IONIZATION OF ATOMS IN AN ALTERNATING ELECTRIC FIELD: II

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The problem studied is that of the ionization of a system bound by short-range forces acted upon by an electromagnetic wave of arbitrary elliptical polarization. For the case of a weak field $F \ll F_0$, $\omega \ll \omega_0$ (F_0 is the intraatomic field and ω_0 is a characteristic atomic frequency), in which many-photon ionization occurs, we derive Eqs. (13) and (14), which give the probability of ionization in the form of a sum of probabilities of many-phonon processes. Equation (23) gives the momentum spectrum of the emerging electrons. We consider the transition to the adiabatic approximation in the case of low frequencies ($\gamma \ll 1$), and also derive asymptotic formulas (32) and (33) for the total probability of ionization in the "antiadiabatic" case ($\gamma \gg 1$). It is shown that with increase of the ellipticity ϵ of the incident light, when other conditions remain the same, there is a decrease of the probability of ionization. In the limiting cases $\epsilon = 0$ (linear polarization) and $\epsilon = \pm 1$ (circular polarization) the formulas go over into the corresponding formulas of [1]. In Sec. 3 we expound a simple quasiclassical method for deriving the main (exponential) factor in the formula for the probability of ionization. In the Appendix we consider some properties of the solutions of the Schrödinger equation for potentials with a Coulomb "tail."

1. INTRODUCTION

IN a previous paper by the writers[1] (hereafter cited as I) a method was proposed for calculating the probability of ionization of a bound system under the action of an alternating external field. In Sec. 2 of the present paper this method is used to study the ionization of a system bound by short-range forces and acted on by a wave with arbitrary elliptical polarization. The main result is contained in Eqs. (13), (14), and (23), which give the probability of ionization of an s level with binding energy $\omega_0 = \kappa^2/2$. For given values of the electric field strength F and the frequency ω the probability of ionization $w(F, \omega, \epsilon)$ decreases with increase of the ellipticity ϵ of the light. The nature of the dependence of the coefficient of the exponential function in $w(F, \omega, \epsilon)$ on ω changes decidedly with changes of ϵ and γ: in the region of Eq. (32) there are threshold oscillations characteristic of the case of linear polarization, whereas in the opposite case, Eq. (33), the coefficient of the exponential is a smooth function of the frequency, as is typical of circular polarization. We also note that for $\epsilon \neq 0$ the most probable momentum of the emitted electrons is different from zero [cf. Eq. (22)].

In Sec. 3 we expound a simple method for deriving the main (exponential) factor in the formula for $w(F, \omega, \epsilon)$; this is an extension of the usual quasiclassical method to the nonstationary case. It brings out some physical features of the penetration of a particle through a potential barrier which changes with time (in particular, we find the dependence of the effective width of the barrier and the time of passage on the frequency of the external field).

In the Appendix we discuss the asymptotic formulas for the wave functions $\psi(\mathbf{r})$ and $\varphi(\mathbf{p})$ in a potential with a Coulomb tail at infinity. In particular, we derive the formula (A.7) connecting the coefficient $C_{\kappa l}$ in the asymptotic form of $\psi(\mathbf{r})$ with the residue of the scattering matrix $S_l(k)$ at the pole $k = i\kappa$ corresponding to the bound state.

2. IONIZATION IN THE FIELD OF AN ELLIPTICALLY POLARIZED WAVE

When the conditions $F \ll F_0$, $\omega \ll \omega_0$ are satisfied, the mean time for ionization is much larger than atomic times,1) and the wave function of the electron is mainly (for $\kappa r \lesssim 1$) of the same form as in the free atom, being decidedly altered only for $\kappa r \sim (F_0/F)^{1/2} \gg 1$. Under the action of the alternating electric field $\mathbf{F}(t)$ the bound level is con-

1)In this paper we use the atomic system of units $e = \hbar = m = 1$; the other notations are the same as in I.

verted into a quasistationary state, whose wave function obeys the integral equation I, (37). For potentials V(r) without a Coulomb tail we can make the replacement I, (40), after which a calculation of the current $\mathbf{j}(\mathbf{r}, t)$ gives

$$\mathbf{j}(\mathbf{r}, t) = \frac{i}{2}(\psi\nabla\psi^* - \psi^*\nabla\psi)$$

$$= \frac{1}{2(2\pi)^3}\int d\mathbf{p}_1 d\mathbf{p}_2 [\boldsymbol{\pi}_1(t) + \boldsymbol{\pi}_2(t)] \cdot I_{lm}^*(\mathbf{p}_1, t) I_{lm}(\mathbf{p}_2, t)$$

$$\times \exp\{i(\mathbf{p}_2 - \mathbf{p}_1)[\mathbf{r} - \boldsymbol{\xi}(t)] - {}^1/_2\, i(p_2^2 - p_1^2)t\}. \qquad (1)$$

Here we have introduced the notations

$$I_{lm}(\mathbf{p}, t) = \int_{-\infty}^{t} dt' \chi_{lm}(\boldsymbol{\pi}(t')) \exp\left\{\frac{i}{2}\left[(\mathbf{p}^2 + \varkappa^2)t' + 2\mathbf{p}\boldsymbol{\xi}(t') + \int_0^{t'} A^2(\tau)\,d\tau\right]\right\}, \qquad (2)$$

$$\chi_{lm}(\mathbf{p}) = {}^1/_2(p^2 + \varkappa^2)\varphi_{lm}(\mathbf{p}), \qquad (3)$$

where $\varphi_{lm}(\mathbf{p})$ is the normalized wave function of the bound state in the $\mathbf{p}$ representation (binding energy $\omega_0 = \kappa^2/2$; l is the orbital angular momentum). The quantities $\boldsymbol{\pi}(t)$, $\boldsymbol{\xi}(t)$, and $\mathbf{A}(t)$ are defined as in I, (39).

Let us consider the most general case of a monochromatic light wave—a wave with elliptical polarization:

$$\mathbf{F}(t) = (F\cos\omega t,\ \varepsilon F \sin\omega t,\ 0) \qquad (4)$$

(the direction of propagation of the wave is along the z axis). The parameter ϵ $(-1 \le \epsilon \le 1)$ is called the ellipticity (see [2]); $\epsilon = 0$ corresponds to linear polarization, and $\epsilon = \pm 1$ to circular polarization. For $\epsilon > 0$ the polarization is left-handed, and for $\epsilon < 0$, right-handed. Using the condition that the field is turned on adiabatically at $-\infty$, we find

$$\mathbf{A}(t) = -\int_{-\infty}^{t} \mathbf{F}(t')\,dt' = \left(-\frac{F}{\omega}\sin\omega t, \frac{\varepsilon F}{\omega}\cos\omega t, 0\right), \qquad (5a)$$

$$\boldsymbol{\xi}(t) = -\int_{-\infty}^{t} \mathbf{A}(t')\,dt' = -\frac{1}{\omega^2}\mathbf{F}(t); \qquad (5b)$$

$\boldsymbol{\xi}(t)$ is the trajectory of the classical particle moving in the uniform field $\mathbf{F}(t)$ with the null initial conditions $\boldsymbol{\xi}(-\infty) = \dot{\boldsymbol{\xi}}(-\infty) = 0$.

