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INELASTIC DIFFRACTION SCATTERING

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A method is proposed for solving the problem of inelastic scattering of particles by nuclei with excitation of collective states. The method is based on the possibility of separating the variables in a central field when the scattering involves large angular momenta. It is shown that as a result the inelastic scattering cross section can be expressed in terms of the elastic scattering phase shifts. The results obtained by Blair and Austern[1] are made more precise and generalized to nuclei with arbitrary spin and to any approximation in the nonsphericity parameter.

1. INTRODUCTION

In a recent paper by Blair and Austern[1] it was shown that the cross section for inelastic scattering ($kR \gg 1$) with excitation of collective nuclear states can be expressed in terms of the S matrix elements for elastic scattering. This result is very attractive, since it shows that the inelastic scattering problem can essentially be reduced to the much simpler elastic scattering problem.

At the same time, there are a number of points in the Blair-Austern theory which require a further development of the theory of inelastic scattering. The authors of this paper employ a number of approximations whose accuracy is very difficult to estimate, i.e., is essentially beyond control. This refers in particular to the higher approximations in the nonsphericity parameter, for new, additional assumptions about the smallness of certain quantities are made, whose correctness is very difficult to ascertain, already in second order in this parameter. Moreover, the principal reason why the problem can be reduced to the elastic scattering problem remains unclear.

In the present paper it is shown that the principal reason for the possibility of reducing the problem under consideration to the elastic scattering problem is the circumstance that for large angular momenta which play the dominant role in diffraction, the problem of the motion of a particle in a noncentral field reduces to the problem of the motion in a central field.

2. APPROXIMATE SEPARATION OF VARIABLES IN THE CASE OF A NONCENTRAL POTENTIAL

In the adiabatic approximation[2] it is first of all necessary to calculate the elastic scattering amplitude in a noncentral field:

$$V(\mathbf{r}) = V(r, R(\vartheta, \varphi)), \tag{1}$$

where

$$R(\vartheta, \varphi) = R_0 + \sum_{\nu\mu} \xi_{\nu\mu}^* Y_{\nu\mu}(\vartheta, \varphi), \tag{2}$$

and the quantities $\xi_{\nu\mu}^*$ are dynamical variables of the target nucleus.

The general expression for the elastic scattering amplitude in an arbitrary coordinate system is given by the expression

$$f(\mathbf{n}, \mathbf{n}_0) = \frac{2\pi}{ik} \sum_{ll'mm'} Y_{l'm'}^*(\mathbf{n}) (l'm'|\hat{S} - 1|lm) Y_{lm}(\mathbf{n}_0), \tag{3}$$

where $\hat{S}$ is the scattering operator, and $\mathbf{n}_0$ and $\mathbf{n}$ are unit vectors defining the direction of motion of the particle before and after the scattering, respectively.

In the diffraction scattering of a particle by a strongly absorbing nucleus all partial waves with $l, l' < L_0 \sim kR \gg 1$ are completely absorbed, i.e.,

$$(l'm'|\hat{S}|lm) = 0, \qquad l, l' < L_0 \gg 1. \tag{4}$$

Thus the problem consists in the determination of the S matrix elements for large values of l and l'. We show that this problem can to a certain extent be reduced to the problem of scattering in a central field.

Let us consider the coordinate function $u_l(r, \vartheta, \varphi)$ defined by the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du_l}{dr}\right) - \frac{l(l+1)}{r^2} u_l + [k^2 - V(r, R(\vartheta, \varphi))] u_l = 0 \tag{5}$$

and the boundary condition

$$u_l = a_l r^l, \qquad r \to 0. \tag{6}$$

It is easy to see that the function

$$\Psi_{lm} = u_l(r, \vartheta, \varphi) Y_{lm}(\vartheta, \varphi) \tag{7}$$

is for $l \to \infty$ asymptotically equal to a particular solution of the Schrödinger equation

$$\Delta\Psi + [k^2 - V(r, R(\vartheta, \varphi))]\Psi = 0. \quad (8)$$

Indeed, writing the Laplace operator in the form

$$\Delta = \Delta_r - r^{-2}\hat{\mathbf{l}}^2, \quad (9)$$

we see that the separation of variables is prevented only by the term with the operator $\hat{\mathbf{l}}^2$ which differentiates the function u_l, as it depends on the angular variables. However, for large quantum numbers, i.e., in our case for large l, the operator can be replaced by a c number, as is well known; in this way one effects the separation of variables. Since the function u_l satisfies Eq. (5), we thus obtain a proof that for large l the function Ψ_{lm} is a particular solution of the Schrödinger equation.

