HYDRODYNAMICS OF A LIQUID WITH INTRINSIC ROTATION

M. I. SHLIOMIS

Perm State University

A complete set of hydrodynamic equations is derived phenomenologically on the basis of the conservation laws for a liquid possessing an intrinsic angular momentum. The total angular momentum per unit volume of such a medium consists of the ordinary angular momentum due to the hydrodynamic (translational) motion of the particles of the liquid and of its characteristic (intrinsic) rotation $\mathbf{M}$. In comparison with the ordinary hydrodynamic equations, the set deduced here contains two additional kinetic coefficients (diffusion of the intrinsic angular momentum and rotational viscosity coefficients) and an equation for the rate of change of $\mathbf{M}$. It follows from the latter equation that the variation of $\mathbf{M}$ is due to two causes: the absence of equilibrium connection between the intrinsic angular momentum and the vortex motion of the liquid (relaxation), and the inhomogeneity of $\mathbf{M}$ (diffusion). The stress tensor is nonsymmetric. It is shown that the interaction between the apparent and latent rotations leads to a dispersion of the first viscosity. The quasistationary transfer of rotation to particles suspended in the liquid is considered. The kinetic coefficients are found by neglecting the diffusion of the particles.

ORDINARILY, the hydrodynamic properties of a liquid are completely determined by two viscosity coefficients—the shear η and the bulk ζ. A more general expression for the forces of internal friction (with additional viscosity coefficients) can be obtained if allowance is made for the fact that the liquid, moving with the hydrodynamic velocity $\mathbf{v}$ the "visible" or "apparent" motion) possesses in addition a certain characteristic ("hidden" or "latent") rotation. Then the total angular momentum per unit volume of the liquid consists of the ordinary angular momentum $\mathbf{L} = \rho \mathbf{r} \times \mathbf{v}$, associated with the apparent motion of the liquid, and of an intrinsic angular momentum $\mathbf{M}$.

The problem of the motion of a liquid with latent angular momentum was first formulated and considered by Sorokin.[1] He found that, even without shear or bulk deformation, an internal friction arises in such a medium, due to rotation of the liquid. In fact, if the rotational state is suddenly changed, then the equilibrium distribution of the momentum between the apparent and latent motions is disrupted. This means that irreversible relaxational processes, associated with additional energy dissipation, are commenced. In these processes, the apparent rotation will frequently go over into the latent form or, conversely, the vortex motion of a liquid will be created at the expense of the intrinsic angular momentum.

In the present research, the complete set of hydrodynamic equations is derived for a liquid with internal rotation. Our derivation, which is based on the use of the conservation laws only, is more general than in [1].

As an example of a system to which the resultant equations are applicable, we choose a suspension, in which the liquid exchanges rotation with the solid particles suspended in it, or a two-phase medium vapor—liquid: the bubbles of vapor can rotate relative to the liquid. In addition, our equations can also have meaning for a "pure" liquid, the molecules of which possess an intrinsic angular momentum (the "hydrodynamics of molecules with spin," according to the terminology of Frenkel'[2]).

1. The hydrodynamic equations of a liquid with intrinsic rotation can be obtained phenomenologically by means of the conservation laws—in the spirit of the hydrodynamic theory of superfluidity.[3]

By defining the density of mass flow in the usual way as the momentum of a unit volume of the liquid, we write down the equations for the conservation of mass, energy, and linear and angular momentum:

$$\partial \rho / \partial t + \mathrm{div}(\rho \mathbf{v}) = 0, \tag{1}$$

$$\partial E / \partial t + \mathrm{div}\mathbf{Q} = 0, \tag{2}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial \Pi_{ik}}{\partial x_k} = 0, \tag{3}$$

$$\frac{\partial}{\partial t}(L_{ik} + M_{ik}) + \frac{\partial G_{ikl}}{\partial x_l} = 0. \tag{4}$$

Here L_{ik} and M_{ik} are antisymmetric tensors which are dual to the pseudovectors of the density of the apparent $L_l = \frac{1}{2} e_{ikl} L_{ik}$ and latent $M_l = \frac{1}{2} e_{ikl} M_{ik}$ momenta, E is the total energy per unit volume of the liquid, and Q, Π_{ik} and G_{ikl} are the flux densities of energy, momentum, and angular momentum to be determined.