To find the probability of ionization $w(F, \omega, \epsilon)$ it is necessary to integrate the radial component j_r of the current over a cylinder of radius R $(R \to \infty)$ with its axis along the z axis (the atom undergoing ionization is at the origin). We denote by $\mathcal{J}(R, t)$ the integral of j_r over the cylinder, by p_z the component of the momentum $\mathbf{p}$ along the z axis, and by $\mathbf{k}$ the component of the momentum $\mathbf{p}$ that lies in the plane of x and y. Using the equations

$$\int_{-\infty}^{\infty} dz \int_0^{2\pi} R\,d\varphi \exp\{i(\mathbf{p}_1 - \mathbf{p}_2)\mathbf{r}\}(\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2)\mathbf{e}_r$$

$$= 4\pi^2 i\delta(p_{1z} - p_{2z})[\pi_1^2(t) - \pi_2^2(t)]\frac{RJ_1(R|\mathbf{k}_1 - \mathbf{k}_2|)}{|\mathbf{k}_1 - \mathbf{k}_2|}, \qquad (6)$$

$$\lim \frac{RJ_1(R|\mathbf{k}_1 - \mathbf{k}_2|)}{|\mathbf{k}_1 - \mathbf{k}_2|} = 2\pi\delta(\mathbf{k}_1 - \mathbf{k}_2) \quad \text{for } R \to \infty \qquad (7)$$

(J_1 is the Bessel function), we get the following expression for $\mathcal{J}(R, t)$:

$$\lim_{R\to\infty} \mathcal{J}(R, t) = \frac{i}{2}\int d\mathbf{p}_1 d\mathbf{p}_2\, \delta(\mathbf{p}_1 - \mathbf{p}_2)[\pi_1^2(t) - \pi_2^2(t)] I_{lm}^*(\mathbf{p}_1, t) I_{lm}(\mathbf{p}_2, t). \qquad (8)$$

because of the presence of the δ function the difference $\pi_1^2(t) - \pi_2^2(t)$ becomes zero, and only singular terms in $I_{lm}(\mathbf{p}, t)$ make finite contributions to the integral.

For the field of the wave (4) the integrand in (2) is periodic and can be expanded in a Fourier series, after which the integration over t' gives

$$I_{lm}(\mathbf{p}, t) = -i\sum_{n=-\infty}^{\infty} F_n(\mathbf{p})\frac{\exp(i\Omega_n t)}{\Omega_n - i\delta},$$

$$\Omega_n = \frac{p^2}{2} + \frac{\varkappa^2}{2}\left(1 + \frac{1 + \varepsilon^2}{2\gamma^2}\right) - n\omega,$$

$$\gamma = \frac{\omega}{\omega_t} = \frac{\varkappa\omega}{F}, \quad \delta \to +0. \qquad (9)$$

The coefficients $F_n(\mathbf{p})$ are very important for what follows [see (14)], and are found from the expansion

$$\sum_{n=-\infty}^{\infty} F_n(\mathbf{p})\exp(-in\omega t)$$

$$= \chi_{lm}(\boldsymbol{\pi}(t))\exp\left\{-i\frac{\omega_0}{\omega}\left[\frac{2k_x}{\varkappa\gamma}\cos\omega t + \frac{2\varepsilon k_y}{\varkappa\gamma}\sin\omega t + \frac{1 - \varepsilon^2}{4\gamma^2}\sin 2\omega t\right]\right\}. \qquad (10)$$

From this we have

$$F_n(\mathbf{p})\,|_{p=p_n} = \frac{\exp\{i\mathbf{p}\boldsymbol{\xi}(0)\}}{2\pi}\int_{-\pi}^{\pi} d\alpha\, \chi_{lm}(\boldsymbol{\pi}(\alpha)) \times \exp\left\{\frac{i}{2\omega}\int_0^{\alpha}[\pi^2(y) + \varkappa^2]\,dy\right\},$$

$$\boldsymbol{\pi}(\alpha) = \left(k_x + \frac{F}{\omega}\sin\alpha,\ k_y - \frac{\varepsilon F}{\omega}\cos\alpha,\ p_z\right). \qquad (11)$$

As we shall proceed to show, for $\omega \ll \omega_0$ it is sufficient for the calculation of $F_n(\mathbf{p})$ to know only the behavior of the wave function $\varphi_{lm}(\mathbf{p})$ near the pole $p^2 = -\kappa^2$.

Using the identity

$$\lim_{p_1\to p_2}\frac{p_1^2-p_2^2}{[(p_1^2+\varkappa^2)/2-n_1\omega-i\delta][(p_2^2+\varkappa^2)/2-n_2\omega+i\delta]} = -2\pi i\delta_{n_1n_2}\delta(p_1^2+\varkappa^2-2\omega n_1), \quad (12)$$

we can transform (8) into a sum of probabilities of many-photon processes:

$$w(F,\omega,\varepsilon)=\sum_{n\geqslant\nu}^{\infty} w(F,\omega,\varepsilon), \quad \nu=\frac{\omega_0}{\omega}\left(1+\frac{1+\varepsilon^2}{2\gamma^2}\right), \quad (13)$$

$$w_n(F,\omega,\varepsilon)=2\pi\int d\mathbf{p}\,\delta\left(\frac{1}{2}p^2-\frac{1}{2}p_n^2\right)|F_n(\mathbf{p})|^2. \quad (14)$$

These formulas are a generalization of I, (43) to the case of elliptical polarization. The quantity ν has the meaning of a threshold for ionization (the minimum number of quanta whose absorption is necessary for ionization is $[\nu]+1$; here $[\nu]$ is the integer part of the number ν). The momentum p_n (the average momentum of the electron at infinity) is found from the law of conservation of energy:

$$\frac{1}{2}p_n^2=n\omega-\frac{1}{2}\varkappa^2\left(1+\frac{1+\varepsilon^2}{2\gamma^2}\right), \quad p_n=[2\omega(n-\nu)]^{1/2} \quad (15)$$

[the term $\kappa^2(1+\epsilon^2)/4\gamma^2 = \frac{1}{2}\overline{A^2(t)}$ is the mean kinetic energy of the vibrational motion of the electron in the field of the light wave (4)].

To get concrete formulas we still have to find $F_n(\mathbf{p})$. For $\omega \ll \omega_0$ the integral (11) can be calculated by the method of steepest descent. The position of the saddle point $\alpha_0 = \alpha_0(\mathbf{p})$ depends on $\mathbf{p}$ and is found from the equation

$$\pi^2(\alpha)\equiv\left(k_x+\frac{\varkappa}{\gamma}\sin\alpha\right)^2+\left(k_y-\varepsilon\frac{\varkappa}{\gamma}\cos\alpha\right)^2+p_z^2=-\varkappa^2. \quad (16)$$

Let us determine the value of $\mathbf{p}$ for which $|F_n(\mathbf{p})|$ is a maximum. To do so we need to find the maximum of the function

$$\exp\left\{\frac{i}{2\omega}\int_0^{\alpha_0(\mathbf{p})}(\pi^2(y)+\varkappa^2)\,dy\right\}\equiv\exp\left\{\frac{\omega_0}{\omega}G(\mathbf{p})\right\},$$

where

$$\operatorname{Re}G(\mathbf{p})=-\left(\frac{p^2}{\varkappa^2}+1+\frac{1+\varepsilon^2}{2\gamma^2}\right)v - \frac{2}{\varkappa\gamma}(k_x\cos u-\varepsilon k_y\sin u)\operatorname{sh}v-\frac{1-\varepsilon^2}{4\gamma^2}\cos 2u\operatorname{sh}2v,$$

$$\alpha_0=\pi/2-u+iv. \quad (17)^*$$

From the conditions

$$\frac{\partial}{\partial p_z}\operatorname{Re}G=\frac{\partial}{\partial k_x}\operatorname{Re}G=\frac{\partial}{\partial k_y}\operatorname{Re}G=0$$

together with (16) we find that $k_x = p_z = 0$, $u = \pm\pi/2$, and the values of k_y and v are found from the equations

$$\pm\varkappa\frac{\operatorname{sh}v}{v}=\frac{\gamma}{\varepsilon}k_y,$$

$$\frac{k_y^2}{\varkappa^2}+\frac{1+\gamma^2}{\gamma^2}\mp\frac{2\varepsilon k_y}{\varkappa\gamma}\operatorname{ch}v-\frac{1-\varepsilon^2}{\gamma^2}\operatorname{ch}^2v=0. \quad (18)^*$$