It is useful to investigate directly how in our case the possibility of replacing the operator by a c number, $l(l+1)$, is realized. To this end we note that the following asymptotic expression for $Y_{lm}(\vartheta, \varphi)$ holds for large l:

$$Y_{lm} \sim \frac{(-1)^m}{\pi\sqrt{\sin\vartheta}} \cos\left[\left(l + \frac{1}{2}\right)\vartheta - \frac{m\pi}{2} - \frac{\pi}{4}\right] e^{im\varphi}. \quad (10)$$

For $\hat{\mathbf{l}}^2$ we have

$$-\hat{\mathbf{l}}^2 = \frac{\partial^2}{\partial\vartheta^2} + \cot\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}. \quad (11)$$

In applying (11) to the function (7) the leading term in l will be

$$-u_l(r, \vartheta, \varphi)\frac{\partial^2}{\partial\vartheta^2} Y_{lm}(\vartheta, \varphi) = \left(l + \frac{1}{2}\right)^2 \Psi_{lm} \approx l(l+1)\Psi_{lm}. \quad (12)$$

Indeed, the differentiation of u_l with respect to the angle ϑ reduces to a variation of u_l with respect to the potential multiplied by the derivative of the potential with respect to the angle. This quantity is of order unity, since the potential, by assumption, depends strongly only on the spherical harmonics of low order. The relation (12) expresses the reduction of the operator $\hat{\mathbf{l}}^2$ to the number $l(l+1)$.

The particular solution obtained for $r \to \infty$ has the form

$$\Psi_{lm} \sim \frac{A_l}{kr}\left\{\exp\left[-i\left(kr - \frac{l\pi}{2}\right)\right] - S_l \exp\left[i\left(kr - \frac{l\pi}{2}\right)\right]\right\} Y_{lm}(\vartheta, \varphi). \quad (13)$$

The quantities $S_l = S_l(R(\vartheta, \varphi))$ are evidently obtained from the S matrix elements $S_l(R_0)$ of the corresponding scattering problem in a central field $V(r, R_0)$ by making, in the latter the replacement $R_0 \to R(\vartheta, \varphi)$.

Let us now write

$$S_l(R(\vartheta, \varphi)) Y_{lm}(\vartheta, \varphi) = \sum_{l'm'} (Y_{l'm'}, S_l Y_{lm}) Y_{l'm'}(\vartheta, \varphi) \quad (14)$$

and substitute this in (13). Then we obtain

$$\Psi_{lm} \sim \frac{A_l}{kr}\left\{\exp\left[-i\left(kr - \frac{l\pi}{2}\right)\right] Y_{lm}(\vartheta, \varphi) - \sum_{l'm'} i^{l'-l}(Y_{l'm'}, S_l Y_{lm}) \exp\left[i\left(kr - \frac{l'\pi}{2}\right)\right] Y_{l'm'}(\vartheta, \varphi).\right. \quad (15)$$

According to the definition of the S matrix, we find from (15)

$$(l'm'|\hat{S}|lm) = i^{l'-l}(Y_{l'm'}, S_l Y_{lm}) \quad (16)$$

and the approximate expression for the scattering amplitude will be

$$f(\mathbf{n}, \mathbf{n}_0) = \frac{2\pi}{ik}\sum_{ll'mm'} Y^*_{l'm'}(\mathbf{n})\, i^{l'-l}(Y_{l'm'}, S_l Y_{lm}) Y_{lm}(\mathbf{n}_0). \quad (17)$$

Let us now consider the amplitude for the time reversed process, i.e., for the scattering $-\mathbf{n} \to -\mathbf{n}_0$. According to (17), this amplitude is equal to

$$f(-\mathbf{n}_0, -\mathbf{n}) = \frac{2\pi}{ik}\sum_{ll'mm'} Y^*_{l'm'}(-\mathbf{n}_0)\, i^{l'-l}(Y_{l'm'}, S_l Y_{lm}) Y_{lm}(-\mathbf{n}). \quad (18)$$

After simple manipulations we obtain from this

$$f(-\mathbf{n}_0, -\mathbf{n}) = \frac{2\pi}{ik}\sum_{ll'mm'} Y^*_{l'm'}(\mathbf{n})\, i^{l'-l}(Y_{l'm'}, S_{l'} Y_{lm}) Y_{lm}(\mathbf{n}_0). \quad (19)$$

For an exact solution, invariance under time rereversal[3] requires

$$f(\mathbf{n}, \mathbf{n}_0) = f(-\mathbf{n}_0, -\mathbf{n}). \quad (20)$$

Our approximate expression for the amplitude does not satisfy this requirement, as is seen from (17) and (19). Evidently we obtain a better approximation, if we take for the scattering amplitude the quantity

$$F(\mathbf{n}, \mathbf{n}_0) = {}^1/_2[f(\mathbf{n}, \mathbf{n}_0) + f(-\mathbf{n}_0, -\mathbf{n})] = \frac{\pi}{ik}\sum_{ll'mm'} Y^*_{l'm'}(\mathbf{n})\, i^{l'-l}(Y_{l'm'}, [S_l + S_{l'}] Y_{lm}) Y_{lm}(\mathbf{n}_0), \quad (21)$$

which has the required symmetry.

