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STABILITY OF VORTEX LATTICES

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The behavior of small perturbations of simple vortex lattices is investigated. It is shown that the triangular lattices and similar ones are stable. The dispersion law for lattice oscillations is found and the shape of the normal oscillations is elucidated. In the limit of long waves they resemble transverse sound in crystals.

1. RECENTLY there has been renewed interest in vortices in an ideal liquid, those discrete vortices whose theory was founded in the last century by Helmholz, Kirchhoff, Stokes, and Routh. For a long time, this theory found no application, and its development was thus hindered. Important results in this field are associated with the name of von Karman, who investigated the effect of a vortex street on the hydrodynamic resistance of a cylinder. His work stimulated many new investigations which, however, have not brought full clarity to the problem of stability of the Karman vortex street. The number of papers on vortices has decreased continually since that time until recently. A considerable number of the aforementioned papers are reviewed in [1-4].

During the last decade, the following topics, which are close in their properties to vortices in an ideal liquid, have become popular: a) vortices of the superfluid component of helium II, b) vortices in superconductors of the second group, c) dislocations (especially screw dislocations) in crystals. Vortices in helium constitute a direct realization of discrete vortices. The flow of the helium is potential everywhere with the exception of vortex filaments, and the circulation of the velocity around the vortex is $\Gamma = h/m \approx 10^{-3}\ \mathrm{cm^2/sec}$ (h—Planck's constant, m—mass of helium atom) and can also be a multiple of this quantity. Vortices in semiconductors differ essentially in that in this case there is a certain depth of penetration λ at which the field of the vortex attenuates exponentially. In many cases the fact that λ is finite is important, even if λ is large. Screw dislocations differ little from the mathematical point of view from vortices in helium.[5]

We shall henceforth concentrate our attention on straight-line vortices parallel to each other. It is precisely these systems that have been the subject of most recent papers.

Investigations of vortices in semiconductors were started by Abrikosov,[6] the study of vortices in helium II was initiated by Feynman,[7] and the planar case of the continual theory of dislocations was considered in detail by Eshelby.[8] Matricon[9] found that in superconductors, out of all the possible simple vortex lattices, the energetically favored lattice is the triangular one. A similar result was obtained for helium by the present author.[5] However, the situation is not so clear with respect to stability of the lattices. The investigations of Abrikosov, Kemoklidze, and Khalatnikov[10] on superconductors and of Pincus and Shapiro[11] on helium have shown that all is not well in connection with stability of lattices against long waves.

In the present paper we undertake an investigation of the stability of a simple vortex lattice in helium.

2. Our problem is essentially a two dimensional one. Therefore it is advantageous to introduce a complex coordinate z in the plane perpendicular to the vortices. In exactly the same manner, the velocity of the liquid will be described by a complex number v, the magnitude and direction of which in the complex z plane coincide with thee velocity of the liquid at the given point.

It is well known[4] that plane potential flow of an incompressible liquid is conveniently described with the aid of a complex potential $\Phi(z)$. The velocity $v(z)$ of the liquid is given by the formula

$$\overline{v(z)} = d\Phi(z)\,/\,dz.$$

The quantity $\overline{v(z)}$ is an analytic function which has on the complex z plane simple poles with equal residues at the points corresponding to the vortices. These are its only singularities in the finite part of the plane. We shall henceforth find it convenient to operate with the function

$$\zeta(z) = \overline{iv(z)}. \qquad (1)$$

For one vortex located at the point $z = 0$, we get $\zeta(z) = 1/z$. If there is an aggregate of Q vortices in an infinite liquid, then we can write for the function ζ the expression

$$\zeta(z) = \sum_{q \in Q} \frac{1}{z - z_q}. \qquad (2)$$

In the case of a simple lattice the vortices are located at the points $z_{mn} = 2m\omega_1 + 2m\omega_2$. Here ω_1 and ω_2 are complex quantities called the semiperiods of the lattice (m and n are integers). The corresponding sum in the right side of (2) diverges when summed over all vortices of the lattice. Nonetheless, a function with identical first-order poles at the points z_{mn} exists, for example the Weierstrass ζ-function. It is obvious that all other such functions differ from the Weierstrass ζ function by an entire function. We are interested in the function

$$\zeta_0(z) = \zeta(z) + \alpha z. \qquad (3)$$

Here the entire function αz is chosen such that the obtained velocity field leads to rotation of the lattice as a whole. The value of α was found in [5].

3. Thus, the investigated motion constitutes rotation of a system of vortices as a whole. The problem consists of determining whether such a motion is stable.

Any statement concerning the stability of a system of vortices encounters the following elementary objection (sometimes called the Earnshaw theorem). Each vortex of the system interacts with with the remaining vortices and the potential of this interaction is harmonic. Therefore the energy of the system is also harmonic in the coordinates of this vortex. It is well known that a harmonic function cannot have a minimum inside any region, and, thus, it is always possible to displace the vortex in such a way that the total energy of the system decreases. The objection consists in the fact that stability in physical systems is frequently connected with an energy minimum, and no energy minimum is attainable for any arrangement of the vortices. The point is, of course, that the connection between the minimum of the energy and the stability is not universal. This system may be stable also in the case when the concept of energy cannot be introduced at all.