It is convenient to make the substitution

$$\operatorname{sh}v=[(s^2+\gamma^2)/(1-s^2)]^{1/2}, \quad (19)$$

after which we get from (18) the following equation for the determination of s:

$$\operatorname{Ar\,th}\left(\frac{s^2+\gamma^2}{1+\gamma^2}\right)^{1/2}=\frac{\varepsilon}{\varepsilon-s}\left(\frac{s^2+\gamma^2}{1+\gamma^2}\right)^{1/2} \quad (20)^\dagger$$

[cf. I, (70) in the case of circular polarization]. This transcendental equation has a unique root $s = s_0(\gamma,\epsilon)$, which always lies in the range $(0,\epsilon)$, and for which $s_0(\gamma,\epsilon) = -s_0(\gamma,-\epsilon)$. For $\gamma \ll 1$ and for $\gamma \gg 1$ Eq. (20) can be solved approximately:

$$s_0(\gamma,\varepsilon)=\begin{cases}\dfrac{\varepsilon}{3}\gamma^2\left[1-\dfrac{1}{3}\left(\dfrac{11}{5}-\dfrac{\varepsilon^2}{3}\right)\gamma^2+\dots\right], & \gamma\ll 1,\\ \varepsilon\left[1-\left(\ln\dfrac{2\gamma}{\sqrt{1-\varepsilon^2}}\right)^{-1}+\dots\right], & \gamma\gg 1,\quad |\varepsilon|\neq 1,\\ \varepsilon[1-(\ln\gamma\sqrt{2\ln\gamma})^{-1}+\dots], & \gamma\gg 1,\quad |\varepsilon|\to 1.\end{cases} \quad (21)$$

The dependence of $s_0(\gamma,\epsilon)$ on γ and ϵ is shown in Fig. 1.

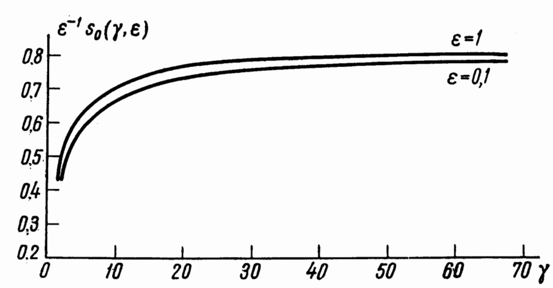

FIG. 1. The root $s_0(\gamma,\epsilon)$ of the transcendental equation (20). The ordinates are values of the quantity $\epsilon^{-1}s_0(\gamma,\epsilon)$.

Knowing $s_0(\gamma,\epsilon)$, we find from (18) the value $\mathbf{p}_0$ of the momentum which corresponds to the maximum $|F_n(\mathbf{p})|$: $\mathbf{p}_0 = (0, \pm k_0, 0)$, where

*sh ≡ sinh.

*ch ≡ cosh.

†Arth ≡ $\tanh^{-1}$.

$$k_0=\frac{\varkappa}{\gamma}(\varepsilon-s_0)\left(\frac{1+\gamma^2}{1-s_0^2}\right)^{1/2}$$

$$=\begin{cases}\varkappa\varepsilon/\gamma, & \gamma\ll 1,\\ \varkappa\dfrac{\varepsilon}{\sqrt{1-\varepsilon^2}}\left(\ln\dfrac{2\gamma}{\sqrt{1-\varepsilon^2}}\right)^{-1}, & \gamma\gg 1, \quad |\varepsilon|\neq 1\end{cases} \quad (22)$$

(see also Fig. 2). For $0<|\epsilon|<1$ the most probable momentum $\mathbf{p}_0$ of the ejected electrons is perpendicular to the maximum electric field strength; for $|\epsilon|=1$ the electrons come out isotropically in the (x, y) plane.

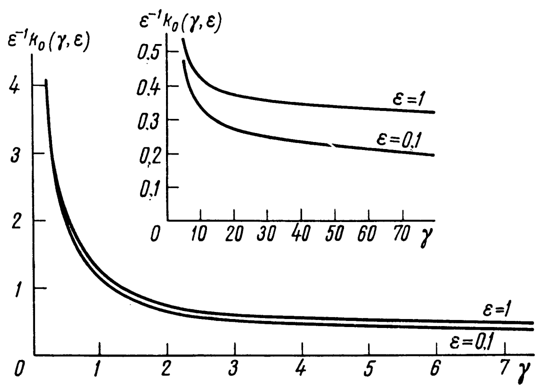

FIG. 2. Dependence of the most probable momentum $k_0(\gamma,\epsilon)$ of the ejected electrons on the parameters γ and ϵ (the unit for measurement of k_0 is the intraatomic momentum κ).

To find the total probability of ionization we must calculate the $|F_n(\mathbf{p})|$ from (11) for values of $\mathbf{p}$ close to $\mathbf{p}_0$. We confine ourselves to the simplest case $l=0$ (ionization of an s level), for which

$$\chi_{lm}(p\to i\varkappa)={}^1/_2(\varkappa/2\pi^2)^{1/2}C_{\varkappa 0}.$$

The extension to arbitrary l is no problem (see I). In the case $l=0$ we get

$$|F_n(\mathbf{p})|^2_{p=p_n}=D(\gamma,\varepsilon)\exp\left\{-\frac{2\omega_0}{\omega}f(\gamma,\varepsilon)\right\}\cdot$$
$$\cdot\exp\left\{-\frac{1}{\omega}[c_1k_x^2+c_2(k_y-k_0)^2+c_3p_z^2]\right\}, \quad (23)$$

where

$$f(\gamma,\varepsilon)=\frac{2}{3}\gamma g(\gamma,\varepsilon)=\left(1+\frac{1+\varepsilon^2}{2\gamma^2}\right)\mathrm{Arsh}\left(\frac{\gamma^2+s_0^2}{1-s_0^2}\right)^{1/2}$$
$$-\frac{1+\varepsilon^2-2\varepsilon s_0}{2\gamma^2(1-s_0^2)}[(1+\gamma^2)(s_0^2+\gamma^2)]^{1/2}, \quad (24)$$

$$f(\gamma,\varepsilon)$$
$$\approx\begin{cases}{}^2/_3\gamma[1-{}^1/_{10}(1-{}^1/_3\varepsilon^2)\gamma^2+\ldots], & \gamma\ll 1\\ \ln(2\gamma/\sqrt{1-\varepsilon^2})-{}^1/_2, & \gamma\gg 1, \quad |\varepsilon|\neq 1.\\ \ln(\gamma\sqrt{2\ln\gamma})-{}^1/_2, & \gamma\gg 1, \quad |\varepsilon|\to 1\end{cases}$$
$$(24')$$

We note that $f(\gamma,\epsilon)$ depends only on $|\epsilon|$. For $\epsilon=0$*

$$f(\gamma,0)=\left(1+\frac{1}{2\gamma^2}\right)\mathrm{Arsh}\,\gamma-\frac{\sqrt{1+\gamma^2}}{2\gamma},$$

which is the same as the result of Keldysh.[3] The other quantities in (23) are

$$c_1=\frac{s_0(1-\varepsilon^2)}{(\varepsilon-s_0)(1-\varepsilon s_0)}\left(\frac{s_0^2+\gamma^2}{1+\gamma^2}\right)^{1/2}$$

$$=\begin{cases}{}^1/_3\gamma^3(1-\varepsilon^2), & \gamma\ll 1\\ \ln\dfrac{2\gamma}{\sqrt{1-\varepsilon^2}}-\dfrac{1}{1-\varepsilon^2}, & \gamma\gg 1, \quad |\varepsilon|\neq 1\end{cases}; \quad (25)$$

$$c_2=\frac{\varepsilon}{\varepsilon-s_0}\left(\frac{s_0^2+\gamma^2}{1+\gamma^2}\right)^{1/2}+\frac{s_0^2}{1-\varepsilon s_0}\left(\frac{s_0^2+\gamma^2}{1+\gamma^2}\right)^{-1/2}$$