The meaning of this symmetrization is that it removes the non-equivalence of the ingoing (lm) and outgoing $(l'm')$ channels, which existed in our original approach.

3. AMPLITUDES AND CROSS SECTIONS FOR INELASTIC SCATTERING

The amplitude for inelastic scattering with excitation of the nucleus from the state I_0, M_0 to the state I, M is equal to

$$F^{IM}_{I_0M_0}(\mathbf{n},\mathbf{n}_0)=(IM|F(\mathbf{n},\mathbf{n}_0)|I_0M_0). \qquad (22)$$

In the calculation the quantization axis is conveniently aligned with $\mathbf{n}_0$. Denoting the angles of $\mathbf{n}$ in this system by ϑ,φ, we obtain

$$F^{IM}_{I_0M_0}(\vartheta,\varphi)=\frac{1}{2ki}\sum_{ll'm'}\sqrt{\pi(2l+1)}\,i^{l'-l}$$

$$\times(Y_{l'm'},(IM|S_l+S_{l'}|I_0M_0)Y_{l_0})Y_{l'm'}(\vartheta,\varphi). \qquad (23)$$

For the further calculations we must make some assumptions about the properties of the quantities S_l. Following Blair and Austern,[1] we assume that for scattering from a noncentral field

$$S_l=\eta(l-l_0)e^{2i\sigma_l}, \qquad (24)$$

where σ_l is the Coulomb phase shift, and $l_0 = kR_0$. We must now write in the case of a noncentral field

$$S_l=\eta(l-l_0-\lambda)e^{2i\sigma_l}, \qquad (25)$$

where according to (2)

$$\lambda=k\sum_{\nu\mu}\xi^*_{\nu\mu}Y_{\nu\mu}(\theta,\phi). \qquad (26)$$

Let us expand $\eta(l-l_0-\lambda)$ in powers of λ:

$$\eta(l-l_0-\lambda)=\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\frac{d^n\eta}{dl^n}. \qquad (27)$$

The quantity λ is a scalar, therefore we can write λ^n in the form

$$\lambda^n=k^n\sum_{Lm}(-1)^m\Phi^{(n)}_{L-m}Y_{Lm}(\theta,\phi), \qquad (28)$$

where the $\Phi^{(n)}_{L-m}$ transform like spherical harmonics V_{L-m} under rotations and reflections and are functions of the quantities $\xi^*_{\nu\mu}$.

For the matrix elements of $\Phi^{(n)}_{L-m}$ we can write[4]

$$(IM|\Phi^{(n)}_{L-m}|I_0M_0)=\frac{(-1)^{I_0-M_0}}{\sqrt{2L+1}}$$

$$\times(I_0I,M,-M_0|L,-m)(I\|\Phi_L^{(n)}\|I_0), \qquad (29)$$

where $(I\|\Phi^{(n)}_L\|I_0)$ is a reduced matrix element, a quantity independent of M, M_0, and m.

The integral of the product of three spherical functions is equal to

$$(Y_{l'm},Y_{Lm}Y_{l0})=(-1)^m\sqrt{\frac{(2l'+1)(2L+1)}{4\pi(2l+1)}}$$

$$\times(Ll'm,-m|l0)(Ll'00|l0). \qquad (30)$$

Substituting in (23) expression (24) for S_l and the analogous expression for $S_{l'}$, we obtain, using (27) to (30),

$$F^{IM}_{I_0M_0}(\vartheta,\varphi)=\frac{1}{4ik}\sum_{Lmn}(-1)^{I_0-M_0}\frac{(-k)^n}{n!}(II_0M,-M_0|L,-m)$$

$$\times(I\|\Phi_L^{(n)}\|I_0)f^{(n)}_{Lm}(\vartheta,\varphi), \qquad (31)$$

where

$$f^{(n)}_{Lm}(\vartheta,\varphi)=\sum_{ll'}e^{2i\sigma_l}\frac{d^n\eta}{dl^n}$$

$$\times\{\sqrt{2l'+1}\,i^{l'-l}(Ll'm,-m|l0)(Ll'00|l0)Y^*_{l'm}(\vartheta,\varphi)$$

$$+\sqrt{2l+1}\,i^{l-l'}(Llm,-m|l'0)(Ll00|l'0)Y^*_{lm}(\vartheta,\varphi)\}. \qquad (32)$$