The complete set of hydrodynamic equations also contain an equation for the rate of change of the intrinsic angular momentum

$$\frac{\partial M_{ik}}{\partial t} + \frac{\partial}{\partial x_l}(v_l M_{ik}) = f_{ik} \tag{5}$$

and the equation for entropy growth

$$\partial S / \partial t + \operatorname{div}(\mathbf{v}S) = R / T, \tag{6}$$

where R is the dissipation function, and f_{ik} is some antisymmetric tensor which, by means of the laws of conservation, can be expressed in terms of other quantities, up to now unknown. For this purpose, we introduce the stress tensor σ_{ik} by the equation

$$\Pi_{ik} = \rho v_i v_k - \sigma_{ik}$$

and, on the basis of (1) and (2), we write down the equation of motion of the liquid in the form

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}\right) = \frac{\partial \sigma_{ik}}{\partial x_k}. \tag{7}$$

It follows from the definition of the tensor $L_{ik} = [\mathbf{r} \times \rho\mathbf{v}]_{ik} = \rho(x_i v_k - x_k v_i)$ and Eq. (7) that

$$\frac{\partial L_{ik}}{\partial t} + \frac{\partial}{\partial x_l}(v_l L_{ik}) = x_i \frac{\partial \sigma_{kl}}{\partial x_l} - x_k \frac{\partial \sigma_{il}}{\partial x_l}$$

$$= \frac{\partial}{\partial x_l}(x_i \sigma_{kl} - x_k \sigma_{il}) + \sigma_{ik} - \sigma_{ki}. \tag{8}$$

From Eqs. (5) and (8), we obtain

$$\frac{\partial}{\partial t}(L_{ik} + M_{ik}) = f_{ik} + \sigma_{ik} - \sigma_{ki}$$

$$- \frac{\partial}{\partial x_l}[v_l(L_{ik} + M_{ik}) - x_i \sigma_{kl} + x_k \sigma_{il}].$$

By comparing the last equation with Eq. (4), which expresses the law of conservation of the total angular momentum, we find

$$f_{ik} = \sigma_{ki} - \sigma_{ik} - \partial g_{ikl} / \partial x_l$$

and Eq. (5) takes the form[1]

[1]For M = 0, the symmetry of the stress tensor follows from Eq. (9) (see [4]). However, this property does not hold if the liquid has an intrinsic angular momentum.

$$\frac{\partial M_{ik}}{\partial t} + \frac{\partial}{\partial x_l}(v_l M_{ik}) = \sigma_{ki} - \sigma_{ik} - \frac{\partial g_{ikl}}{\partial x_l} \tag{9}$$

The flux-density tensor of the intrinsic angular momentum g_{ikl} is connected with the tensor G_{ikl}, introduced above, by the relation

$$g_{ikl} = G_{ikl} - v_l(L_{ik} + M_{ik}) + x_i \sigma_{kl} - x_k \sigma_{il}.$$

Equations (2)–(9) acquire meaning only after the forms of the unknown terms Q, σ_{ik}, g_{ikl}, and R are made clear.

2. We introduce a frame of reference (K') in which the velocity of a given element of the liquid is equal to zero. This system is rotated relative to the original frame (K) with an angular velocity $\Omega = \frac{1}{2}$ curl $\mathbf{v}$. We designate by E and E' the energy per unit volume of the liquid in the frames of reference mentioned. These quantities are connected by the well known relation[5]

$$E = E' + (\mathbf{L} + \mathbf{M})\Omega. \tag{10}$$

For what follows, it is convenient to write E in still another way. For this purpose, we introduce into consideration in the system K', along with the total energy density E', the internal energy of the liquid $\widetilde{E}$, including the energy of latent rotation. Then the internal energy in the system K will be $\widetilde{E} + \mathbf{M} \cdot \mathbf{\Omega}$ and, by defining E as the sum of the internal and kinetic energy of the liquid, we get

$$E = \frac{1}{2}\rho v^2 + \mathbf{M}\mathbf{\Omega} + \widetilde{E}. \tag{11}$$

For $\mathbf{M} = 0$, this expression transforms into the usual

$$E = \frac{1}{2}\rho v^2 + E_{00}(S, \rho). \tag{12}$$

In the thermodynamics of a liquid with intrinsic rotation, the latent angular momentum $\mathbf{M}$ must be regarded as an independent thermodynamic variable, the value of which (as a function of $\mathbf{\Omega}$) is determined by the condition for equilibrium.[2]