$$=\begin{cases}\gamma, & \gamma\ll 1\\ \ln\dfrac{2\gamma}{\sqrt{1-\varepsilon^2}}+\dfrac{\varepsilon^2}{1-\varepsilon^2}, & \gamma\gg 1, \quad |\varepsilon|\neq 1\end{cases}; \quad (26)$$

$$c_3=\frac{\varepsilon}{\varepsilon-s_0}\left(\frac{s_0^2+\gamma^2}{1+\gamma^2}\right)^{1/2}$$

$$=\begin{cases}\gamma, & \gamma\ll 1\\ \ln\dfrac{2\gamma}{\sqrt{1-\varepsilon^2}}, & \gamma\gg 1, \quad |\varepsilon|\neq 1\end{cases}; \quad (27)$$

$$D=\frac{|C_{\varkappa 0}|^2}{8\pi^3}\frac{\omega\gamma^2}{\varkappa}\frac{1-s_0^2}{1-\varepsilon s_0}[(s_0^2+\gamma^2)(1+\gamma^2)]^{-1/2}$$

$$=\begin{cases}\dfrac{C_{\varkappa 0}^2}{8\pi^3}\dfrac{\omega\gamma}{\varkappa}, & \gamma\ll 1\\ \dfrac{C_{\varkappa 0}^2}{8\pi^3}\dfrac{\omega}{\varkappa}, & \gamma\gg 1\end{cases}. \quad (28)$$

In the exponent in (23) we have dropped terms of higher order than $\kappa^{-2}(p^2-p_0^2)$; owing to the factor $\omega_0/\omega\gg 1$ this expansion is legitimate. Equation (23) gives the momentum spectrum of the emerging electrons.

In the adiabatic case ($\gamma\ll 1$) the coefficients c_i go to zero and a huge number n of photons is effectively involved in the ionization. When we replace the summation in (13) with an integration, we find

$$w(F,\omega,\varepsilon)=\frac{|C_{\varkappa 0}|^2}{4\sqrt{\pi}\varkappa}\frac{\omega^{3/2}\gamma}{\sqrt{c_1c_2c_3}}\exp\left\{-\frac{2}{3}\frac{F_0}{F}g(\gamma,\varepsilon)\right\}$$
$$=|C_{\varkappa 0}|^2\frac{\omega_0}{2}\left[\frac{3}{\pi(1-\varepsilon^2)}\left(\frac{F}{F_0}\right)^{3/2}\right]^{1/2}$$
$$\times\exp\left\{-\frac{2}{3}\frac{F_0}{F}\left[1-\frac{1}{10}\left(1-\frac{\varepsilon^2}{3}\right)\gamma^2+\ldots\right]\right\}, \quad (29)$$

where the function $g(\gamma,\epsilon)$ is connected with $f(\gamma,\epsilon)$ by Eq. (24) [cf. I, (12) and I, (4) for $l=\lambda=0$]. This formula is good for all ϵ except in narrow ranges near $\epsilon=\pm 1$. For $\epsilon^2\to 1$ the coefficient c_1 goes

*Arsh ≡ $\sinh^{-1}$.

to zero, and higher-order terms are important in the expansion (23). It can be verified (see a paper by the writers[4]) that inclusion of these terms leads to replacement of the factor $[3F/\pi(1-\epsilon^2) \times F_0]^{1/2}$ in (29) by a function $A(F, \omega)$, which was derived in I by an independent treatment of the adiabatic approximation. When in (29) we neglect the correction terms $\sim\gamma^2$ in the exponent we get complete agreement with the formulas of the adiabatic approximation.

It can be seen already from (29) that with increase of $|\epsilon|$ the function $f(\gamma, \epsilon)$ increases monotonically, and the probability of ionization decreases. This conclusion is valid for all values of γ. In Fig. 3 we show curves of the function $f(\gamma, \epsilon)$ for several values of ϵ, as obtained by numerical computation from (20) and (24). In the region $\gamma \sim 30-50$, which is of practical importance for experiments, the difference between $\epsilon = 0$ and $\epsilon = 1$ is extremely important. The threshold for ionization with illumination with linearly polarized light is lower than that for the case of circularly polarized light. The physical reason for this is that in the field of an elliptically polarized wave the trajectory of the electron is "twisted up" and its emergence through the barrier is made more difficult (for details on this see Sec. 3, point C).

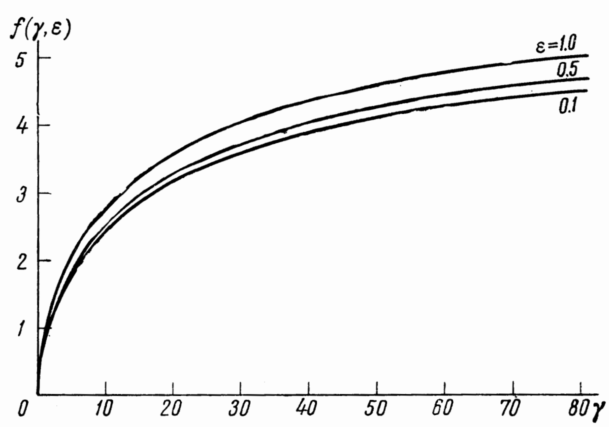

FIG. 3. The function $f(\gamma,\epsilon)$, the argument of the exponential in the formula (29) for the probability of ionization. The numbers on the curves indicate values of ϵ.

Let us now proceed to the region of high frequencies, $\gamma \gg 1$. The most probable number of quanta absorbed is found from the condition

$$n_0 - \nu = \frac{1}{2\omega} k_0^2 \sim \frac{\omega_0}{\omega} \frac{\varepsilon^2}{1-\varepsilon^2}\left(\ln\frac{2\gamma}{\sqrt{1-\varepsilon^2}}\right)^{-2}. \quad (30)$$

The effective width of the distribution in n is

$$\Delta n \sim \left[\frac{\varepsilon^2}{1-\varepsilon^2} \frac{\omega_0}{\omega(\ln\gamma)^3}\right]^{1/2}. \quad (31)$$

We consider two limiting cases, $\Delta n \ll 1$ and $\Delta n \gg 1$.

1. In the first case

$$\Delta n \ll 1, \quad \ln\gamma \gg \left(\frac{\varepsilon^2}{1-\varepsilon^2} \frac{\omega_0}{\omega}\right)^{1/3}.$$

The main contribution to the probability of ionization is made by one value of n ($n = n_0$). If the somewhat stronger condition

$$\ln\gamma \gg \left(\frac{\varepsilon^2}{1-\varepsilon^2} \frac{\omega_0}{\omega}\right)^{1/2},$$

is satisfied, then $n_0 - \nu \ll 1$ and the sum over n reduces to the first term of the series. In this case the total probability of ionization can be put in the form[2)]

$$w(F, \omega, \varepsilon) = \frac{2\omega_0}{\pi}|C_{\varkappa 0}|^2\left(\frac{F}{F_0}\gamma\right)^{3/2}\sqrt{1-\varepsilon^2}$$
$$\times \exp\left\{-2\delta\left(\ln\frac{2\gamma}{\sqrt{1-\varepsilon^2}} - \frac{1}{1-\varepsilon^2}\right)\right\} w\left(\left(\frac{2\delta}{1-\varepsilon^2}\right)^{1/2}\right)$$
$$\times \exp\left\{-\frac{2\omega_0}{\omega} f(\gamma, \varepsilon)\right\}. \quad (32a)$$

Here $\delta = [\nu] + 1 - \nu$. For $\epsilon \to 0$ Eq. (32a) goes over into the formula for linear polarization with $l = 0$ and $\gamma \gg 1$, Eq. I, (54). The behavior of the factor before the exponential in (32a) shows threshold singularities at the frequencies ω_n that satisfy the condition $\nu = n$, an integer, and is in general similar to that in the case of linear polarization (see Fig. 3 in I).