A very important circumstance is that the summation over l' in (32) can be carried out completely up to terms of order $(kR)^{-1}$, i.e., with the same accuracy with which the theory has been constructed from the very beginning. The method for calculating this sum is given in the appendix. The result depends on the region of the scattering angles considered. In the case of small scattering angles satisfying $kR\vartheta\le 1$, we obtain

$$f^{(n)}_{Lm}(\vartheta,\varphi)=2\sqrt{\frac{4\pi}{2L+1}}Y^*_{Lm}\left(\frac{\pi}{2},\varphi\right)$$

$$\times\sum_l\sqrt{2l+1}\,e^{2i\sigma_l}\frac{d^n\eta}{dl^n}Y^*_{lm}\left(\vartheta,-\frac{\pi}{2}\right). \qquad (33)$$

For scattering angles determined by the inequality $kR\vartheta\gg 1$, we have

$$f^{(n)}_{Lm}(\vartheta,\varphi)=(-1)^m\sqrt{\frac{4\pi}{2L+1}}$$

$$\times\left[Y^*_{Lm}\left(\frac{\pi}{2}+\vartheta,\varphi\right)+Y^*_{Lm}\left(\frac{\pi}{2},\varphi\right)\right]$$

$$\times\sum_l\sqrt{2l+1}\,e^{2i\sigma_l}\frac{d^n\eta}{dl^n}Y^*_{lL}\left(\vartheta,\frac{\pi}{2}\right). \qquad (34)$$

Averaging the square moduli of the amplitudes over the initial states and summing over the final states, we obtain the differential cross section

$$\sigma_{II_0}(\vartheta)=\frac{1}{16k^2(2I_0+1)}$$

$$\times\sum_{Lm}\left|\sum_n\frac{(-k)^n}{n!}(I\|\Phi_L^{(n)}\|I_0)f^{(n)}_{Lm}(\vartheta,\varphi)\right|^2. \qquad (35)$$

In the region of angles of most practical interest, $kR\vartheta\gg 1$, in which the diffraction minima and maxima are located, expression (35) simplifies:

$$\sigma_{II_0}(\vartheta)=\frac{1}{16k^2(2I_0+1)}\sum_L[1+P_L(\cos\vartheta)]\left|\sum_n\frac{(-k)^n}{n!}\right.$$

$$\left.\times(I\|\Phi_L^{(n)}\|I_0)f_L^{(n)}(\vartheta)\right|^2, \qquad (36)$$

where

$$f_L^{(n)}(\vartheta) = \sum_l \sqrt{2l+1}\, e^{2i\sigma_l} \frac{d^n\eta}{dl^n} Y_{lL}(\vartheta, 0). \quad (37)$$

Formulas (31) and (33) to (37) constitute the main results of the present paper.

4. DISCUSSION OF THE FORMULAS OBTAINED

Let us consider some general properties of the expressions obtained for the amplitudes and cross sections.

Let us show first of all that the Glendenning-Kromminga-MacCarthy rule[5-6] follows from expressions (31) and (33): the cross sections vanish for $\vartheta = 0$ if the parity of the nucleus in the excited state is opposite to the parity of the ground state. This rule was proved earlier on the basis of the distorted wave method in first approximation in the deformation parameter,[6] and also in the adiabatic approximation on the basis of symmetry considerations.[7]

For $\vartheta = 0$ it follows from (33) that only the terms with $m = 0$ are nonvanishing in (31). Therefore the following factors appear in the sum over L in (31):

$$P_L(\pi/2) = 0, \quad \text{if L is odd} \quad (38)$$

On the other hand, in transitions with change of parity,

$$(I\|\Phi_L^{(n)}\|I_0) = 0, \quad \text{if L is even}, \quad (39)$$

since the $\Phi_L^{(n)}$ transform like spherical harmonics of order L under inversions. Comparison of (38) and (39) gives the required result.