As is well known,[5] the total angular momentum is connected with the energy E' by the relation

$$\mathbf{L} + \mathbf{M} = -\partial E' / \partial\Omega.$$

The derivative is taken here for fixed values of the intrinsic parameter $\mathbf{M}$, the entropy and the density of the liquid. Integrating, we get

$$E' = E_0(M) - \mathbf{M}\mathbf{\Omega} - \int \mathbf{L}d\mathbf{\Omega}. \tag{13}$$

By computing the derivative $\partial E / \partial M$ from (10) and (11), and equating them, we get the equation $\partial\widetilde{E}/\partial M = \partial E'/\partial M$ which, after substitution in it of E' from (13), takes the form

[2]The situation is similar in a ferromagnet: there, the magnetization $\mathbf{M}$ and magnetic field $\mathbf{H}$ correspond to our $\mathbf{M}$ and $\mathbf{\Omega}$.

$$\frac{\partial \tilde{E}}{\partial \mathbf{M}} = \frac{\partial E_0(M)}{\partial \mathbf{M}} - \boldsymbol{\Omega}.$$

The latter equation can be integrated. Taking (11) and (12) into account, and eliminating E_{00} in $E_0(M)$, we get

$$\tilde{E} = E_0(M) - \mathbf{M}\boldsymbol{\Omega}. \tag{14}$$

The expansion of the scalar function $E_0(M)$ in powers of the pseudovector $\mathbf{M}$ obviously contains only terms of even powers. With accuracy up to M^2, we have

$$\tilde{E} = E_{00} + {}^1\!/_2 \alpha M^2 - \mathbf{M}\boldsymbol{\Omega}. \tag{15}$$

The equilibrium value of $\mathbf{M}$ is determined from the condition $\partial \tilde{E}/\partial \mathbf{M} = 0$:

$$\mathbf{M} = \boldsymbol{\Omega}/\alpha. \tag{16}$$

Thus, for $\boldsymbol{\Omega} = 0$, the minimum energy $\tilde{E}$ should correspond to $\mathbf{M} = 0$. It then follows that $\alpha > 0$.

The following thermodynamic identity holds for the internal energy:

$$d\tilde{E} = \rho T ds + w d\rho + (\alpha \mathbf{M} - \boldsymbol{\Omega}) d\mathbf{M} - \mathbf{M} d\boldsymbol{\Omega}. \tag{17}$$

Here T is the absolute temperature, $s = S/\rho$ and w are the entropy and the heat function per unit mass, respectively. From the identity (17) and the definition of the heat function $w = (\tilde{E} + p)/\rho$, we get the following expression for the differential of the pressure p:

$$dp = -\rho T ds + \rho dw - (\alpha \mathbf{M} - \boldsymbol{\Omega}) d\mathbf{M} + \mathbf{M} d\boldsymbol{\Omega}, \tag{18}$$

which in a liquid entirely at rest ($\mathbf{M} = \boldsymbol{\Omega} = 0$) goes over into the ordinary thermodynamic relation $dw = T\,ds + dp/\rho$.

3. To determine the unknown terms in the hydrodynamic equations, we differentiate Eq. (11) with respect to time. By using (17) and substituting the time derivatives of ρ and $\mathbf{v}$ from Eqs. (1) and (7), we get

$$\frac{\partial E}{\partial t} = -\rho \mathbf{v}\nabla\frac{v^2}{2} - \mathbf{v}\nabla p + v_i \frac{\partial \sigma_{ik}'}{\partial x_k}$$
$$- \left(\frac{v^2}{2} + w\right)\operatorname{div}(\rho\mathbf{v}) + \rho T\frac{\partial s}{\partial t} + \alpha \mathbf{M}\frac{\partial \mathbf{M}}{\partial t}, \tag{19}$$

where $\sigma_{ik}' = \sigma_{ik} - p\delta_{ik}$. The second term in the right side of (19) can be rewritten with the help of the identity (18) in the form

$$-\mathbf{v}\nabla p = \rho T\mathbf{v}\nabla s - \rho\mathbf{v}\nabla w - \mathbf{v}\nabla(\mathbf{M}\boldsymbol{\Omega}) + \alpha\mathbf{M}(\mathbf{v}\nabla)\mathbf{M}$$
$$= \rho T\mathbf{v}\nabla s - \rho\mathbf{v}\nabla w - \operatorname{div}[\mathbf{v}(\mathbf{M}\boldsymbol{\Omega})]$$
$$+ \alpha M_i \frac{\partial}{\partial x_k}(v_k M_i) - (\alpha M^2 - \mathbf{M}\boldsymbol{\Omega})\delta_{ik}\frac{\partial v_i}{\partial x_k}.$$