A close connection between stability and the energy extremum is realized in the case when the energy is the first integral of the equations of motion, and furthermore the only first integral. On the other hand, if there are other first integrals, for example the angular momentum M and the momentum P, then we can speak of the extremum of the quantity $E - \Omega M - VP$, where Ω and V are Lagrange multipliers. This quantity is no longer a harmonic function of the coordinates of the vortices, and Earnshaw's theorem has no relation to such systems.

We shall refer henceforth to stability in the usual sense, that is, we shall investigate whether a sufficiently small perturbation of the motion will remain small in the course of time.

4. A small perturbation is a rather undefined concept. It can be regarded to be, for example, a perturbation such that the displacements of the vortices ϵ_{mn} are sufficiently small; we can add to this the requirement that the changes in the vortex velocities ν_{mn} be small. However, it is easy to indicate in the most trivial manner a perturbation satisfying these two requirements, yet with respect to which there is no stability; if we impart to all vortices a small additional velocity, then in the course of time they will move arbitrarily far from their undisturbed positions. In the present paper we shall investigate the behavior of perturbations that satisfy at the initial instant of time the conditions (η is a sufficiently small positive number)

$$\sum |\varepsilon_{mn}|^2 \leqslant \eta, \qquad (4)$$

$$\sum |\nu_{mn}|^2 < \infty. \qquad (5)$$

Thus, a small change in the lattice parameters ω_1 and ω_2 does not satisfy these conditions. Nor are these conditions satisfied by any periodic perturbation. Nonetheless, the class of perturbations satisfying (4) and (5) is quite large—this is the class of perturbations that can be expanded in an integral in terms of plane waves.

The majority of perturbations are best assigned a complex linear structure, that is, we define in them the addition of perturbations and their multiplication by a complex number (this is done in obvious fashion), obtaining thereby an infinite-dimensional vector space. In light of a condition (4) it is reasonable to introduce a norm $\|\epsilon\| = (\Sigma|\epsilon_{mn}|^2)^{1/2}$ of the perturbation $\boldsymbol{\epsilon} = \{\epsilon_{mn}\}$. We thus obtain immediately the Hilbert space of the sequences l_2.[12] It contains all the perturbations for which $\Sigma|\epsilon_{mn}|^2$ converges.

We note here that the Hilbert space will soon become too narrow for us.

5. We proceed now to study the equations of motion. The velocity field of the liquid in the case

of an unperturbed lattice is known[5]—it is described by a slightly modified Weierstrass ζ function

$$\zeta_0(z) = \zeta(z) + \alpha z = \frac{1}{z} + \alpha z - \frac{g_2}{60} z^3 - \frac{g_3}{140} z^5 \ldots \quad (6)$$

Let us find the velocity field for a small change of the lattice. In the vicinity of zero we immediately separate the term $1/(z - \epsilon_{00})$ corresponding to a displaced central vortex. The influence of the displacement of the vortex located at z_{mn} on $\zeta(z)$ can be written in the vicinity of $z = 0$ in the form

$$\beta_{mn}(z) = \frac{1}{z - z_{mn} - \varepsilon_{mn}} - \frac{1}{z - z_{mn}} = \frac{\varepsilon_{mn}}{z_{mn}^2} + o\left(\frac{\varepsilon_{mn}}{z_{mn}^2}\right) + O\left(\frac{\varepsilon_{mn} z}{z_{mn}^3}\right).$$

For the perturbed function $\zeta_b(z)$ in the vicinity of zero we have

$$\zeta_b(z) = \frac{1}{z - \varepsilon_{00}} + \sum_{mn \neq 0} \beta_{mn} + \alpha z + O(z^3).$$

Putting $z \to \epsilon_{00}$, we obtain

$$\lim_{z \to \varepsilon_{00}} \left[\zeta_b(z) - \frac{1}{z - \varepsilon_{00}}\right] = \alpha\varepsilon_{00} + \sum_{mn \neq 0} \frac{\varepsilon_{mn}}{z_{mn}^2} + o\left(\sum_{mn \neq 0} \frac{\varepsilon_{mn}}{z_{mn}^2}\right) + \sum_{mn \neq 0} O\left(\frac{\varepsilon_{mn}\,\varepsilon_{00}}{z_{mn}^3}\right) + O(\varepsilon_{00}^3).$$

It is easy to see that the three last terms of the right side are of higher order of smallness than the first two as $\Sigma|\epsilon_{mn}|^2 \to 0$. The left side contains a quantity which differs only by a coefficient from the complex-conjugate velocity of the central vortex. For the velocity ν_{00} of the vortex we thus have, neglecting small quantities of higher order (the bar denotes complex conjugation),

$$\frac{d\bar{\varepsilon}_{00}}{dt} = \bar{\nu}_{00} = -i\alpha\varepsilon_{00} - i\sum_{mn \neq 0} \frac{\varepsilon_{mn}}{z_{mn}^2}. \quad (7)$$

For the change of the velocities of the remaining vortices we have analogous expressions.

