^2}, \quad (38)$$

$$\frac{d^2f}{dz^2}=-\frac{n_2E_0^2}{n_0r_0^2f^{2+m}}+\frac{2n_4E_0^4}{n_0r_0^2f^{3+2m}}+\frac{1}{k^2r_0^4f^3}. \quad (39)$$

Considering boundary conditions (15), we can write the first integral of (39). For the most interesting case of the three-dimensional beam (m = 1), the first integral has the form

$$\left(\frac{df}{dz}\right)^2=\frac{1}{r_0^2f^2}\left(\frac{n_2E_0^2}{n_0}-\frac{1}{k^2r_0^2}\right)-\frac{n_4E_0^4}{n_0r_0^2f^4}+C,$$

$$C=\frac{1}{R^2}-\frac{1}{r_0^2}\left(\frac{n_2E_0^2}{n_0}-\frac{1}{k^2r_0^2}\right)+\frac{n_4E_0^4}{n_0r_0^2}. \quad (40)$$

Using (40), we can readily verify that when $n_4 = 0$ and

$$E_0^2 > n_0/n_2k^2r_0^2 \quad (41)$$

the paths of the rays, computed by taking diffraction into account, do not differ qualitatively from those determined in the approximation of geometrical optics; the expanding force due to diffraction is not able to compensate for the focusing force associated with the term containing n_2 (see also [8,9]). From now on, the beam intensity corresponding to the equality, $E_0^2 = n_0/n_2k^2r_0^2$ (equilibrium between the expanding and focusing forces), will be called the critical intensity.[4)]

The presence of a defocusing effect (however weak) due to saturation substantially changes the picture. Weakly converging (or weakly diverging) beams at z = 0 (c < 0), such that

$$R^2>\frac{r_0^2}{n_2E_0^2/n_0-1/k^2r_0^2}$$

become self-trapped in a medium with $n_2 > 0$ and $n_4 < 0$. The diameter of such a self-trapped beam oscillates; its maximum and minimum values are determined by the following expressions:

$$r^2=\frac{1}{2|C|}\left\{\left(\frac{n_2E_0^2}{n_0}-\frac{1}{k^2r_0^2}\right)\pm\left[\left(\frac{n_2E_0^2}{n_0}-\frac{1}{k^2r_0^2}\right)^2-\frac{4|C|n_4E_0^4r_0^2}{n_0}\right]^{1/2}\right\}. \quad (42)$$

Beams strongly focused at the boundary of the nonlinear medium cannot be self-trapped; the minimum cross section of such a beam in a nonlinear medium corresponds to df/dz = 0, and is determined by

$$[r_f^{(nl)}]^2=\frac{1}{2C}\left\{-\left(\frac{n_2E_0^2}{n_0}-\frac{1}{k^2r_0^2}\right)+\left[\left(\frac{n_2E_0^2}{n_0}-\frac{1}{k^2r_0^2}\right)^2+\frac{4Cn_4E_0^4r_0^2}{n_0}\right]^{1/2}\right\}. \quad (43)$$

For the sake of comparison, we note that the diameter of the focal spot obtained when the beam is focused in a linear medium is

$$r_f^{(l)}=R/kr_0=\lambda/2\pi\alpha, \quad (44)$$

where $\alpha = r_0/R$ is the angular aperture.

Using (43) and (44), we can determine the ratio

4)It should be noted that the intensity determined by $E_0^2 = n_0/n_2k^2r_0^2$, corresponds to the steady-state intensity of a self-trapping beam with $R = \infty$, computed in[3].

of focal sections of a three-dimensional beam focused in a linear and in a nonlinear medium (in the case of the strongly focused beam under consideration, one can assume with a sufficient degree of accuracy that $C = 1/R^2$):

$$\Phi = \left[\frac{r_f^{(l)}}{r_f^{(nl)}}\right]^2 = 2\left\{1 - \frac{P}{P_{cr}} + \left[\left(\frac{P}{P_{cr}} - 1\right)^2 + 4\alpha^2 \frac{n_4}{n_2}\frac{n_0}{n_2}\left(\frac{P}{P_{cr}}\right)^2\right]^{1/2}\right\}^{-1}, \tag{45}$$

Here, P/P_{cr} is the ratio of beam power to critical power, defined by (41).