Substituting this in Eq. (19) and separating terms

of the form div, we find

$$\frac{\partial E}{\partial t} = -\operatorname{div}\left[\rho\mathbf{v}\left(\frac{v^2}{2} + w\right) - (\mathbf{v}\sigma') + \mathbf{v}(\mathbf{M}\boldsymbol{\Omega})\right]$$
$$+ \rho T\left(\frac{\partial s}{\partial t} + \mathbf{v}\nabla s\right) + \alpha M_i\left[\frac{\partial M_i}{\partial t} + \frac{\partial}{\partial x_k}(v_k M_i)\right]$$
$$- [\sigma_{ik}' + \mathbf{M}(\alpha\mathbf{M} - \boldsymbol{\Omega})\delta_{ik}]\frac{\partial v_i}{\partial x_k}. \tag{20}$$

According to Eqs. (1) and (6), the second term on the right side is equal to R, and the third term can be transformed by means of Eq. (9):

$$\alpha M_i\left[\frac{\partial M_i}{\partial t} + \frac{\partial}{\partial x_k}(v_k M_i)\right] = \frac{\alpha}{2}M_{ik}\left[\frac{\partial M_{ik}}{\partial t} + \frac{\partial}{\partial x_l}(v_l M_{ik})\right]$$
$$= -\alpha M_{ik}\sigma_{ik}' - \frac{1}{2}M_{ik}\frac{\partial g_{ikl}}{\partial x_l}.$$

Then Eq. (20) takes the form

$$\frac{\partial E}{\partial t} + \operatorname{div}\left[\rho\mathbf{v}\left(\frac{v^2}{2} + w\right) + \alpha\mathbf{v}M^2 - (\mathbf{v}\lambda) - (\mathbf{M}g)\right]$$
$$= \lambda_{ik}\left(\frac{\partial v_i}{\partial x_k} + \alpha M_{ik}\right) - g_{ik}\frac{\partial M_i}{\partial x_k} - R, \tag{21}$$

where the new tensor

$$\lambda_{ik} = \sigma_{ik}' + \mathbf{M}(\alpha\mathbf{M} - \boldsymbol{\Omega})\delta_{ik} \tag{22}$$

is introduced in place of σ_{ik}' and the following notation is used:

$$(\mathbf{v}\lambda) = v_k\lambda_{ik}, \qquad (\mathbf{M}g) = M_k g_{ik}.$$

Equating (21) with the equation of energy conservation (2), we find

$$\mathbf{Q} = \rho\mathbf{v}(v^2/2 + w) + \alpha\mathbf{v}M^2 - (\mathbf{v}\lambda) - (\mathbf{M}g), \tag{23}$$

$$R = \lambda_{ik}\left(\frac{\partial v_i}{\partial x_k} + \alpha M_{ik}\right) - g_{ik}\frac{\partial M_i}{\partial x_k}$$
$$= \lambda_{ik}\left[{}^1\!/_2\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) + (\alpha M_{ik} - \Omega_{ik})\right]$$
$$- g_{ik}\frac{\partial M_i}{\partial x_k}, \tag{24}$$

where

$$\Omega_{ik} = {}^1\!/_2(\partial v_k/\partial x_i - \partial v_i/\partial x_k).$$

By virtue of the law for entropy growth, the dissipation function R must be positive. The most general form of the tensors λ_{ik} and g_{ik} follows from this requirement. Limiting ourselves to terms linear in the gradient, we obtain

$$\lambda_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)$$
$$+ \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l} + {}^1\!/_2\gamma(\alpha M_{ik} - \Omega_{ik}), \tag{25}$$

$$g_{ik} = -\mu\partial M_i/\partial x_k, \qquad (g_{ikl} = -\mu\partial M_{ik}/\partial x_l). \tag{26}$$

Thus there are altogether four independent kinetic coefficients: η—the coefficient of first viscosity, ζ—the second, γ—the third (rotational), and μ—the diffusion coefficient of the intrinsic angular momentum. In the stress tensor

$$\sigma_{ik} = [p + \mathbf{M}(\Omega - \alpha\mathbf{M})]\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_h} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)$$

$$+ \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l} + \frac{\gamma}{2}(\alpha M_{ik} - \Omega_{ik}) \qquad (27)$$

the contribution made by the intrinsic rotation consists of two parts: the renormalization of the pressure $\mathbf{M}\cdot(\Omega - \alpha\mathbf{M})$ and a dissipative term proportional to γ. The latter is an antisymmetric tensor, so that as a whole the stress tensor is seen to be asymmetric (see note [1]).