Blair and Austern[8] obtained the following selection rule:

$$F_{00}{}^{IM}(\vartheta, \varphi) = 0, \quad (40)$$

where M is odd and the amplitudes are calculated in a coordinate system with a z axis directed along the vector $\mathbf{q} = \mathbf{k} - \mathbf{k}_0$, i.e., the vector of the transferred momentum. This property plays an important role in the consideration of the correlation function of the γ quanta emitted by the excited nucleus. Let us show that our results contain this selection rule. We set $I_0 = M_0 = 0$, $L = I$, $m = -M$ in (31), (33), and (34). We further specify the choice of coordinate system by equating the reaction plane with the xz axis. Then $\varphi = 0$ and

$$F_{00}{}^{IM}(\vartheta, 0) = \sum_l \chi(l, \vartheta)\, Y^*_{l-M}\left(\frac{\pi}{2}, 0\right) Y^*_{l-M}\left(\vartheta, -\frac{\pi}{2}\right),$$

$$kR\vartheta \lesssim 1; \quad (41)$$

$$F_{00}{}^{IM}(\vartheta, 0) = \left[Y^*_{IM}\left(\frac{\pi}{2} + \vartheta, 0\right) + Y^*_{IM}\left(\frac{\pi}{2}, 0\right)\right]\psi(\vartheta),$$

$$kR\vartheta \gg 1, \quad (42)$$

where χ and ψ are functions independent of M.

It follows from the definition of the amplitude (22) that F_{00}^{IM} transforms like Y^*_{IM} under rotations of the coordinate system. Thus the amplitude $F_{00}^{IM'}$ in a system with a z axis along the vector $\mathbf{q}$ is expressed through the amplitudes of the original system in the following way:

$$F_{00}{}^{IM'} = \sum_M D^{I*}_{MM'}\left(0, \frac{\pi + \vartheta}{2}, 0\right) F_{00}{}^{IM}, \quad (43)$$

since the angle between the vectors $\mathbf{q}$ and $\mathbf{k}_0$ is equal to $(\pi + \vartheta)/2$.

From this we obtain for $kR\vartheta \gg 1$, using (42),

$$F_{00}{}^{IM'} = [Y^*_{IM'}(1/2\vartheta, 0) + Y^*_{IM'}(1/2\vartheta, \pi)]\psi(\vartheta)$$
$$= [1 + (-1)^{M'}]\, Y^*_{IM'}(1/2\vartheta, 0)\psi(\vartheta), \quad (44)$$

which yields the required selection rule.

In the case $kR\vartheta \gg 1$ one may neglect the angle ϑ in the argument of the D function in (43), since $\vartheta \sim 1/kR$. We note, moreover, that (41) has the consequence that

$$F_{00}{}^{IM} = (-1)^M F_{00}{}^{I-M};$$
$$F_{00}{}^{IM} = 0, \text{ if } I + M \text{ is odd}. \quad (45)$$

We can now write (43) in the form

$$F_{00}{}^{IM'} = D^{I*}_{0M'}\left(\frac{\pi}{2}\right) F_{00}{}^{I0} + \sum_{M=1}\left[D^{I*}_{MM'}\left(\frac{\pi}{2}\right) + (-1)^M D^{I*}_{-MM'}\left(\frac{\pi}{2}\right)\right] F_{00}{}^{IM}. \quad (46)$$

The first term in (46) vanishes for odd M′, since according to (45), $F_{00}^{I0} = 0$ if I is odd and, on the other hand, $D^I_{0M'}(\pi/2) = 0$ if $I + M'$ is odd. As far as the second term in (46) is concerned, one easily obtains from the symmetry properties of the D functions the relation

$$D^I_{-MM'}\left(\frac{\pi}{2}\right) = (-1)^{I+M'} D^I_{MM'}\left(\frac{\pi}{2}\right),$$

taking account of the fact that $I + M$ must be even according to (45), the sum in (46) is expressed in the form

$$\sum_M = \sum_{M=1}^{I} [1 + (-1)^{M'}] D^{I*}_{MM'} F_{00}{}^{IM}, \quad (47)$$

which concludes the required proof.

Let us now compare our results with the results of Blair and Austern.[1] Restricting ourselves to the case $I_0 = 0$, which was also considered in [1], we compare the results for $n = 1$, i.e., in first order in the parameter of nonsphericity. Then we obtain from (31) and (33) for $kR\vartheta \lesssim 1$

$$F_{00}{}^{IM}(\vartheta,0)=\frac{i}{2}\sqrt{2I+1}c_1(I)[I:M]$$

$$\times\sum_l\sqrt{2l+1}e^{2i\sigma_l}\frac{d\eta}{dl}Y_{l-M}(\vartheta,0), \qquad (48)$$

where

$$c_1(I)=\frac{(-1)^M}{\sqrt{2I+1}}(I\|\Phi_I{}^{(1)}\|0)=(IM|\xi_{IM}|00), \qquad (49)$$

$$[I:M]=i^{-M}\sqrt{\frac{4\pi}{2I+1}}Y_{IM}\left(\frac{\pi}{2},0\right). \qquad (50)$$