The function $w(x)$ which appears in the right member of (32a) is defined in I, (56) (for $m = 0$). Using for $f(\gamma, \epsilon)$ the asymptotic form (24′) for $\gamma \gg 1$, we can transform (32a) to a more convenient form,

$$w(F, \omega, \varepsilon) \approx A(\delta, \varepsilon)\,\omega\left(\frac{\omega}{\omega_0}\right)^{1/2}\left(\frac{F}{F_1}\right)^{2[\nu+1]}, \quad (32b)$$

where

$$A(\delta, \varepsilon) = \frac{1}{\pi}|C_{\varkappa 0}|^2\left(\frac{1-\varepsilon^2}{2}\right)^{1/2} \times \exp\left(\delta\frac{1+\varepsilon^2}{1-\varepsilon^2}\right) w\left(\left(\frac{2\delta}{1-\varepsilon^2}\right)^{1/2}\right), \quad (32c)$$

Because of the factor ω/ω_0 the field F_1 is about an order of magnitude smaller than the intraatomic field F_0. With increase of the ellipticity ϵ the quantity F_1 increases, and this leads to a sharp decrease of the probability of ionization. The factor $A(\delta, \epsilon)$ is of the order of magnitude of unity

2)For the derivation of this formula see[4].

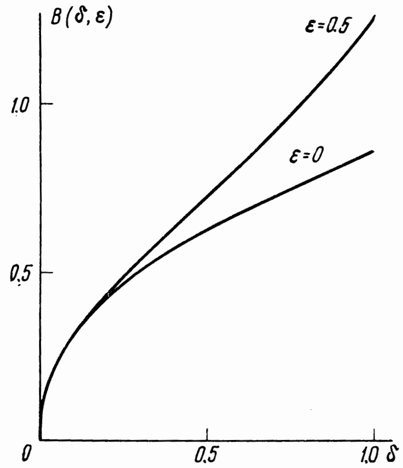

FIG. 4. Curves of the function $B(\delta,\epsilon)$; the function $A(\delta,\epsilon)$ in (32b) differs from $B(\delta,\epsilon)$ only by a constant factor of the order of unity: $A(\delta,\epsilon) = \pi^{-1}2^{-1/2}|C_{\kappa 0}|^2 B(\delta,\epsilon)$.

(see Fig. 4), except in a narrow range of frequencies close to the threshold frequency, $\omega \to \omega_n$. In this region $\delta \to 0$, and the second term $(n = \nu + 1)$ of the series in (13) must be included. The result is that for $\delta \to 0$ the quantity $A(\delta, \epsilon)$ takes a small value $\sim (F/F_1)^2$ [but not zero, so that Eq. (32b) does not hold near the threshold]. This frequency range is very small, however: $\Delta\omega/\omega_n \sim n^{-1}(F/F_1)^4 \ll 1$. If ω is not very close to ω_n and ϵ is not close to unity, then $A(\delta, \epsilon) \sim 1$; in this case it follows from (32b) that $w(F) \sim F^{2[\nu+1]}$ (for fixed frequency of the laser).

2. In the other limiting case

$$\Delta n \gg 1, \quad \ln\gamma \ll \left(\frac{\varepsilon^2}{1-\varepsilon^2}\,\frac{\omega_0}{\omega}\right)^{1/3}, \quad \gamma \gg 1.$$

Replacing the sum over n by an integral, we have

$$w(F, \omega, \varepsilon) = \frac{\omega_0}{2\sqrt{\pi}}|C_{\varkappa 0}|^2\left(\frac{F}{F_0}\right)^{3/2}\left[\frac{\gamma}{\ln(2\gamma/\sqrt{1-\varepsilon^2})}\right]^{3/2} \times \exp\left\{-\frac{2\omega_0}{\omega}f(\gamma, \varepsilon)\right\}. \quad (33a)$$

The factor before the exponential is a smooth function of the frequency and has no threshold singularities. We can simplify this expression in the same way as before:

$$w(F, \omega, \varepsilon) \approx \frac{|C_{\varkappa 0}|^2}{4\sqrt{2\pi}}\,\omega\left(\frac{\omega}{\omega_0}\right)^{1/2}\left[\ln\left(\frac{F_1}{F}\right)\right]^{-3/2}\left(\frac{F}{F_1}\right)^{2\nu}. \quad (33b)$$

For ionization of gases by ruby-laser light $\omega_0/\omega \sim 10-15$, and $\gamma \sim 30-50$; under these conditions, if ϵ is not very close to unity we have the first case. The second case always occurs, however, when we go over to circular polarization $(|\epsilon| \to 1)$.

3. THE QUASICLASSICAL APPROXIMATION IN THE PROBLEM OF IONIZATION

The formulas derived above are rather complicated [even if we confine ourselves to the exponential factor in $w(F, \omega, \epsilon)$]. We shall now give a simple method for deriving the exponential in the formula for $w(F, \omega, \epsilon)$; it reveals the physical content of the exact method we have used to calculate $w(F, \omega, \epsilon)$. We get an elucidation of a number of important features of the passage of particles through a potential barrier varying with the time.

Under the conditions $F \ll F_0$, $\omega \ll \omega_0$ the wave function satisfies the equation [cf. I, (37) and I, (40)]:

$$\psi(\mathbf{r}, t) = -i\int_{-\infty}^{t} dt' \int d\mathbf{r}'\, G(\mathbf{r}t; \mathbf{r}'t')\, V(\mathbf{r}')\varphi_0(\mathbf{r}')\exp(-iE_0t'). \quad (34)$$

In the quasiclassical approximation

$$G(\mathbf{r}, t; \mathbf{r}', t') \sim \exp\{iS(\mathbf{r}, t; \mathbf{r}', t')\}, \quad (35)$$

where $S(\mathbf{r}, t; \mathbf{r}', t')$ is the classical action[3)] (cf., e.g.,[5]). From this we have

$$\psi(\mathbf{r}, t) \sim e^{-iE_0t}\int_{-\infty}^{t} dt' \int d\mathbf{r}' \exp\{i\tilde{S}(\mathbf{r}t, \mathbf{r}'t')\}\, V(\mathbf{r}')\varphi_0(\mathbf{r}'), \quad (36)$$

where $\tilde{S}$ is the so-called contracted action [$L(\tau)$ is the Lagrangian]

$$\tilde{S}(\mathbf{r}t; \mathbf{r}'t') = S(\mathbf{r}t; \mathbf{r}'t') + E_0(t - t') = \int_{t'}^{t} [L(\tau) + E_0]\, d\tau. \quad (37)$$

For $\omega \ll \omega_0$ the exponential in (36) is a rapidly oscillating function, and the main contribution to the integral over t' is determined solely by the saddle point t_0; moreover, the quantity $V(\mathbf{r}') \times \varphi_0(\mathbf{r}')$ decreases exponentially for $\kappa r' \gg 1$, i.e., it is small values of $\mathbf{r}'$ that are important in the integral over $\mathbf{r}'$. We then have

$$\psi(\mathbf{r}, t) \sim \exp\{i[\tilde{S}(\mathbf{r}, t; 0, t_0) - E_0t]\}. \quad (38)$$

The saddle point t_0 is found from the condition

$$\left.\frac{\partial\tilde{S}(\mathbf{r}, t; 0, t')}{\partial t'}\right|_{t'=t_0} = 0 \quad \text{or} \quad H(0, t_0) = E_0 = -\frac{\varkappa^2}{2} \quad (39)$$

[here $H(\mathbf{r}, t)$ is the Hamiltonian]. For motions that include passage through a potential barrier

3)The Green's function for motion in a uniform electric field is of the form

$$G(\mathbf{r}\, t; \mathbf{r}'\, t') = \frac{\theta(t-t')}{[2\pi i(t-t')]^{3/2}}\exp\{iS(\mathbf{r}\, t; \mathbf{r}'\, t')\}.$$

This is one of the few cases in which the exact quantum-mechanical Green's function is the same as the quasiclassical approximation (see[6]).