Finally, the hydrodynamic equations of a liquid with intrinsic rotation are written as follows:

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}\right] = -\nabla(p + \mathbf{M}\Omega - \alpha M^2) + \left(\eta + \frac{\gamma}{4}\right)\Delta\mathbf{v}$$

$$+ \left(\zeta + \frac{\eta}{3} - \frac{\gamma}{4}\right)\nabla\,\text{div}\,\mathbf{v} + \frac{\alpha\gamma}{2}\,\text{rot}\,\mathbf{M}, \qquad (28)*$$

$$\partial\mathbf{M}/\partial t + (\mathbf{v}\nabla)\mathbf{M} = \gamma(\Omega - \alpha\mathbf{M}) + \mu\Delta\mathbf{M} - \mathbf{M}\,\text{div}\,\mathbf{v}. \qquad (29)$$

To them one must add the equation of continuity (1) and the equation for entropy increase (6). In the latter, the dissipation function is equal to

$$R = \frac{\eta}{2}\left(\frac{\partial v_i}{\partial x_h} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)^2 + \frac{\gamma}{2}(\alpha M_{ik} - \Omega_{ik})^2$$

$$+ \zeta(\text{div}\,\mathbf{v})^2 + \mu\left(\frac{\partial\mathbf{M}}{\partial x_i}\right)^2. \qquad (30)$$

From the requirement that R be positive it follows that

$$\eta, \zeta, \gamma, \mu \geqslant 0. \qquad (31)$$

The equations are somewhat simplified if the liquid can be regarded as incompressible:

$$\text{div}\,\mathbf{v} = 0, \qquad (32)$$

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}\right] = -\nabla(p + \mathbf{M}\Omega - \alpha M^2)$$

$$+ \left(\eta + \frac{\gamma}{4}\right)\Delta\mathbf{v} + \frac{\alpha\gamma}{2}\,\text{rot}\,\mathbf{M}, \qquad (33)$$

$$\partial\mathbf{M}/\partial t + (\mathbf{v}\nabla)\mathbf{M} = \gamma(\Omega - \alpha\mathbf{M}) + \mu\Delta\mathbf{M}. \qquad (34)$$

As is seen from the latter equation, there are two reasons for the change in the intrinsic angular momentum: the absence of an equilibrium connection between $\mathbf{M}$ and Ω (relaxation) and the inhomogeneity of $\mathbf{M}$ (diffusion).

4. As an example, we consider the motion of an incompressible liquid, brought about by small har-

monic oscillations of a solid immersed in it. The set of equations (32)–(34) can be linearized in the given case:

$$\rho\dot{\mathbf{v}} = -\nabla p + \left(\eta + \frac{1}{4\alpha\tau}\right)\Delta\mathbf{v} + \frac{1}{2\tau}\,\text{rot}\,\mathbf{M}, \qquad (35)$$

$$\dot{\mathbf{M}} = -\frac{1}{\tau}\left(\mathbf{M} - \frac{\Omega}{\alpha}\right) + \mu\Delta\mathbf{M}. \qquad (36)$$

In place of γ we have introduced here the new constant $\tau = (\alpha\gamma)^{-1}$ which has the dimensions of time. Equations (35) and (36) achieve a simple interpretation if one can neglect the diffusion of the intrinsic angular momentum ($\mu = 0$). Then, setting all the quantities proportional to $\exp(-i\omega t)$, where ω is the frequency of oscillation of the solid, and eliminating $\mathbf{M}$ from the equations, we get

$$-i\omega\rho\mathbf{v} = -\nabla p + \left(\eta - \frac{i\omega\tau}{1 - i\omega\tau}\frac{1}{4\alpha\tau}\right)\Delta\mathbf{v}.$$

According to the ordinary scheme, the coefficient of $\Delta\mathbf{v}$ should be regarded as the first viscosity. Thus the presence of a latent rotation is equivalent in the given case to replacing the ordinary viscosity η by an effective viscosity

$$\tilde{\eta} = \eta - \frac{i\omega\tau}{1 - i\omega\tau}\frac{1}{4\alpha\tau}. \qquad (37)$$

In the limiting cases of low ($\omega\tau \ll 1$) and high ($\omega\tau \gg 1$) frequencies, this formula gives, respectively, $\tilde{\eta} = \eta$ and $\tilde{\eta} = \eta + (4\alpha\tau)^{-1}$. For the intermediate frequencies, $\tilde{\eta}$ exhibits dispersion. This circumstance can be used for the experimental determination of the rotational viscosity $\gamma = (\alpha\tau)^{-1}$.