Expression (48) for the amplitude agrees exactly with the expression found in [1]. However, in [1] this expression was assumed valid for all scattering angles, i.e., also in the region $kR\vartheta \gg 1$. Let us see to what results the extrapolation of (48) into the region $kR\vartheta \gg 1$ leads. In this region, using $Y_{IM}(\pi/2) = 0$ if $I + M$ is odd and the property (10), we may in (48) or (34) make the replacement

$$Y_{l-M}(\vartheta,0)\to i^{M-I}Y_{l-I}(\vartheta,0). \qquad (51)$$

Then we obtain from (48)

$$F_{00}^{IM}(\vartheta,0)=i^{I+1}(-1)^M\sqrt{\pi}c_1(I)Y_{IM}\left(\frac{\pi}{2},0\right)$$

$$\times\sum_l\sqrt{2l+1}\,e^{2i\sigma_l}\frac{d\eta}{dl}Y_{l-I}(\vartheta,0). \qquad (52)$$

On the other hand, (31) and (34) lead in this case to the following result:

$$F_{00}{}^{IM}(\vartheta,0)=\frac{1}{2}i^{I+1}(-1)^M\sqrt{\pi}\,c_1(I)$$

$$\times\left[Y_{IM}\left(\frac{\pi}{2}+\vartheta,0\right)+Y_{IM}\left(\frac{\pi}{2},0\right)\right]$$

$$\times\sum_l\sqrt{2l+1}\,e^{2i\sigma_l}\frac{d\eta}{dl}Y_{l-I}(\vartheta,0). \qquad (53)$$

The comparison of (52) and (53) shows that the difference between the results of Blair and Austern and ours consists in the replacement of the factor $Y_{IM}(\pi/2, 0)$ by the factor

$$\tfrac{1}{2}[Y_{IM}(\pi/2+\vartheta,0)+Y_{IM}(\pi/2,0)].$$

As we have seen, the last factor guarantees, in particular, the satisfaction of the Blair-Wilets selection rule, i.e., makes the structure of the amplitude considerably more precise. In the differential cross section this improvement leads to the appearance of the factor $[1 + P_I(\cos\vartheta)]/2$, which can have an appreciable effect on the magnitude of the cross section. Thus, for example, for $I = 2$ and $\vartheta = \pi/2$ this factor is equal to $1/4$, i.e., reduces the cross section a factor of one fourth as compared to the corresponding expression obtained from (51).

We note that (53) can be obtained also in the Blair-Austern method. To this end one must approximate better than was done in [1] the radial integrals

$$I(l,l')=\int_0^\infty f_l(kr)\frac{\partial V}{\partial R}f_{l'}(kr)\,dr. \qquad (54)$$

In [1] the quantities $I(l, l')$ were approximated essentially by

$$I(l,l')\approx I(l',l')=a\,d\eta_{l'}/dl', \qquad (55)$$

where a is some coefficient. A more accurate approximation is

$$I(l,l')\approx\frac{1}{2}[I(l,l)+I(l',l')]=\frac{1}{2}a\left[\frac{d\eta_l}{dl}+\frac{d\eta_{l'}}{dl'}\right]. \qquad (56)$$

Indeed, this approximation is exact up to terms which are quadratic in the difference $\tau = l - l'$, while (55) is valid only up to terms linear in τ. Substituting (56) in the corresponding expressions with subsequent summation according to the method explained in the appendix, we find our result (53). Nevertheless, the method described here is to be preferred owing to its greater simplicity and the possibility it offers of considering any order of perturbation theory without any complications, and also of including the spin of the nucleus.

5. PROPERTIES OF THE CROSS SECTIONS. PHASE RULE

Let us briefly consider the basic properties of expressions (35) and (36) for the differential cross sections. Expression (35) shows that for scattering on a nucleus with nonvanishing spin the cross section is composed, as it were, of the cross sections for scattering on a nucleus with spin zero with transitions into states with spins L which run through the values from $|I - I_0|$ to $|I + I_0|$. The quantity L is naturally called the multipolarity of the corresponding transition. For simplicity we shall consider only the first nonvanishing approximation, discarding all higher terms, i.e., we shall consider the quantities

$$\sigma_{Lm}{}^{(n)}(\vartheta)=\frac{k^{2n-2}}{16(2I_0+1)(n!)^2}$$

$$\times|(I\|\Phi_L{}^{(n)}\|I_0)|^2\,|f_{Lm}{}^{(n)}(\vartheta,\varphi)|^2. \qquad (57)$$