A similar analysis can be performed for the case of a two-dimensional beam. Setting $m = 0$ in (38) and (39) and considering boundary conditions (15), one can write the first integral as

$$\left(\frac{df}{dz}\right)^2 = \frac{2n_2E_0^2}{n_0r_0^2f} - \frac{1}{f^2}\left(\frac{2n_4E_0^4}{n_0r_0^2} + \frac{1}{k^2r_0^4}\right) + C,$$
$$C = \frac{1}{R^2} - \frac{2n_2E_0^2}{n_0r_0^2} + \frac{2n_4E_0^4}{n_0r_0^2} + \frac{1}{k^2r_0^4}. \tag{46}$$

Here, in contrast with the case of the three-dimensional beam, it is not necessary to account for $n_4 \neq 0$; f(z) nowhere turns to zero when the diffraction term is taken into account.

The behavior of the cylindrical beam in a nonlinear medium is determined by the boundary conditions (the C parameter). Self-trapping of the beam occurs when $C < 0$, i.e., when the initial divergence (or convergence) of the beam is not very high,

$$R^2 > n_0r_0^2 / 2n_2E_0^2.$$

The width of the self-trapped beam oscillates within the limits

$$r = \frac{1}{|C|r_0}\left\{\frac{n_2E_0^2}{n_0} \pm \left[\left(\frac{n_2E_0^2}{n_0}\right)^2 - \frac{|C|}{k_0^2}\right]^{1/2}\right\}. \tag{47}$$

Strongly focused (or strongly defocused) beams, for which $C > 0$, are not subject to self-trapping. The factor Φ, characterizing the decrease in the cross-section of the focal spot when a two-dimensional beam is focused in a nonlinear medium, is expressed by

$$\sqrt{\Phi} = kR\left\{\frac{n_2E_0^2}{n_0} + \left[\left(\frac{n_2E_0^2}{n_0}\right)^2 + \frac{1}{k^2R^2}\right]^{1/2}\right\} \tag{48}$$

(as before, we let $C \approx R^{-2}$ for a strongly focused beam). A marked decrease of the cross section of the focal region in a two-dimensional beam will be obtained when

$$Pn_2 / n_0 \approx c\lambda\alpha / 8\pi^2. \tag{49}$$

5. DISCUSSION

The results of the developed theory thus allow us to reveal many important features concerning the behavior of finite light beams in a medium whose refractive index depends upon the wave intensity. It should be remembered at the same time that, as a rule, analytical results can be obtained only for the near-axial portion of the beam, using the first terms in the expansion with respect to r (see (15a), (39), etc.). A complete analysis of the behavior of peripheral rays and complex beams is possible, in general, only with the aid of computers. Nevertheless, it is possible to obtain some notion on the behavior of beams of complex structure by means of the perturbation method, i.e., assuming that the variation in the amplitude and phase wave front, due to nonlinearity, is small.

As an illustration, we shall consider such an analysis in the approximation of geometrical optics for the case of the two-dimensional beam. It will be convenient to use (28) for this purpose. Setting $\rho = \rho_0 + \rho'$, where $\rho' \sim \mu'$, and $u = \partial s/\partial x \sim \mu'$ (μ' is a small parameter characterizing the perturbation), we arrive at linearized equations for perturbations (ρ_0 = const) in place of (28):

$$\frac{\partial u}{\partial z} - \gamma\frac{\partial \rho}{\partial x} = 0, \qquad \frac{\partial \rho'}{\partial z} + \rho_0\frac{\partial u}{\partial z} = 0, \tag{50}$$

which can be reduced to the corresponding elliptical equations. Setting $\rho' = f(x)$ and $u = \psi(x)$ at $z = 0$, we can find ρ' and ψ at any section z. For example, we have for ρ',

$$\rho'(z, x) = \frac{1}{2}[f(x + i\Gamma z) + f(x - i\Gamma z)] - \frac{\rho_0}{2i\Gamma}[\psi(x + i\Gamma z) - \psi(x - i\Gamma z)], \tag{51}$$

where $\Gamma^2 = \rho_0\gamma$; f and ψ are analytic functions.

Analysis of the solutions (51) shows that if f(x) and ψ(x) are oscillating functions no self-focusing of the beam as a whole occurs, and the beam separates into filaments instead. Since the dimensions of inhomogeneities in f(x) or ψ(x) are usually small (according to Bakhudarova et al.,[13] these dimensions for a ruby laser are $\sim 100\mu$) individual filaments are self-focused over distances which are smaller than the self-focusing of the beam as a whole (compare with Eq. (34)).[5)] However, the most interesting is the fact that a Gaus-

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5)This may explain the stratification of unfocused laser beams in solids and liquids, observed in some experiments.

sian beam with a plane phase front reveals a tendency toward stratification:

$$\psi(x) = 0, \quad f(x) = \rho_0 \exp(-x^2/r_0^2), \quad \rho_0 = \text{const.} \quad (52)$$

Substituting (52) into (51), we get

$$\rho'(x,z) = \rho_0 \exp\left[-\frac{x^2 + \Gamma^2 z^2}{r_0^2}\right] \cos 2\frac{\Gamma x z}{r_0^2}. \quad (53)$$

Thus, consideration of effects on the periphery of the beam may disclose new phenomena.