(impossible in classical mechanics) the point t_0 moves off into the complex plane. Equation (38) determines the probability of ionization (apart from the factor in front of the exponential):[4)]

$$w \sim \lim_{\substack{r\to\infty \\ t\to\infty}} |\psi(\mathbf{r}, t)|^2 \sim \exp\{-\lim_{r\to\infty} 2\,\mathrm{Im}\,\tilde{S}(\mathbf{r}t;\, 0t_0)\}. \quad (40)$$

We now show that (40) gives the correct exponential in the formula for $w(F, \omega, \epsilon)$. Here we can assume that the motion through the barrier occurs under the action of the external field $F(t)$ alone (since we confine ourselves to potentials without a Coulomb "tail").

A. Ionization by a Constant Field F

Let us first consider the one-dimensional problem. The classical "trajectory" is determined by Newton's equation $\ddot{x} = F$ and the initial conditions $x(t_0) = 0$, $\dot{x}(t_0) = i\kappa$. Choosing the origin for time at the instant when $\dot{x} = 0$ (the time of emergence of the particle from the barrier), we have

$$\dot{x} = Ft, \quad x = \frac{F}{2}(t^2 - t_0^2), \quad t_0 = \frac{i\kappa}{F} = \frac{i}{\omega_t}. \quad (41)$$

In the process of the particle's motion the "time" t varies along the curve shown in Fig. 5, a; for the motion through the barrier the "time" is imaginary. The variations of x and $\dot{x}$ are shown in Fig. 5, b and c. At the time $t = 0$ ($\dot{x} = 0$, $x = x_0 = \kappa^2/2F$) the particle emerges beyond the barrier, and the further part of the trajectory has meaning also in classical mechanics.

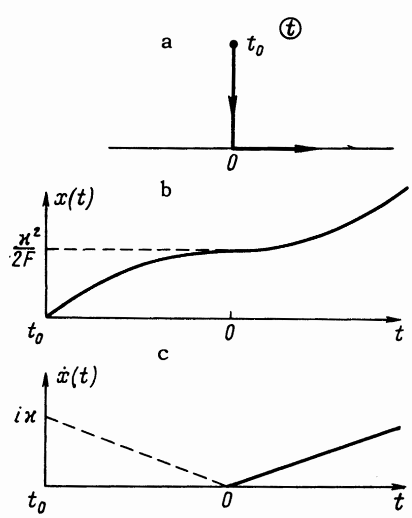

FIG. 5. Classical trajectory corresponding to passage of a particle through a potential barrier. The figure shows the changes during the motion of the following quantities: a – the time; b – the coordinate; c – the velocity. Pure imaginary quantities are shown with dashed lines.

The trajectory we have found is an analytic solution of the equations of classical mechanics (in particular, the point $t = 0$ is not a point of discontinuity). When we go over into the quantum domain it acquires physical meaning: the action $\tilde{S}$, calculated along this trajectory, determines the wave function $\psi(x, t)$ (in the quasiclassical approximation). From (41) we have

$$\tilde{S}(t, t_0) = \int_{t_0}^{t} \dot{x}^2(t')\,dt' = \frac{F^2}{3}(t^3 - t_0^3). \quad (42)$$

The variations of Re $\tilde{S}$ and Im $\tilde{S}$ during the motion are shown in Fig. 6; after the emergence from the barrier Im $\tilde{S}$ remains constant, and from this we get

$$w_{\mathrm{stat}}(F) \sim \exp\{-2\,\mathrm{Im}\,\tilde{S}(0, t_0)\} = \exp\left(-\frac{2}{3}\frac{F_0}{F}\right). \quad (43)$$

Accordingly, apart from a constant factor (40) gives the correct value for $w_{\mathrm{stat}}(F)$ in the one-dimensional case.

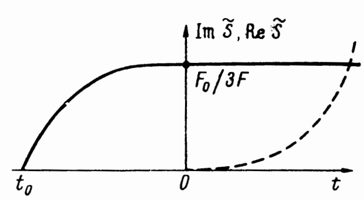

FIG. 6. Change of the contracted action $\tilde{S}$ during the process of passage through a barrier. The solid curve shows the variation of Im$\tilde{S}$, and the dashed curve, that of Re$\tilde{S}$.

Proceeding to the three-dimensional case, we note that the field does not change the transverse momentum $\mathbf{p}_\perp$, which therefore behaves classically ($p_\perp^2 > 0$). There is motion through the barrier only in the direction of the field; that is, the problem reduces to the one-dimensional case. The only change is in the initial condition: we now have $\dot{x}(t_0) = i(\kappa^2 + p_\perp^2)^{1/2}$, and this means that we make the replacement

$$\kappa \to \kappa' = (\kappa^2 + p_\perp^2)^{1/2} = \kappa(1 + {}^1\!/_2(p_\perp/\kappa)^2 + \ldots). \quad (44)$$

Making this change in (43), we have

$$w(F, \mathbf{p}_\perp) \sim \exp\left\{-\frac{2}{3}\frac{F_0}{F} - \frac{F_0}{F}\left(\frac{p_\perp}{\kappa}\right)^2 + \ldots\right\},$$

from which we get

$$w_{\mathrm{stat}}(F) = \int d\mathbf{p}_\perp\, w_{\mathrm{stat}}(F, \mathbf{p}_\perp) \sim \frac{F}{F_0}\exp\left(-\frac{2}{3}\frac{F_0}{F}\right). \quad (45)$$

The exact formula for the probability of ionization

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4)In the stationary case a quasiclassical formula analogous to (40) holds for the probability of passage through a potential barrier (cf. [7], page 220, and also [8]); in this formula for the stationary case the ordinary action appears instead of the contracted action $\tilde{S}$.

of an s level in a short-range potential is as follows:[9]

$$w_{\text{stat}}(F)=\frac{\omega_0}{2}|C_{\varkappa 0}|^2\frac{F}{F_0}\exp\left(-\frac{2}{3}\frac{F_0}{F}\right). \tag{46}$$

A comparison of these formulas shows that apart from a numerical factor of the order of unity the value of $w_{stat}(F)$ in a constant field can be found from Eq. (40).

B. Ionization in the Field of a Wave with Linear Polarization

Let the field $F(t) = F\cos\omega t$ be directed along the x axis. The classical trajectory x(t) is found from the equations:

$$\ddot{x}=F\cos\omega t,\quad x(t_0)=0,\ \dot{x}(t_0)=i\varkappa'=i(\varkappa^2+p_\perp^2)^{1/2} \tag{47}$$

[the transverse momentum is taken into account by using (44)]. We get

$$x(t)=p_x(t-t_0)-F\omega^{-2}(\cos\omega t-\cos\omega t_0),$$
$$\dot{x}(t)=p_x+F\omega^{-1}\sin\omega t, \tag{48}$$

where $p_x = i\kappa' - F\omega^{-1}\sin\omega t_0$; p_x is the average momentum of the particle (in the direction of the field) during its motion to infinity. The particle emerges from the barrier at the time t = 0, when the field reaches its maximum value (the amplitude). The "initial time" t_0 is found from the equation

$$\sin\omega t_0=\frac{\omega}{F}(i\varkappa'-p_x)=\gamma\left[i\left(1+\frac{p_\perp{}^2}{\varkappa^2}\right)^{1/2}-\frac{p_x}{\varkappa}\right], \tag{49}$$

which can be put in the form

$$\left(p_x+\frac{F}{\omega}\sin\omega t_0\right)^2+p_\perp{}^2\equiv\pi^2(t_0)=-\varkappa^2. \tag{50}$$

Both the trajectory $\mathbf{r}(t)$ and the action $\tilde{S}$ are dependent on $\mathbf{p}$. Let us choose the $\mathbf{p}$ for which $\operatorname{Im}\tilde{S}$ is a minimum, i.e., the probability of ionization is a maximum. This trajectory corresponds to $\mathbf{p}_\perp = 0$, $p_x = 0$, $\omega t_0 = i\sinh^{-1}\gamma$. For the intuitive description of the motion through the barrier it is convenient to change to a real time $\tau = it$ $(-\tau_0 \le \tau \le 0)$; the equation of the extremal trajectory is then

$$x(\tau)=\frac{F}{\omega^2}(\operatorname{ch}\omega\tau_0-\operatorname{ch}\omega\tau),\quad \frac{dx}{d\tau}=-\frac{F}{\omega}\operatorname{sh}\omega\tau. \tag{51}$$

