# AN INVESTIGATION OF THE STRUCTURE OF SPACE TIME BY ANALYTIC CONTINUATION OF THE EQUATIONS OF GEODESICS

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A study is made of the structure of spaces in general relativity theory which are of a special type (1), without using the transition to a regular coordinate system employed in [5]; use is made of analytic continuation of the equations of isotropic geodesic lines through the hypersurface of fictitious singularity of the metric (1).

1. QUITE a number of spaces in general relativity theory admit systems of coordinates for which the metric tensor is of the form

$$ds^2 = -\varphi(r)\,dt^2 + dr^2/\varphi(r) + r^2 d\Omega^2. \tag{1}$$

Such spaces include the Schwarzschild space ($\varphi = 1 - \mu/r$), the De Sitter world ($\varphi = 1 - kr^2$) (cf., e.g., [1]), the Kottler space[2] ($\varphi = 1 - \mu/r - kr^2$) which is a generalization of these two, and also the Nordström space[3] ($\varphi = 1 - \mu/r + \epsilon^2/r^2$) caused by a charged mass.

As is well known (cf., e.g., [4–6]), any space of this type is not confined to the ordinary region where $\varphi(r) > 0$ (which we shall here call the R region, as in [7]), but extends beyond the hypersphere $\varphi(r) = 0$ on which there is a fictitious singularity of the metric (1). For the study of the entire structures of some of these spaces, work by Kruskal[4] and by Graves and Brill[5] employs a more complicated coordinate system which overlaps the hypersurface in question and has no singularity on it. In the present paper it is shown that this sort of study can be carried out on the basis of the coordinate system with the metric (1) without going over to other coordinates, and the structure of the spaces (1) is derived in various cases.

2. We shall start from the fact that the space in question is the maximally smooth continuation of (some) basic R region, i.e., of a connected region for which $\varphi(r) > 0$ and t takes values $-\infty < t < \infty$.

It is assumed mathematically that we can put in correspondence with the space as a whole an atlas of local maps, that the regions of the maps cover the whole space, and that in the region of any map the metric tensor is given by analytic functions of the local coordinates [these functions being defined for the basic region by the metric (1)]. To fulfill these conditions in a region covered by several maps, the transformation from one to another set of local coordinates must be expressible in analytic functions. But this makes it possible, by an analytic continuation of the coordinate transformations through the boundaries of the regions of the maps, to extend to the entire space the coordinate system of the basic region with the metric (1), so that the space as a whole is imaged (in general not in a one-to-one way) on some range of variation (not necessarily real) of the coordinates r and t.

It is not necessary, however, to make actual use of the atlas we have mentioned in order to determine such a range of variation of the coordinates r and t. Instead, we can simply use the equations of geodesic lines in the coordinates r and t in the whole space, under the condition that they be continued analytically through the boundaries of the regions and that the canonical parameter on them be real (see also [8]). For a metric of type (1) this method is especially convenient, because, as can easily be verified, a canonical parameter on isotropic radial geodesics (which are all that will be used) is the coordinate r itself. The equations of these geodesics are of the following form:

$$t = t_0 \pm \int_{r_0}^{r} \frac{dr}{\varphi(r)} = t_0 \pm \Phi(r) \mp \Phi(r_0), \tag{2}$$

where $\Phi(r)$ denotes the primitive function.

3. The study of the structure of the space consists of constructing the image of its radius-time part. The guiding principles for this are as follows. Through each point of the diagram there must pass two isotropic geodesic lines (the light "cone"), on which a direction is indicated (from past to future). Depending on the sign in Eq. (2)

all such lines are divided into two families (opposed rays), and lines belonging to the same family do not intersect each other. Two adjacent lines of the same family are easily compared as to direction and go in the same direction. Of two adjacent lines of the same family we can always indicate which is the later line, so that the set of lines of a family are ordered. Each line intersects the lines of the other family everywhere in the same direction (for example, from left to right)—from the past to the future. The coordinate r varies monotonically along the lines. By definition, the coordinate r increases along an e line and decreases along an i line. The i lines and the e lines belonging to the same family are separated by lines $r = \zeta$ [the $\zeta$ are the roots of the equation $\varphi(r) = 0$] which, being isotropic geodesics, themselves belong to the family.