The mapping $\boldsymbol{\epsilon} \to \bar{\boldsymbol{\nu}}$ can be regarded as a linear operator in Hilbert space. It can be written in the form

$$d\bar{\boldsymbol{\varepsilon}} / dt = \bar{\boldsymbol{\nu}} = -iA^*\boldsymbol{\varepsilon}. \quad (8)$$

The operator A^* in matrix form is

$$A^*_{mn,\,kl} = (z_{mn} - z_{kl})^{-2} \quad \text{for} \quad (m, n) \neq (k, l),$$

$$A^*_{mn,\,mn} = \alpha. \quad (9)$$

A study of the operator A^* will show later that it is bounded, and is therefore also a continuous operator. The boundedness of the operator A^* denotes the existence of a number N such that

$$\|A^*\boldsymbol{\varepsilon}\| \leqslant N\|\boldsymbol{\varepsilon}\|$$

for all $\epsilon \in l_2$. We see therefore that from $\|\boldsymbol{\epsilon}\| < \eta$ it follows by virtue of (8) that

$$\|\mathbf{v}\| = \|A^*\boldsymbol{\varepsilon}\| \leqslant N\|\boldsymbol{\varepsilon}\| \leqslant N\eta.$$

This relation agrees with the condition (5).

We now call attention to the fact that the velocity field of the liquid is determined by specifying the coordinates of the vortices not uniquely, but only accurate to a certain entire function f(z). By virtue of the condition $\Sigma|\nu_{mn}|^2 < \infty$, the function f(z) is bounded and must therefore be only a constant. From the same condition it follows also that this constant is equal to zero.

We have thus defined uniquely the equations of motion of the vortices (8) under conditions (4) and (5).

6. The operator A^* depends on the time t. This is connected with the fact that the system of vortices rotates with angular velocity Ω so that

$$z_{mn}(t) = z^0_{mn} e^{i\Omega t}.$$

The coefficient α also depends on the rotation of the system:[5]

$$\alpha(t) = \alpha^0 e^{-2i\Omega t}.$$

Here Ω is the angular velocity of the rotation. From the definition (9) of the operator A^* we conclude immediately that

$$A^*(t) = Ae^{-2i\Omega t}, \quad (10)$$

where A is an operator independent of the time.

In order to go over from (8) to an equation with an operator that does not depend on the time, we go over to a rotating reference frame. This is done by means of the transformation

$$\boldsymbol{\varepsilon} e^{-i\Omega t} = \mathbf{c}, \quad (11)$$

where $\mathbf{c} = \{c_{mn}\}$.

In the new variables, Eq. (8) becomes

$$d\mathbf{c} / dt = -i\Omega\mathbf{c} + \overline{iA\mathbf{c}} = H\mathbf{c}. \quad (12)$$

7. We have expressed the derivative $d\mathbf{c}/dt$ as a result of the action of a certain operator H on $\mathbf{c}$. When regarded as an operator in complex space, this is a semilinear operator in the sense that

$$H(\gamma\mathbf{c}) \neq \gamma H\mathbf{c}$$

for complex γ. As an operator in real space, it it will be linear. Equation (12) admits of several first integrals, but their analysis does not clarify the stability of the system. We shall therefore not write them out.

8. In order to investigate the behavior of the solutions of (12), it is necessary to carry out a spectral analysis of the operator H. Since H is not a linear operator on a field of complex numbers, we first obtain a spectral analysis of the operator A. We write it in matrix form (9)

$$A_{mn,kl} = [2(m-k)\omega_1 + 2(n-l)\omega_2]^{-2}$$
$$\text{for} \quad (m,n) \neq (k,l),$$
$$A_{mn,mn} = \alpha. \tag{13}$$

The spectrum of the operator A is the aggregate of the numbers γ for which the operator $A - \gamma E$ has no inverse (E is the identity operator). In other words, those are the values of γ for which the "determinant" of the infinite matrix $|A - \gamma E|$ vanishes. We shall not seek γ directly from this condition. In addition to the spectrum, we shall need also generalized eigenvectors, generalized because they do not belong to the Hilbert space l_2.

The complementation of the Hilbert space to a linear space l_A in which there already are eigenvectors of the operator A is given in the Appendix. It is perfectly analogous to the Dedekind construction of irrational numbers. The role of the rational numbers is played by vectors from the Hilbert space.

The vortex displacements corresponding to the eigenvector of the operator A can be written in the form

$$a_{mn} = e^{i(m\varphi + n\psi)}, \tag{14}$$

where φ and ψ are real numbers. The action of the operator A on the perturbation a specified by means of the components a_{mn}, is not defined. If we nevertheless attempt to calculate the components $b = Aa$, then we encounter the divergent expressions

$$b_{kl} = \sum_{m,n} \frac{e^{i(m\varphi+n\psi)}}{[2(m-k)\omega_1 + 2(n-l)\omega_2]^2} + \alpha e^{i(k\varphi+l\psi)}. \tag{15}$$

In the space l_A, however, the operator A transforms the eigenvector into a fully defined vector, to which there correspond uniquely several displacements of the vortices b_{kl}. It turns out that these displacements coincide with those calculated by formulas (15), if their right side is summed as a Fourier series. We introduce the notation

$$B(\varphi,\psi) = \sum_{nm\neq 0} \frac{e^{i(m\varphi+n\psi)}}{(2m\omega_1 + 2n\omega_2)^2} + \alpha. \tag{16}$$

The right side should be regarded here as the Fourier series of the function $B(\varphi, \psi)$. Then $b_{kl} = Ba_{kl}$ and B is the eigenvalue of the operator A.