We omit for simplicity the Coulomb phases $\exp(2i\sigma_l)$, which in the case of small values of the quantity $z_1z_2e^2/\hbar v$, leads only to insignificant corrections in the region of angles $kR\vartheta \gg 1$. Expression (57) leads in the region $kR\vartheta \lesssim 1$ to the usual expressions obtained on the basis of Fraunhofer diffraction theory.[9] Of more interest is the region $kR\vartheta \gg 1$, where the diffraction maxima and minima are located. In this region

$$\sigma_L^{(n)}(\vartheta) = \sum_m \sigma_{Lm}^{(n)}(\vartheta) = A_L^{(n)}[1 + P_L(\cos\vartheta)] \left| \sum_l \sqrt{2l+1} \frac{d^n\eta}{dl^n} Y_{lL}(\vartheta, 0) \right|^2, \quad (58)$$

where

$$A_L^{(n)} = k^{2n-2} |(I\|\Phi_L^{(n)}\|I_0)|^2 / 16(2I_0+1)(n!)^2.$$

We replace the summation in (58) by an integration from $-\infty$ to $+\infty$, since the quantity $d^n\eta/dl^n$ differs from zero only in a small region near $l = l_0$. We may also replace $\sqrt{2l+1}$ by $\sqrt{2kR}$ in (58) and integrate by parts $n-1$ times, taking account of (10) and keeping only the leading terms in l_0. Then we obtain

$$\sigma_L^{(n)}(\vartheta) = \frac{2kRA_L^{(n)}}{\pi^2 \sin\vartheta} \vartheta^{2n-2}(1 + P_L(\cos\vartheta)) \times \left| \int_{-\infty}^{+\infty} \frac{d\eta}{dl} \cos\left[\left(l + \frac{1}{2}\right)\vartheta - \frac{1}{2}(L+n)\pi + \frac{\pi}{4}\right] dl \right|^2. \quad (59)$$

This same formula, with $L = n = 0$, will be valid for the principal term in the elastic cross section.

Formula (59) gives the possibility of explaining a number of general properties of the cross sections. First of all, it is seen that the integral in (59) is a rapidly oscillating function of the angle, since the quantity $d\eta/dl$ differs from zero only in the narrow region $\Delta l \ll l_0$ near l_0. In the limiting case of a black nucleus $U(l - l_0) \equiv d\eta/dl = \delta(l - l_0)$. The presence of a boundary region leads to a smearing out of the function $U(l - l_0)$, which moreover becomes complex. After simple transformations we obtain from (59)

$$\sigma_L^{(n)}(\vartheta) = \frac{2kRA_L^{(n)}}{\pi^2 \sin\vartheta} \vartheta^{2n-2}(1 + P_L(\cos\vartheta)) |F(\vartheta)|^2 \times \left\{ b^2(\vartheta) + \cos^2\left[\left(l_0 + \frac{1}{2}\right)\vartheta - \frac{1}{2}(L+n)\pi + \frac{\pi}{4} + \gamma(\vartheta)\right]\right\}, \quad (60)$$

where

$$F(\vartheta) = \sqrt{A^2 + B^2}, \quad A = \int_{-\infty}^{+\infty} U(x)\cos(x\vartheta)\,dx, \quad B = \int_{-\infty}^{+\infty} U(x)\sin(x\vartheta)\,dx,$$

$$\cos\psi = A/F, \quad \sin\psi = BF, \quad \gamma(\vartheta) = \operatorname{Re}\psi, \quad b(\vartheta) = \operatorname{sh}\lambda, \quad \lambda = \operatorname{Im}\psi. \quad (61)$$

We see that the quantities $F(\vartheta)$, $\gamma(\vartheta)$, and $b(\vartheta)$ are independent of L and n and vary comparatively little as ϑ is increased. Expression (10) contains the phase rule which was obtained by Blair[9] on the basis of Fraunhofer scattering theory. Here we can formulate it in general form: for the cross section for a transition of multipolarity L and order n the phase shift of the oscillations relative to the elastic scattering cross section ($n = L = 0$) is equal to $(L + n)\pi/2$.

We note that the independence of the form factor $F(\vartheta)$ of L and n is in agreement with the results of [10], where the diffuseness of the nuclear boundary was taken into account within the framework of the Fraunhofer theory.

Another consequence of (60) is the dependence of the cross section on ϑ for different n: this dependence is given by the factor ϑ^{2n}. For example, the cross section for double excitation ($n = 2$) contains the factor ϑ^4 relative to the elastic cross section ($n = 0$), which leads to a much smoother decrease of the envelope of the oscillations with increasing angle.