The total time τ_0 of the motion through the barrier decreases monotonically with increase of γ:

$$\tau_0=\frac{1}{\omega_t}\frac{\operatorname{Arsh}\gamma}{\gamma}\approx\frac{1}{\omega_t}\begin{cases}1-{}^1/_6\gamma^2+\ldots, & \gamma\ll 1\\ \gamma^{-1}\ln\gamma, & \gamma\gg 1\end{cases}. \tag{52}$$

With increase of frequency there is also a decrease of the classical turning point (the length of the barrier):

$$x_0=\frac{F}{\omega^2}(\operatorname{ch}\omega\tau_0-1)=\frac{\varkappa^2}{F}(1+\sqrt{1+\gamma^2})^{-1}$$
$$\approx\begin{cases}\dfrac{\varkappa^2}{2F}\left(1-\dfrac{\gamma^2}{4}\right), & \gamma\ll 1\\ \dfrac{\varkappa^2}{F\gamma}, & \gamma\gg 1\end{cases}. \tag{53}$$

This is the reason for the increase of $w(F,\omega)$ for $\omega \gg \omega_t$; as γ increases the barrier gets "shorter" because of the decrease of x_0.

It is easy to see that for $t \to \infty$ only the motion through the barrier contributes to $\operatorname{Im}\tilde{S}(t, t_0)$, so that it suffices to find $\tilde{S}(0, t_0)$:

$$\tilde{S}(0,t_0)=\int_{t_0}\left\{\frac{1}{2}\dot{x}^2+Fx(t')\cos\omega t'-\frac{\varkappa^2}{2}\right\}dt'$$
$$=i\frac{\omega_0}{\omega}\left\{\left(1+\frac{1}{2\gamma^2}\right)\operatorname{Arsh}\gamma-\frac{\sqrt{1+\gamma^2}}{2\gamma}\right\}. \tag{54}$$

By means of (40) we find that the exponent in the formula for $w(F,\omega)$ is of the form

$$\exp\left(-\frac{2\omega_0}{\omega}f(\gamma)\right),$$
$$f(\gamma)=\left(1+\frac{1}{2\gamma^2}\right)\operatorname{Arsh}\gamma-\frac{\sqrt{1+\gamma^2}}{2\gamma}, \tag{55}$$

which is the same as I, (75).

Trajectories corresponding to $\mathbf{p}\neq 0$ make smaller contributions to $w(F,\omega)$. From (49) we find for $\mathbf{p}^2 \ll \kappa^2$:

$$\omega t_0=i\operatorname{Arsh}\gamma-\frac{\gamma}{\sqrt{1+\gamma^2}}p_x$$
$$+\frac{i\gamma}{2\sqrt{1+\gamma^2}}\left(\mathbf{p}_\perp{}^2+\frac{\gamma^2}{1+\gamma^2}p_x{}^2\right)+\ldots, \tag{56}$$

i.e., the "initial time" t_0 is shifted from the imaginary axis into the complex plane. Calculating

$$\operatorname{Im}\tilde{S}_{\mathbf{p}}-\operatorname{Im}\tilde{S}_{\mathbf{p}=0}=\delta\operatorname{Im}\tilde{S}_{\mathbf{p}}$$

to and including terms $\sim \mathbf{p}^2/\kappa^2$, we find

$$\delta\operatorname{Im}\tilde{S}=\frac{\omega_0}{\omega}\left\{\left(\operatorname{Arsh}\gamma-\frac{\gamma}{\sqrt{1+\gamma^2}}\right)\frac{p_x{}^2}{\varkappa^2}+\operatorname{Arsh}\gamma\frac{p_\perp{}^2}{\varkappa^2}\right\}. \tag{57}$$

Owing to the factor $\omega_0/\omega \gg 1$, only trajectories with small $\mathbf{p}$ are important. This justifies the expansion in powers of $\mathbf{p}^2/\kappa^2$ which has been made in (57). Equations (55) and (57) give the main terms in the expression for $|F_n(\mathbf{p})|^2$ [cf. I, (53)]; the term in I, (53) containing

$$\cos\left(4\frac{\omega_0}{\omega}\frac{\sqrt{1+\gamma^2}}{\gamma}\frac{p_x}{\varkappa}\right)$$

cannot be derived in the framework of the quasi-classical approximation, but its contribution to the total probability of ionization is negligibly small.

C. Wave with Elliptical Polarization

Here the electric field $\mathbf{F}(t)$ is of the form (4). We shall look for only the extremal classical trajectory, which minimizes $\operatorname{Im}\tilde{S}$. In analogy with the preceding case we assume that $p_z = 0$ and that the particle emerges from the barrier at the time $t = 0$ when the field reaches a maximum value, so that $\dot{x}(0) = 0$. Using these conditions, we find for the velocity of the motion

$$\dot{x}(t) = \frac{F}{\omega}\sin\omega t, \quad \dot{y}(t) = \pm k_0 - \frac{\varepsilon F}{\omega}\cos\omega t, \quad \dot{z} = 0. \tag{58}$$

The average momentum of the particle at infinity is $\pm k_0$ and is directed along the y axis. The condition $\mathbf{p}^2(t_0) = -\kappa^2$ gives

$$\left(\pm k_0 - \frac{\varepsilon F}{\omega}\operatorname{ch}\omega\tau_0\right)^2 - \frac{F^2}{\omega^2}\operatorname{sh}^2\omega\tau_0 = -\varkappa^2 \quad (\tau_0 = it_0.) \tag{59}$$

Integrating (58) and using the initial condition $x(t_0) = y(t_0) = 0$, we get

$$x(t) = F\omega^{-2}(\cos\omega t_0 - \cos\omega t),$$

$$x(t) = \varepsilon F\omega^{-2}(\sin\omega t_0 - \sin\omega t) \pm k_0(t - t_0). \tag{60}$$

It is assumed that the "time" t varies in the same way as in Fig. 5, a.

In order for the coordinate $y(t)$ to be real at $t \to \infty$, the following condition must be satisfied:

$$\operatorname{Im}(\varepsilon F\omega^{-2}\sin\omega t_0 \mp k_0 t_0) = 0$$

or

$$k_0 = \pm\varkappa\frac{\varepsilon}{\gamma}\frac{\operatorname{sh}\omega\tau_0}{\omega\tau_0}, \quad \tau_0 = it_0. \tag{61}$$

Setting $\omega\tau_0 = v$ and $k_0 = k_y$, we see that (59) and (61) are the same as the equations (18) for determining the saddle point. We bring $\tilde{S}(0, t_0)$ into the following form:

$$\tilde{S}(0, t_0) = \int_{t_0}^{0}\left\{L(t) - \frac{\varkappa^2}{2}\right\}dt = -\frac{1}{2}\int_{t_0}^{0}(\dot{\mathbf{r}}^2 + \varkappa^2)\,dt + (\dot{\mathbf{r}}\mathbf{r})\Big|_{t=t_0}^{t=0}.$$

Owing to the conditions $\dot{\mathbf{r}}(t_0) = 0$, $\mathbf{r}(0) = 0$, the last term is zero, and we have

$$-2\operatorname{Im}\tilde{S}(0, t_0) = -\frac{2\omega_0}{\omega}\operatorname{Im}\int_0^{\omega t_0}\left(1 + \frac{\mathbf{r}^2}{\varkappa^2}\right)dt. \tag{62}$$

Since $\dot{\mathbf{r}}(t) = \pi(\omega t)$, $\exp[-2\operatorname{Im}\tilde{S}(0, t_0)]$ is the same as the exponential in (11).