5. In conclusion, we shall show how the coefficients α and τ can be found for a liquid in which the carriers of the intrinsic angular momentum are minute solid particles suspended in it. We shall carry out the calculations for the case of low concentrations of spherical particles.

As an auxiliary problem, we shall determine the friction torque acting on a sphere of radius a rotating uniformly with angular velocity ω in an unbounded liquid with a given vortex velocity Ω at infinity. For low Reynolds numbers, we get the following expression for the velocity of the liquid in spherical coordinates (ω and Ω are directed along the polar axis):

$$v_\varphi = v = r\sin\vartheta[\Omega - (\Omega - \omega)a^3/r^3].$$

The frictional force acting per unit area on the sphere is

$$\eta\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\Big|_{r=a} = 3\eta(\Omega - \omega)\sin\vartheta,$$

and the total torque of this force is

---

*rot $\equiv$ curl.

$$8\pi\eta a^3(\Omega - \omega). \tag{38}$$

We now proceed to the problem of the determination of the rotational viscosity of the suspension and consider the following nonstationary problem. The liquid rotates as a whole with constant angular velocity Ω. At a certain time, the sphere is placed at the origin of the coordinates. The friction torque exerted on it by the liquid leads to a change in the angular velocity ω of rotation of the sphere from its zero value at the initial instant to a maximum value Ω (total drag of the sphere by the liquid) after the lapse of a time of the order of τ. If

$$\tau \gg a^2/v \qquad (v = \eta/\rho), \tag{39}$$

then one can neglect the derivative $\partial v/\partial t \sim v/\tau$ in the equation of motion of the liquid in comparison with the viscous term $v\Delta v \sim (v/a^2)v$. We assume that the condition of quasistationarity (39) is satisfied. Then, for calculation of the time derivative of the rotational momentum $m = {}^8/_{15}\pi a^5\rho'\omega$ of the sphere (ρ' is its density), we can use Eq. (38):

$$\frac{dm}{dt} = 8\pi\eta a^3(\Omega - \omega) = -\frac{15v}{a^2}\frac{\rho}{\rho'}\left(m - \frac{8\pi}{15}a^5\rho'\Omega\right).$$

In view of the assumed low concentration n of the suspension, the intrinsic angular momentum M per unit volume of the liquid is equal to the product of the angular momentum of the individual sphere by n. Then

$$\frac{dM}{dt} = -\frac{15v}{a^2}\frac{\rho}{\rho'}\left(M - \frac{8\pi}{15}a^5\rho'n\Omega\right). \tag{40}$$

Comparing (40) and (36), we find the desired coefficients:

$$\alpha = \left(\frac{8\pi}{15}a^5\rho'n\right)^{-1}, \quad \tau = \frac{a^2}{15v}\frac{\rho'}{\rho}. \tag{41}$$

As was to have been expected, α^{-1} is identical with the moment of inertia of the spheres contained in unit volume. From the relations (39) and (41), it is seen that the condition of quasistationarity is satisfied only for heavy particles $(\rho' \gg \rho)$.

It should be noted that the applicability of Eqs. (41) is still limited by one condition. The fact is that we did not take the diffusion of the suspended particles into account in the calculations. This is legitimate only in the case in which the time $\tau_D \sim a^2/D$, during which the sphere is moved a distance a by Brownian motion, is much larger than τ. Thus τ is limited by (39) and above:

$$a^2/D \gg \tau \gg a^2/v. \tag{42}$$

The conditions of the compatability of the inequalities (42) follow from the Einstein relations for the diffusion coefficient $D \sim kT/\eta a$ and Eqs. (41);

$$\rho' \gg \rho, \qquad a \gg kT\rho'/\eta^2. \tag{43}$$

[1] V. S. Sorokin, JETP **13**, 306 (1943).

[2] I. I. Frenkel, Kinetic Theory of Liquids, Dover, 1949.

[3] I. M. Khalatnikov, Vvedenie v teoriyu sverkhtekuchesti (Introduction to the Theory of Superfluidity), Fizmatgiz, 1965.

[4] L. D. Landau and E. M. Lifshitz, Teoriya uprugosti (Theory of Elasticity), Fizmatgiz, 1965.

[5] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Fizmatgiz, 1964.