APPENDIX

CALCULATION OF THE SUM OVER l'

In the sum (32) the quantity $d^n\eta/dl^n$ differs from zero only for large l, $l \sim KR \gg 1$. We shall assume that the multipolarities of the transitions are small, i.e., we assume $L/l \ll 1$. Under these conditions we can use the asymptotic formula for the Clebsch-Gordan coefficients with two large values and one small value of the angular momentum:[4]

$$(\varkappa, l-\tau, \mu, m-\mu | lm) \approx (-1)^{\varkappa-\tau} D_{\mu\tau}^{\varkappa}(\theta), \quad (A.1)$$

where $D_{\mu\tau}^{K}$ is the finite rotation matrix, and the angle θ is defined by $\cos\theta = m/l$.

Let us consider the sum entering in the first term of (32):

$$\sum = \sum \sqrt{2l'+1}\, i^{l'-l}(Ll'm, -m|l0) \times (Ll'00|l0)\, Y_{l'm}(\vartheta, \varphi). \quad (A.2)$$

Introducing the quantity $\tau = l - l'$, using (A.1) and noting that up to terms of order $1/kR$ we can write $\sqrt{2l'+1} = \sqrt{2l+1}$, we write this sum in the form

$$\sum = \sqrt{2l+1} \sum i^{-\tau} D_{m\tau}^{L}\left(\frac{\pi}{2}\right) D_{0\tau}^{L}\left(\frac{\pi}{2}\right) Y^{*}_{l-\tau, m}(\vartheta, \varphi). \quad (A.3)$$

Noting that $i^{-\tau} = e^{-i\pi\tau/2}$ and also using the simple properties of the D functions, we can rewrite (A.3) as

$$\sum = \sqrt{2l+1} \sum_{\tau} D_{m\tau}^{L}\left(0, \frac{\pi}{2}, 0\right) \times D_{\tau 0}^{L}\left(-\frac{\pi}{2}, \frac{\pi}{2}, 0\right) Y_{l-\tau, m}^{*}(\vartheta, \varphi). \quad (A.4)$$

We shall now distinguish two cases:

1) $l\vartheta \approx kR\vartheta \approx 1$, i.e., $\vartheta \ll 1$;

2) $kR\vartheta \gg 1$, i.e., $\vartheta \sim 1$.

In the first case the dependence of $Y_{l-\tau, m}$ on τ can be neglected. This is seen most easily if one uses the asymptotic expression of $Y_{l-\tau, m}$ for large l and small ϑ in terms of the Bessel function. Then

$$\sum = \sqrt{2l+1}\, Y_{lm}^{*}(\vartheta, \varphi) \times \sum_{\tau} D_{m\tau}^{L}\left(0, \frac{\pi}{2}, 0\right) D_{\tau 0}^{L}\left(-\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$$
$$= \sqrt{2l+1}\, Y_{lm}^{*}(\vartheta, \varphi) D_{m0}^{L}\left(-\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}\right)$$
$$= \sqrt{4\pi \frac{2l+1}{2L+1}}\, Y_{Lm}^{*}\left(\frac{\pi}{2}, \varphi\right) Y_{lm}^{*}\left(\vartheta, -\frac{\pi}{2}\right). \quad (A.5)$$

We use the following definition of the Euler angles α, β, γ:[4] a rotation by α about the z axis transfers the system S to the position S′; a rotation by β about the y axis of the system S′ transfers this system to the position S″; a rotation by γ about the z axis of the system S″ transfers the body to the final position. A right-hand system of axes of the system S is used, and a rotation which corresponds to the motion of a right-hand screw along the positive direction of an axis is called a positive rotation about that axis.

In the second case we can use the asymptotic expression for the spherical functions with large l:

$$Y_{l-\tau, m}^{*}(\vartheta, \varphi) \approx \frac{1}{\pi\sqrt{\sin\vartheta}} \cos\left[\left(l - \tau + \frac{1}{2}\right)\vartheta + \frac{m\pi}{2} - \frac{\pi}{4}\right] e^{-im\varphi}. \quad (A.6)$$

Expanding the cosine into exponentials and including the factors $e^{\pm i\tau\vartheta}$ in the D function, we can again carry out the summation over τ with the result

$$\sum = 2i^{L} \sqrt{\frac{2l+1}{\pi(2L+1)\sin\vartheta}} \cos\left[\left(l + \frac{1}{2}\right)\vartheta - (L+1)\frac{\pi}{2} + \frac{\pi}{4}\right] Y_{L-m}\left(\frac{\pi}{2} + \vartheta, \varphi\right)$$
$$= \sqrt{4\pi \frac{2l+1}{2L+1}} (-1)^{m} Y_{Lm}^{*}\left(\frac{\pi}{2} + \vartheta, \varphi\right) Y_{lL}^{*}\left(\vartheta, \frac{\pi}{2}\right). \quad (A.7)$$

The sum entering in the second term of (32) is calculated in an analogous fashion.

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