An examination of the classical trajectory (60) elucidates a number of features of the process of ionization by the field of an elliptically polarized wave. By means of (60) and (19) we find that at the moment when it emerges from the barrier the particle is at a distance $x_0 = x_0(\gamma, \epsilon)$ from the atom

$$x_0 = \frac{\varkappa^2}{F\gamma^2}\left[\left(\frac{1+\gamma^2}{1-s_0^2}\right)^{1/2} - 1\right] = \begin{cases} \dfrac{\varkappa^2}{2F}\left[1 - \left(\dfrac{1}{4} - \dfrac{\varepsilon^2}{9}\right)\gamma^2 + \ldots\right], & \gamma \ll 1 \\ \dfrac{\varkappa^2}{F\gamma\sqrt{1-\varepsilon^2}}, \quad \gamma \gg 1, & |\varepsilon| \neq 1 \end{cases} \tag{63}$$

The larger $|\epsilon|$, the more slowly x_0 decreases with increase of γ (see figure in [4]). This shows why the probability of ionization decreases with increase of $|\epsilon|$. Furthermore, in its motion in the field (4) the particle describes the ellipse (58). At the time of emergence from the barrier $\dot{x} = 0$, and therefore the average momentum of the ejected electrons is perpendicular to the direction of maximum field strength.

Thus the idea of motion of the particle through the barrier along a complex trajectory which formally satisfies the classical equations of motion is extremely useful. In the writers' next paper this idea will be used to take into account the Coulomb interaction between the electron and the atomic residue.

APPENDIX

SOME PROPERTIES OF THE SOLUTIONS OF THE SCHRÖDINGER EQUATION FOR A POTENTIAL WITH A COULOMB "TAIL"

As can be seen from Eqs. I, (4) and I, (54), in the case $F \ll F_0$ the probability of ionization is determined by the "tail" of the wave function of the atomic electron. In the investigation of the asymptotic properties of $\psi(\mathbf{r})$ it is usually assumed that the potential has a finite range. In this case

$$\psi_{\varkappa lm}(\mathbf{r}) \sim C_{\varkappa l}\varkappa^{3/2}\frac{e^{-\varkappa r}}{\varkappa r}Y_{lm}\left(\frac{\mathbf{r}}{r}\right), \quad \varkappa r \gg 1. \tag{A.1}$$

The Coulomb interaction, with its long-range action, distorts the shape of the wave function at large distances. We shall here discuss some properties of the wave functions for potentials which have a Coulomb "tail" at infinity:

$$V(r) \sim -\varkappa_C / r, \quad \varkappa r \gg 1 \tag{A.2}$$

(here κ_C is the Coulomb momentum; in atomic units $\kappa_C = Z$). The behavior of V(r) at small distances can be arbitrary.

It follows from the Schrödinger equation that the asymptotic behavior of the wave function of a bound level with binding energy $\kappa^2/2$ in the potential (A.2) is of the form

$$\psi_{\varkappa lm}(\mathbf{r}) \sim C_{\varkappa l}\varkappa^{3/2}(\varkappa r)^{\lambda-1}e^{-\varkappa r}Y_{lm}(\mathbf{r}/r), \qquad (A.3)$$

where $\lambda = \kappa_C/\kappa$. The larger λ, the more slowly $|\psi|^2$ falls off at infinity, and the more "crumbly" is the system. The dimensionless coefficient $C_{\kappa l}$ determines the probabilities of peripheral processes, which depend on the behavior of the "tail" of the wave function. The determination of $C_{\kappa l}$ requires exact solution of the Schrödinger equation in all space, which can be accomplished only in the simplest cases. For example, for the hydrogen atom in a state with principal quantum number n

$$\lambda = n, \quad C_{nl} = (-1)^{n-l-1}\cdot 2^n[n(n+l)!(n-l-1)!]^{-1/2} \qquad (A.4)$$

and the factor before the exponential in (A.3) is of the form $(\kappa r)^{n-1}$ which agrees with the exact solution [cf. Eq. (36.15) in [7]]. For the s level in the three-dimensional δ potential $C_{\kappa 0} = 2^{1/2}$.

The asymptotic behavior of $\psi(\mathbf{r})$ for $r \to \infty$ determines the behavior of the wave function $\varphi(\mathbf{p})$ in the p representation near the pole $p^2 = -\kappa^2$. By a Fourier transformation we get from (A.3)

$$\varphi_{lm}(\mathbf{p}) = \varphi_l(p)Y_{lm}(\mathbf{p}/p);$$

$$\varphi_l(p) \sim \xi_l C_{\varkappa l}\frac{\Gamma(\lambda+1)}{(2\pi\varkappa^3)^{1/2}}\left(\frac{2\varkappa^2}{p^2+\varkappa^2}\right)^{\lambda+1} \qquad (A.5)$$

where $\xi_l = 1$ for $p \to i\kappa$ and $\xi_l = (-1)^l$ for $p \to -i\kappa$.

For short-range potentials $\lambda = 0$ and the singularity at $p^2 = -\kappa^2$ is a simple pole. When there is a Coulomb "tail" the wave function $\varphi_l(p)$ in general has branch points at $p = \pm i\kappa$; for integer values $\lambda = n$ there is a pole of n-th order. The tail of the potential V(r) also determines the position and character of the nearest singularity of the scattering amplitude f(E, z). For the potential (A.2) this singularity lies at the edge of the physical region:

$$f(E,z) \sim \frac{\varkappa_C}{k^2}\cdot 2^{-i\varkappa_C/k}\frac{\Gamma(1-i\varkappa_C/k)}{\Gamma(1+i\varkappa_C/k)}(1-z)^{-1+i\varkappa_C/k}, \quad z\to 1 \qquad (A.6)$$

($E = k^2/2$, $z = \cos\theta$). It is a branch point, and its index $(-1 + i\kappa_C/k)$ depends on the energy E of the incident particles. This is typical of the Coulomb potential; if V(r) falls off more rapidly than r^{-1} for $r \to \infty$, the nature of the nearest singularity of f(E, z) is determined only by the asymptotic behavior of the potential (see [10]).

The important physical significance of the constant $C_{\kappa l}$ has been emphasized in papers by Heisenberg,[11] Möller,[12] and N. Hu,[13] in which a connection was established between $C_{\kappa l}$ and the residue of the scattering matrix $S_l(k)$ at the pole $k = i\kappa$ which corresponds to the bound state. Here it is essential to use the form of the asymptotic behavior, Eq. (A.1). These results can be extended also to potentials of the type (2); to do this, following the method of Hu's paper,[13] we consider the completeness relation for the radial wave functions for $r \to \infty$. The result is the following formula[5)]

$$|C_{\varkappa l}|^2 = e^{i\pi(l+\lambda+1/2)}\frac{2^{2\lambda}}{\varkappa}r_{\varkappa l}, \qquad (A.7)$$

in which $r_{\kappa l}$ is the residue of the scattering matrix $S_l(k)$ at the pole $k = i\kappa$,

$$S_l(k) \underset{k\to i\varkappa}{=} \frac{r_{\varkappa l}}{k-i\varkappa} + \text{ finite part} \qquad (A.8)$$

For potentials that decrease more rapidly at infinity than the Coulomb potential (i.e., for $\lambda = 0$), (A.7) corresponds to Eq. (22) in [13].

In the case of a pure Coulomb potential

$$S_l(k) = \frac{\Gamma(l+1-i\varkappa_C/k)}{\Gamma(l+1+i\varkappa_C/k)} \xrightarrow[k\to i\varkappa_n]{} \frac{r_{nl}^C}{k-i\varkappa_n}$$

$$\varkappa_n = \frac{\varkappa_C}{n}, \qquad r_{nl}^C = \frac{i(-1)^{n-l-1}\varkappa}{n^2(n+l)!(n-l-1)!}$$

By means of (A.4) we can verify that (A.7) is satisfied identically.

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