9. In calculating B we encounter difficulties of purely technical order. We rewrite B in the form

$$B = \frac{1}{(2\omega_1)^2} \sum_{mn\neq 0} \frac{e^{i(m\varphi+n\psi)}}{(m+n\tau)^2} + \alpha, \tag{17}$$

where $\tau = \omega_2/\omega_1$.

We obtain the sum

$$\sum_{m=-\infty}^{+\infty} \frac{e^{im\varphi}}{(m+n\tau)^2}.$$

Analogous sums were calculated by Lamb[1] in the analysis of the stability of the Karman street. Using his result, we obtain $(0 \leq \varphi < 2\pi)$

$$\sum_{m=-\infty}^{+\infty} \frac{e^{im\varphi}}{(m+n\tau)^2} = \frac{\pi^2 e^{-in\tau\varphi}}{\sin^2 \pi n\tau} + i\pi\varphi \frac{e^{in\tau(\pi-\varphi)}}{\sin \pi n\tau},$$

$$\sum_{m=-\infty}^{\infty} \left[\frac{e^{i(m\varphi+n\psi)}}{(m+n\tau)^2} + \frac{e^{i(m\varphi-n\psi)}}{(m-n\tau)^2} \right]$$
$$= 2\pi^2 \frac{\cos n(\tau\varphi - \psi)}{\sin^2 \pi n\tau} - 2\pi\varphi \frac{\sin[n\tau(\pi-\varphi) + n\psi]}{\sin \pi n\tau}. \tag{18}$$

The sum of the terms (17) with n = 0 can be readily obtained:

$$\sum_{m\neq 0} \frac{e^{im\varphi}}{(2m\omega_1)^2} = \frac{2}{(2\omega_1)^2} \sum_{m=1}^{\infty} \frac{\cos m\varphi}{m^2}$$
$$= \frac{1}{(2\omega_1)^2} \left[\frac{\pi^2}{3} - \frac{(2\pi-\varphi)\varphi}{2} \right]. \tag{19}$$

We now have for B, taking (18) and (19) into account $(0 \leq \varphi < 2\pi)$

$$B = \frac{1}{(2\omega_1)^2} \left[\frac{\pi^2}{3} - \frac{(2\pi-\varphi)\varphi}{2} + 2\pi^2 \sum_{n=1}^{\infty} \frac{\cos n(\tau\varphi-\psi)}{\sin^2 \pi n\pi} \right.$$
$$\left. + 2\pi\varphi \sum_{n=1}^{\infty} \frac{\sin n(\tau\varphi - \psi - \tau\pi)}{\sin \pi n\tau} \right] + \alpha. \tag{20}$$

Inasmuch as the sum for B diverges at the point $\varphi = 0 = \psi$, we make a change of variables $\xi + \pi = \varphi$, $\eta + \pi = \psi$. We then obtain for B an expression which, after introducing the notation assumes the form

$$\pi\varkappa = \varphi\omega_2 - \psi\omega_1, \quad u = \varkappa + \omega_1 - \omega_2 = \omega_1(\tau\xi - \eta)/\pi,$$
$$q = e^{i\pi\tau}$$

assumes the form

$$B = \frac{1}{(2\omega_1)^2} \left[-\frac{\pi^2}{6} + \frac{\xi^2}{2} \right.$$
$$+ 2\pi^2 \sum_{n=1}^{\infty} (-1)^n \frac{\cos(\pi nu/\omega_1)\cos \pi n\tau}{\sin^2 \pi n\tau}$$
$$\left. + 2\pi\xi \sum_{n=1}^{\infty} (-1)^n \frac{\sin(\pi nu/\omega_1)}{\sin \pi n\tau} \right] + \alpha. \tag{21}$$

For the second sum of the right side we know an expression in terms of the θ function[13]

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin(\pi nu/\omega_1)}{\sin\pi n\tau}=-\frac{i}{2}\frac{\theta_3'(\pi u/2\omega_1)}{\theta_3(\pi u/2\omega_1)}. \quad (22)$$

Integrating (22) with respect to u, we obtain

$$-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\cos(\pi nu/\omega_1)}{\sin\pi n\tau}$$

$$=-i\ln\theta_3\left(\frac{\pi u}{2\omega_1},\ \tau\right)+i\ln G(\tau), \quad (23)$$

where $G(\tau)$ is a quantity that does not depend on u, which we now define, putting $u = 0$:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{2iq^n}{1-q^{2n}}=-i\ln\theta_3(0,\ \tau)+i\ln G(\tau). \quad (24)$$

Using the formulas of Appendix I of [5] and the representation of the θ functions in the form of products,[14] we get from (24)

$$G(\tau)=\prod_{n=1}^{\infty}(1-q^{2n})=\left[\frac{\theta_1'(0,\tau)}{2q^{1/4}}\right]^{1/3}.$$

The first sum of the right side of (21) obviously coincides, apart from a coefficient, with the derivative of the left side of (23) with respect to τ. We therefore have

$$\pi\sum_{n=1}^{\infty}(-1)^n\frac{\cos(\pi nu/\omega_1)\cos\pi n\tau}{\sin^2\pi n\tau}$$

$$=-i\frac{\partial\theta_3(\pi u/2\omega_1,\tau)/\partial\tau}{\theta_3(\pi u/2\omega_1,\tau)}+\frac{i}{3}\frac{\partial\theta_1'(0,\tau)/\partial\tau}{\theta_1'(0,\tau)}-\frac{i}{12}i\pi$$

$$=-\frac{\pi}{4}\frac{\theta_3''(\pi u/2\omega_1)}{\theta_3(\pi u/2\omega_1)}+\frac{\pi}{12}\frac{\theta_1'''(0)}{\theta_1'(0)}+\frac{\pi}{12}. \quad (25)$$