To construct the representation it is essential to know which kind of lines (e or i) intersect some chosen line. The following argument (which we carry through completely here for definiteness only for the Nordström space) elucidates this matter. Let the equation of the i line which passes through a point $r = r_0$, $t = 0$, which for definiteness we suppose belongs to the basic region, be

$$t = -\Phi(r) + \Phi(r_0). \tag{3}$$

The equation of a line infinitesimally later than it is

$$t = -\Phi(r) + \Phi(r_0 + dr_0). \tag{4}$$

On the line (3) we choose an arbitrary point $r_1$, $t_1 = -\Phi(r_1) + \Phi(r_0)$. An elementary displacement from this point tangent to the world line of the other family has components $dr_1$, $dt_1 = dr_1/\Phi(r_1)$. If we require that this displacement lead to a point on the delayed geodesic (4), then the displacement will be directed toward the future, and (in the case of Nordström space) we find for it successively

$$t_1 + dt_1 = -\Phi(r_1 + dr_1) + \Phi(r_0 + dr_0),$$
$$2r_1^2 dr_1 / (r_1 - \zeta_1)(r_1 - \zeta_2) = dr_0 / \varphi(r_0),$$

where $\zeta_1$, $\zeta_2$ ($\zeta_2 \leq \zeta_1$) are the roots of the equation $\varphi(r) = 0$, which are distinct (case a) or coincident (case b). Accordingly, in case a, since $dr_1$ changes sign when the point $r_1, t_1$ goes through $r_1 = \zeta$, both i lines and e lines belong to a single family. In case b the symbols i and e belong to the families as wholes.

This enumeration of the properties of the lines makes no pretense to exhaustive rigor (possible questions can be cleared up easily by physical considerations). The indicated properties, however, allow us to construct representations which satisfy all the necessary requirements. These representations for the various cases, together with the corresponding plots of the functions $\varphi(r)$, are shown in Figs. 1–8. The lines in the representations are radial isotropic geodesics, and the heavy lines are simultaneously the boundaries of the regions at $r = \zeta$.

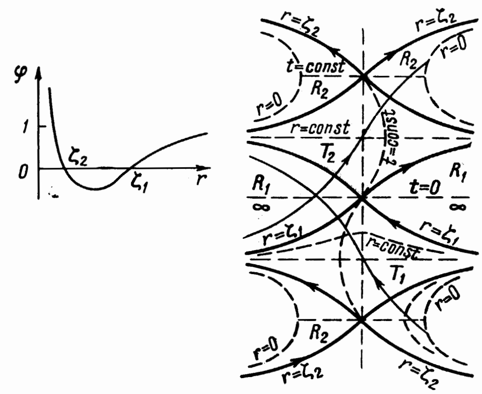


FIG. 1. Nordström space. Case a (distinct roots).

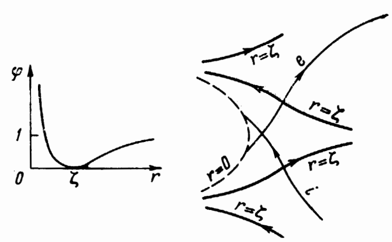


FIG. 2. Nordström space. Case b (equal roots).

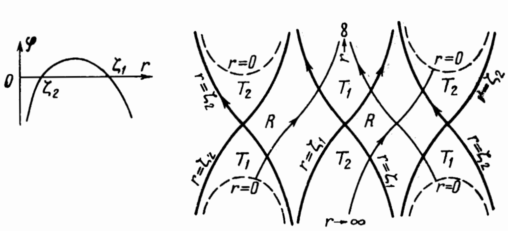


FIG. 3. Kottler space, $k > 0$, $\mu > 0$.

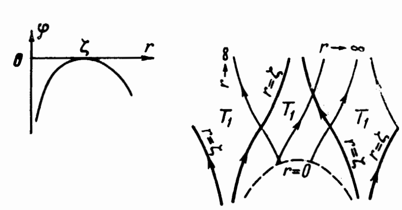


FIG. 4. Kottler space, $k > 0$, $\mu > 0$.

4. We shall make some remarks about the individual diagrams.

Fig. 1. In case a the Nordström space contains

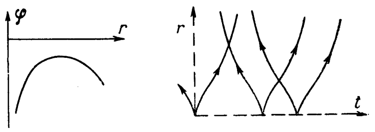


FIG. 5. Kottler space, k > 0, μ > 0.