In conclusion, it should be emphasized that the present work was limited to stationary (time-independent) solutions; consequently, in a comparison of these results with experiment, low-inertia effects, leading to a dependence $n = n(\overline{E})$, should be first taken into account. The lowest inertia is possessed by the mechanism of the nonlinear electronic polarization ($\tau_r \sim 10^{-14}-10^{-15}$ sec) and the high-frequency Kerr effect ($\tau_r \approx 10^{-12}$ sec).

Electronic polarization is characterized by a general relation of the type (see, for example, [14]),

$$P_N^{(nl)}/P_{N-1}^{(nl)} = E/E_{\text{at}}, \quad (54)$$

where P_N^{nl} is the nonlinear polarization of the N-th order.

Equation (45) can be simplified by using (54), since it follows directly from (54) that $(n_4/n_2) \times (n_0/n_2) \approx 1$ (the signs of n_2 and n_4 depend in this case upon the relationship between the wave frequency ω and the resonance frequency ω_0 of the medium; when $\omega_0/2 > \omega > \omega_0/3$ the conditions of Sec. 4 are realized.[6])

When $P = P_{cr}$ (for liquids, $P_{cr} \approx 10-40$ MW, powers typical of modern experimental nonlinear optics) we have $\Phi = \alpha^{-1}$. When $r_0 \approx 1$ cm and $R \approx 5-10$ cm, we get $\Phi = 5-10$; thus, the change in focal spot cross section, due to the self-focusing effect, can be considerable. When $P/P_{cr} \gg 1$ we get $\Phi \sim \alpha^{-2} P_{cr}/P$. Plots of the function $\Phi = \Phi(P/P_{cr})$ are given in Fig. 4 for various values of parameter α. On the other hand, the change in the focal cross section of the two-dimensional beam under the same conditions (see (48) and (49)) is small. This means that the self-focusing effects must be considered in the interpretation of experiments in nonlinear optics involving focused beams; from the viewpoint of revealing the self-focusing effect, the comparison of the results of experiments with cylindrical and spherical lenses would be of considerable interest.

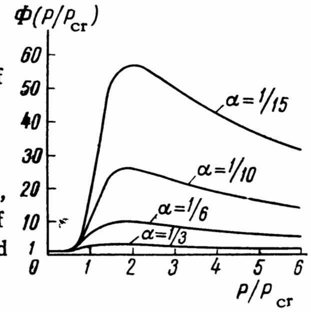

FIG. 4. Plots of the parameter Φ defining the effect of self-focusing upon the size of focal cross-section of the three-dimensional beam in nonlinear medium, as a function of P/P_{cr}. The curve parameter is $\alpha = r_0/R$, denoting one-half of the angle of convergence of the beam focused by a spherical lens.

A correct allowance for the effect of inertial mechanisms (these consist primarily of electrostriction and thermal effects, although the high-frequency Kerr effect is also significant in the case of sufficiently narrow beams) requires the solution of the nonstationary problem. The material equation (5) should be replaced in this case by an appropriate differential equation. We have reported the results of an investigation of nonstationary self-trapping in [14].

Note added in proof (May 3, 1966). The analysis of the nonlinear polarization saturation effect, carried out to the first approximation in this paper, seems to furnish, at least qualitatively, an explanation of the results of recent experimental work on self-focusing of light in liquids[15,16]. The appearance of narrow wave channels in the propagation of unfocused laser beams in liquids having anisotropic molecules, observed in these experiments, is in agreement with the results of analysis which accounts for the presence of the $n_4 < 0$ term in (2) (see (42)). Equations (42) also describe the oscillations of the radius of the self-trapped beam observed by C. Townes and his associates. In this connection, it would be of interest to develop a theory of self-focusing that would completely account for the saturation effect of nonlinear polarization (for example, in the form suggested by Zel'dovich and Raĭzer[7] for the Kerr effect). Finally, we may note that valuable experimental data on the role of the Kerr effect in the self-focusing of light can be obtained by additionally impressing a strong static field upon the liquid.

6)In the case of the high-frequency Kerr effect, always $n_4 < 0$, which is due to saturation occurring when the molecules are "aligned" over the field.

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185