We have used here the differential equation of the θ functions.[14] We can now express B in terms of θ functions:

$$B=\frac{1}{(2\omega_1)^2}\left[\frac{\xi^2}{2}-\frac{\pi^2}{2}\frac{\theta_3''(\pi u/2\omega_1)}{\theta_3(\pi u/2\omega_1)}+\frac{\pi^2}{6}\frac{\theta_1'''(0)}{\theta_1'(0)}\right.$$

$$\left.-\pi\xi i\frac{\theta_3'(\pi u/2\omega_1)}{\theta_3(\pi u/2\omega_1)}\right]+\alpha. \quad (26)$$

This expression can be given a symmetrical form by replacing the θ functions by a ζ_0 function and the elliptic function $\wp$ of Weierstrass. However, we prefer

$$\wp_0(\varkappa)=\wp(\varkappa)-\alpha=-\zeta_0'(\varkappa).$$

We write also the expression for α in terms of the θ function:[5]

$$\alpha=\Omega\frac{\bar{\omega}_1}{\omega_1}+\frac{\pi^2}{12\omega_1^2}\frac{\theta_1'''(0)}{\theta_1'(0)}.$$

Using the equation

$$\zeta_0(\varkappa)=\zeta_0(u-\omega_1+\omega_2)$$

$$=\frac{\Omega\bar{\omega}_1}{\omega_1}u-\Omega\bar{\omega}_1+\Omega\bar{\omega}_2+\frac{\pi}{2\omega_1}\frac{\theta_3'(\pi u/2\omega_1)}{\theta_3(\pi u/2\omega_1)},$$

which is similar to one contained in [5], and the derivative of this equation with respect to u, we obtain after laborious algebraic transformations

$$B(\varkappa)=\frac{\wp_0(\varkappa)-\zeta_0^2(\varkappa)}{2}+\Omega\bar{\varkappa}\zeta_0(\varkappa)-\frac{\Omega^2\bar{\varkappa}^2}{2}+\frac{3}{2}\alpha. \quad (27)$$

We can check by direct calculation that (27) defines a periodic function with two periods, $2\omega_1$ and $2\omega_2$. Whereas the preceding expressions for B were valid only for $0 \le \varphi < 2\pi$, the latter, by virtue of its periodicity, is valid for all complex κ.

10. The perturbation of $\mathbf{a}(\kappa)$, specified by (14), can be regarded as a circularly polarized plane wave, the length of which is λ (complex). It is easy to find a connection between κ and λ:

$$\varkappa\bar{\lambda}=i\pi/\Omega, \quad (28)$$

Since $\Omega = \pi/4\mathrm{Im}\,(\bar{\omega}_1\omega_2)$.[5]

The function $B(\kappa)$, as we shall presently see, is bounded. Indeed, its only singular point (apart from a shift by one period) is $\kappa = 0$. In the vicinity of this point, we have, taking (6) into account ($\kappa = \rho e^{i\sigma}$, $\rho > 0$, and σ are all real),

$$B(\varkappa)=B(\rho e^{i\sigma})=\Omega e^{-2i\sigma}-{}^1\!/_2\Omega^2\rho^2e^{-2i\sigma}+\alpha\Omega\rho^2$$

$$+(g_2/24-\alpha^2/2)\rho^2e^{2i\sigma}. \quad (29)$$

Thus $\lim_{\varkappa\to 0}|B(\kappa)| = \Omega$ and consequently $B(\kappa)$ is bounded for all κ.

We now call attention to the fact that by constructing the eigenvectors $\mathbf{a}(\kappa)$ for all κ we have obtained a complete system. In fact, for any vector $\mathbf{c}\in l_2$ we have

$$c_{mn}=\frac{1}{(2\pi)^2}\int\int C(\varphi,\ \psi)\,e^{-i(m\varphi+n\psi)}\,d\varphi\,d\psi, \quad (30)$$

where $C(\varphi, \psi) = \sum_{m,n} c_{mn}\exp[i(m\varphi + n\psi)]$. We shall write out the Parseval equality, which we shall need later:

$$\sum|c_{mn}|^2=\frac{1}{(2\pi)^2}\int\int|C(\varphi,\ \psi)|^2\,d\varphi\,d\psi. \quad (31)$$

From the completeness of the constructed system it follows that the region of values of $B(\kappa)$ constitutes the entire spectrum of the operator A. From the boundedness of the spectrum we now obtain[15] the boundedness of the operator A referred to already in Sec. 5.

We can thus regard the spectral analysis of the operator A as completed.