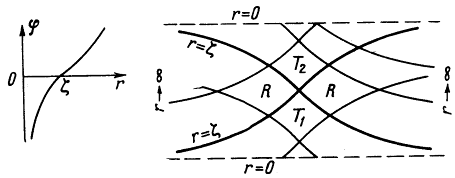


FIG. 6. Kottler space, k < 0, μ > 0.

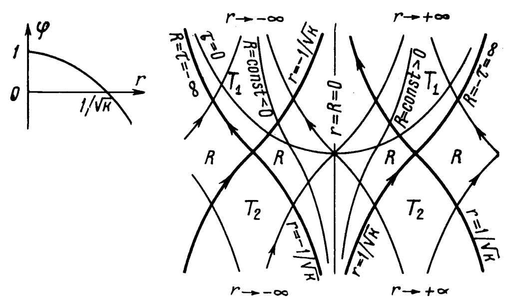


FIG. 7. De Sitter world, k > 0.

(in agreement with [5]) a denumerable set of regions of four different types. The $R_1$ are ordinary regions with $r > \zeta_1$; $R_2$ are ordinary regions with $r < \zeta_2$; $T_2$ are contracting T-regions (in Novikov's terminology[7]) containing trapped surfaces (in the terminology of Penrose[9]); $T_1$ are expanding T-regions.

Some of the coordinate lines of the r, t grid are shown by dashed lines in this figure; in the T-regions and the $R_2$ regions the coordinate t (which is not necessarily timelike) takes complex values, the respective imaginary parts in the two cases being $i\pi\zeta_1^2/(\zeta_1 - \zeta_2)$ and $i\pi\mu$, in accordance with Eq. (2), which gives for the Nordström space in case a

$$t = t_0 \pm \left( r - r_0 + \frac{\zeta_1^2}{\zeta_1 - \zeta_2} \ln \frac{r - \zeta_1}{r_0 - \zeta_1} + \frac{\zeta_2^2}{\zeta_2 - \zeta_1} \ln \frac{r - \zeta_2}{r_0 - \zeta_2} \right).$$

Fig. 2. In case b the Nordström contains only R-regions of the two types.

Fig. 3. In the case of Kottler space we can identify the similar regions shown at the edges of the diagrams. The space is then closed in the intrinsically spacelike direction.

Figures 4 and 5 show the cases in which the roots are equal and in which there are no roots of

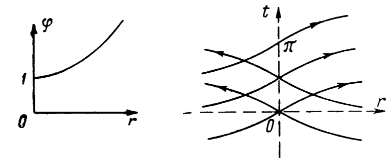


FIG. 8. De Sitter world, k < 0.

the equation $\varphi(r) = 0$. There is another possible version of these diagrams, containing only contracting T-regions.

The structure of the space of Fig. 6 is reminiscent of that of Schwarzschild space.

A representation for the case of the pure De Sitter world of constant positive curvature (with signature +2) is shown in Fig. 7, and is a deformed unrolling of a hyperboloid of one sheet, with which there is represented the radius-time part of the De Sitter world as imbedded in a pseudoeuclidean three-space (with signature +1).[1] In this case it is essential to identify regions at the edges of the diagram.

A very intuitive illustration can be given of the relation between the coordinate system (1) and other systems used to describe the De Sitter world, and also of Zel'manov's idea of the relativity of the extent of space.[10] As an example we show with solid lines in Fig. 7 some coordinate lines of the Robertson-Lemaître system, which has the metric

$$ds^2 = -d\tau^2 + (dR^2 + R^2 d\Omega^2) \exp(\tau / \sqrt{k}),$$

and which is obtained if the De Sitter world is regarded as a Friedman world with a flat (Euclidean) intrinsic space for $\epsilon = -p > 0$ (see [11], Secs. 105 and 106). It can be directly verified that to the region $-\infty < R < \infty$, $-\infty < \tau < \infty$ of variation of the coordinates R, τ there corresponds not the entire De Sitter world, but only one R-region and two T-regions which are in relation to it.

Fig. 8. If, in spite of the violation of the causality principle, we make an identification of the regions in this case also, we get a deformed unrolling of a hyperboloid of one sheet which is imbedded by means of the equations

$$x = r, \quad y = \sqrt{r^2 - 1/k} \cos \sqrt{-k}\, t,$$

$$z = \sqrt{r^2 - 1/k} \sin \sqrt{-k}\, t$$

in a three-space with the metric $ds^2 = dx^2 - dy^2 - dz^2$.

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