11. We turn to Eq. (12). Differentiation with respect to time yields

$$\frac{d^2\mathbf{c}}{dt^2} = -i\Omega\frac{d\mathbf{c}}{dt} + i\bar{A}\frac{d\bar{\mathbf{c}}}{dt}. \qquad (32)$$

Equation (32) can be reduced to a linear equation over a field of complex numbers, by expressing $d\bar{\mathbf{c}}/dt$ and $d\mathbf{c}/dt$ with the aid of (12). A similar procedure was used by Milne-Thompson[4] in the analysis of the stability of the Karman street. We then obtain in lieu of (32)

$$d^2\mathbf{c}/dt^2 + (\Omega^2 - \bar{A}A)\mathbf{c} = 0. \qquad (33)$$

Here $\bar{A}$ is an operator which is complex conjugate to A. The solutions of (12) satisfy (33), and would be bounded were we to have $\Omega^2 > r^2 + \delta$, where δ is a positive number, for any eigenvector of the operator $\bar{A}A$ with eigenvalue r^2.

Unfortunately, δ is not positive, so that the analysis of the solutions of (12) becomes somewhat more complicated. In fact, besides the eigenvector $\mathbf{a}(\kappa)$, $\overline{\mathbf{a}(\kappa)} = \mathbf{a}(-\kappa)$ is also an eigenvector, to which the same eigenvalue $B(\kappa) = B(-\kappa)$ corresponds. Consequently, the eigenvectors of the operator $\bar{A}$ will be $\mathbf{a}(\kappa)$ with eigenvalues $\overline{B(\kappa)}$ and the eigenvalues of the operator $\bar{A}A$ will be $|B(\kappa)|^2$. Since $\lim_{\kappa\to 0} |B(\kappa)|^2 = \Omega^2$, δ is not positive.

Using the elements of homotopy theory, we can prove that for any lattice there exists a κ_0 such that $B(\kappa_0) = 0$. It follows therefore that the maximum frequency of the oscillation spectrum is always equal to Ω.

Let us find the eigenvectors of the operator H or, what is the same, the normal vibrations of the vortex lattice. We put $B(\kappa) = re^{2i\chi}$, $r > 0$, χ real, and consider lattice oscillations with wavelength $\pm\lambda$. We seek the solution of (12) in the form

$$\mathbf{c}(t) = e^{-i\chi}(\mathbf{d}\cos\mu t + \mathbf{b}\sin\mu t). \qquad (34)$$

Here $\mu = \sqrt{\Omega^2 - |B|^2}$, $\mathbf{d}$ and $\mathbf{b}$ are linear combinations with complex coefficients of the vectors $\mathbf{a}(\kappa)$ and $\mathbf{a}(-\kappa)$. Let us find the connection between $\mathbf{d}$ and $\mathbf{b}$, imposed by Eq. (12):

$$-\mu\mathbf{d}\sin\mu t + \mu\mathbf{b}\cos\mu t = -i\Omega(\mathbf{d}\cos\mu t + \mathbf{b}\sin\mu t) + ir(\bar{\mathbf{d}}\cos\mu t + \bar{\mathbf{b}}\sin\mu t).$$

Thus

$$\mu\mathbf{b} = -i\Omega\mathbf{d} + ir\bar{\mathbf{d}}, \qquad -\mu\mathbf{d} = -i\Omega\mathbf{b} + ir\bar{\mathbf{b}}.$$

Putting $\mathbf{d} = \mathbf{g} + i\mathbf{h}$, where $\mathbf{g}$ and $\mathbf{h}$ are real vectors, we get

$$\mathbf{b} = \frac{\Omega + r}{\mu}\mathbf{h} - i\frac{\Omega - r}{\mu}\mathbf{g}. \qquad (35)$$

Let us consider in greater detail the case $\mathbf{d} = \mathrm{Re}\,\mathbf{a}(\kappa)$. Then $\mathbf{b} = -i\mu^{-1}(\Omega - r)^{-1}\,\mathrm{Re}\,\mathbf{a}(\kappa)$, and for the normal oscillations we have

$$c_{mn}(t) = e^{-i\chi}\Big(\cos\mu t\cdot\cos(m\varphi + n\psi) - i\frac{\Omega - r}{\mu}\sin\mu t\cdot\cos(m\varphi + n\psi)\Big). \qquad (36)$$

We see that each vortex describes an ellipse with an axis ratio $(\Omega - r)/\mu = \sqrt{(\Omega - r)/(\Omega + r)}$. The length of the axis varies sinusoidally with the period of the wavelength, and the orientation remains unchanged. This is a typical normal oscillation, similar to those in which any arbitrary perturbation satisfying (4) and (5) is expanded.

12. We call special attention to the study of the triangular lattice, since such a lattice and lattices close to it are stable. In the case of a triangular lattice $\alpha = 0$, $g_2 = 0$, and

$$\zeta_0(u) = \frac{1}{u} - g_3\frac{u^5}{140} - \frac{g_3^2u^{11}}{112112}\cdots \qquad (37)$$

for $B(\kappa)$ we have by virtue of (27) in the vicinity of $\kappa = 0$

$$B(\rho e^{i\sigma}) = \Big(\Omega - \frac{\Omega^2\rho^2}{2}\Big)e^{-2i\sigma} + g_3\Big(\frac{1}{40} - \frac{\Omega\rho^2}{140}\Big)\rho^4e^{4i\sigma} + g_3^2\Big(\frac{1}{30800} - \frac{\Omega\rho^2}{112112}\Big)\rho^{10}e^{10i\sigma}\cdots \qquad (38)$$

We put $g_3 = 1$. Then[13] $\omega_1 = 1.52995$ and $\Omega = 0.387405$. Calculations of B by means of this formula at the points ω_1 and $(^2/_3)(\omega_1 + \omega_2)$ yield 0.31495 and 0.00013. The exact values, as can be shown on the basis of the Weierstrass theory of elliptic functions, will be $2^{-5/3} = 0.3150$ and 0. The accuracy of formula (38) increases with decreasing ρ. We shall not stop to prove in detail that $B(\kappa) < \Omega$ everywhere except $\kappa = 0$ (accurate to one period).

Obviously, a similar statement is true also for lattices which are sufficiently close to triangular. For all these lattices the instability can be revealed only in the long-wave region ($\kappa \sim 0$). We shall investigate this region in detail in the next section.

Let us demonstrate here the instability of a quadratic lattice. For such a lattice $\alpha = 0$ and $g_3 = 0$. Let us put $g_2 = 1$, then $\omega_1 = K(2^{-1/2}) = 1.8541$ and $\Omega = 0.228467$. From the Weierstrass theory it follows in this case that $B(\omega_1) = {}^1/_4$. Inasmuch as this quantity is larger than Ω, the quadratic lattice is unstable.

13. Let us consider now the time behavior of small perturbations. In the case of a Karman street, no such analysis has been carried out before, and this has led to different misconceptions with respect to its stability (see [16]). Let us write out again the Fourier expansion of the small perturbation:

$$c_{mn}(t) = \frac{1}{(2\pi)^2} \iint C(\varkappa, t) e^{-i(m\varphi+n\psi)} d\varphi\, d\psi. \quad (39)$$

According to (12), we have for $C(\kappa, t)$

$$\frac{d}{dt} C(\varkappa, t) = -i\Omega C(\varkappa, t) + i\overline{B(\varkappa) C(-\varkappa, t)}. \quad (40)$$

If we put $C(\kappa, 0) = C(\kappa)$, then the solution of (40) will be

$$C(\varkappa, t) = C(\varkappa)\cos\mu t - i\frac{\Omega}{\mu} C(\varkappa)\sin\mu t + i\frac{\overline{B(\varkappa) C(-\varkappa)}}{\mu}\sin\mu t. \quad (41)$$

Formulas (39) and (41) give the solution of (12). As follows from Sec. 5, the solution will be unique if conditions (4) and (5) are satisfied.

For a triangular lattice

$$\mu = \sqrt{\Omega^2 - |B(\varkappa)|^2} \sim \Omega^{3/2}|\varkappa| = \Omega^{3/2}\rho \quad (42)$$

at small values of $|\kappa|$ (long waves). For sufficiently small μ, the values of $C(\kappa, t)$ can be arbitrarily large for large t. It turns out, however, that

$$\lim_{t\to\infty} c_{mn}(t) = 0. \quad (43)$$

The satisfaction of (43) follows from (42) and from the integrability of $\iint |C(\kappa)|^2\, d\varphi\, d\psi$. This can be proved by using the theory of the Lebesgue integral and the Dirichlet integral. The integrability of $\iint |C(\kappa)|^2\, d\varphi\, d\psi$ follows in turn from (4) by virtue of the Parseval equality (31). It must be noted that uniformity of the approach of $c_{mn}(t)$ to zero with respect to the indices m and n cannot be proved. It follows from (43) that any small perturbation goes off to infinity in the course of time, but whether it increases or decreases cannot be stated on the basis of (43). We therefore investigate further the behavior of

$$\|\mathbf{c}(t)\|^2 = \sum |c_{mn}(t)|^2 = \frac{1}{(2\pi)^2}\iint |C(\varkappa, t)|^2 d\varphi\, d\psi.$$

We confine ourselves to the case of a continuous function $C(\kappa)$, which will assuredly be continuous if only a finite number of vortices is displaced during the initial instant. The growth of $\|\mathbf{c}(t)\|$ as $t\to\infty$ is determined by the behavior of $C(\kappa)$ in the vicinity of $|\kappa| = 0$. In this vicinity we have

$$C(\varkappa, t) \sim C_0\cos\mu t - i\frac{\sin\mu t}{\mu}(\Omega C_0 - \Omega e^{-2i\sigma}\bar{C}_0). \quad (44)$$

Here $C_0 = C(0)$ and $\kappa = \rho\, e^{i\sigma}$. The principal term in $\|\mathbf{c}(t)\|^2$ as $t\to\infty$ will be

$$\frac{1}{(2\pi)^2}\iint_{|\varkappa|<\delta} \frac{\sin^2\mu t}{\mu^2}\Omega^2(|C_0|^2 - C_0^2 e^{2i\sigma} - \bar{C}_0^2 e^{-2i\sigma} + |C_0|^2)\, d\varphi\, d\psi = \frac{2\Omega^2|C_0|^2}{(2\pi)^2}\int_{|\varkappa|<\delta}\frac{\sin^2\mu t}{\mu^2}\rho\, d\rho = \frac{2|C_0|^2}{(2\pi)^2\Omega}\int_{|\varkappa|<\delta}\frac{\sin^2\mu t}{\mu^2}\mu\, d\mu.$$

As $t\to\infty$, an asymptotic estimate for the latter integral is $[\ln(\Omega t)]/2$ and we ultimately have

$$\|\mathbf{c}(t)\| \sim \frac{|C_0|}{2\pi\sqrt{\Omega}}\sqrt{\ln\Omega t_0}. \quad (45)$$

Thus, the perturbation going off to infinity has a rather slow growth. The slowness of the growth allows us to speak of stability of a triangular lattice. A similar stability takes place also for lattices which are sufficiently close to triangular. This follows from the continuity of the spectrum with respect to a small change of the lattice parameters.

14. Summarizing the foregoing, we can state that we have succeeded in the investigating of the behavior of small perturbations of vortex lattices. It turns out that the long-wave oscillations are not determined by the vortex density, but depend on the structure of the lattice and are anisotropic in all lattices except triangular ones. This should not surprise us, for we encounter similar phenomena in real crystalline bodies. Triangular lattices and those close to them can be characterized as stable—small perturbations in these lattices go off to infinity, and their norm $\|\mathbf{c}(t)\|$ increases sufficiently slowly. All other lattices, including quadratic, are absolutely unstable—some of their small perturbations grow exponentially. We have obtained an exact expression for the oscillation spectrum of the lattices and determined their normal oscillations. All these results were obtained with the aid of the Weierstrass theory of elliptic functions, developed a century ago.

We shall not discuss in this article the possibility of experimentally observing vortex lattices in rotating helium and the influence of the normal component of helium on the lattice oscillations. We note only that in the continual approximation long-wave excitations of a triangular lattice satisfy the wave equation $(z = x + iy)$

$$\frac{1}{s^2}\frac{\partial^2 c(x, y; t)}{\partial t^2} = \Delta c(x\ y; t) \quad (46)$$

with velocity $s = (1/2)\sqrt{\Omega}$ in dimensionless units or $(1/2)\sqrt{\hbar\Omega/m}$ cm/sec.

We note, finally, that the results have a bearing not only on helium, inasmuch as we have essentially obtained the dispersion law for a plane lattice with screened Coulomb interaction. If necessary, similar methods can be used to investigate complex lattices or lattices with defects.

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APPENDIX

Let us consider a sequence of vectors $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$ $\mathbf{a}^{(3)}$, ... , in an infinite-dimensional vector space, that converges weakly[15, 17] to a vector $\mathbf{a}^{(\infty)}$. This denotes the convergence of each component separately. If $A^{(s)} \in l_2$, this still does not mean that $\mathbf{a}^{(\infty)} \in l_2$. Let us consider now the sequence $A^k\mathbf{a}^{(1)}$, $A^k\mathbf{a}^{(2)}$, and $A^k\mathbf{a}^{(3)}$, ... It may or may not have a limit in weak topology. In the case when such a limit exists for arbitrary natural k, we shall say that the sequence $[\mathbf{a}^{(s)}]$ converges in A-topology. Different sequences $[\mathbf{a}^{(s)}]$ and $[\mathbf{b}^{(s)}]$, which have identical limits in weak topology, can correspond to different limits in A-topology, if such limits exist at all.

Let us consider the set of all possible sequences that converge in A-topology. We introduce in it, in natural fashion, a complex linear structure, after which the set becomes a complex linear space. Two elements of this space, $[\mathbf{a}^{(s)}]$ and $[\mathbf{b}^{(s)}]$, will be regarded as equal if $\lim_{s\to 0} A^k(\mathbf{a}^{(s)} - \mathbf{b}^{(s)}) = 0$ in weak topology of space l_2 for all natural k. Following such an identification, we obtain a complex linear space which we denote by l_A. The space l_A contains in natural fashion all the elements from l_2. Indeed, to each element $\mathbf{b} \in l_2$ we can set in correspondence a sequence $[\mathbf{b}^{(s)}]$, where all $\mathbf{b}^{(s)} = \mathbf{b}$. Moreover, if A is a bounded operator, then all the sequences $\mathbf{b}^{(s)}$ whose norm converges to $\mathbf{b}$ represent in l_A the same element, which we shall denote simply by $\mathbf{b}$. The initial region in which the operator A is defined (the space l_2) can be expanded to the space l_A. The operator A transforms an arbitrary element $\mathbf{a} = [\mathbf{a}^{(s)}] \in l_A$ $\times$ $(\mathbf{a}^{(s)} \in l_2)$ into $A\mathbf{a} = [A\mathbf{a}^{(s)}]$. As is easily seen, $A\mathbf{a}$ does not depend on the concrete choice of the sequence $[\mathbf{a}^{(s)}]$ which represents $\mathbf{a}$.

Let the operator A be now defined in accordance with (13), and let us consider the sequence and the vectors $\mathbf{a}^{(s)} = \{a^{(s)}_{mn}\}$, where

$$a_{mn}{}^{(s)} = e^{i(m\varphi+n\psi)} \quad \text{for} \quad |m| \leqslant M_s, \quad |n| \leqslant N_s,$$
$$a_{mn}{}^{(s)} = 0 \quad \text{for} \quad |m| > M_s, \quad |n| > N_s.$$

The integers M_s and N_s tend to infinity as $s \to \infty$, while φ and ψ are real numbers. With the aid of the theory of trigonometric series[18] it is easy to prove that this sequence converges to a certain element $\mathbf{a} \in l_2$ in A-topology, with $\mathbf{a}$ independent of the manner in which M_s and N_s tend to infinity. Moreover, we can let $M \to \infty$ at fixed N, and then let $N \to \infty$, obtaining the same element $\mathbf{a}$ (the so-called "unbounded summability").

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