A COMPLETE SET OF ANGULAR FUNCTIONS FOR THE THREE BODY PROBLEM FOR AN ARBITRARY ORBITAL ANGULAR MOMENTUM

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A complete set of independent angular functions (harmonic polynomials) which realize an irreducible representation of the rotation group in three dimensional space and an irreducible representation of the permutation group for three particles is proposed. The functions obtained are eigenfunctions of the total orbital angular momentum of the system L and its projection M on the z axis. The degree of the polynomials K is the eigenvalue of the square of the global momentum in six-dimensional space. Symmetry relative to permutations is characterized by the symbol ν.

I. INTRODUCTION

In this paper a method is proposed for obtaining in the coordinate representation a complete set of independent wave functions which constitute an irreducible representation of the rotation group in three dimensions and an irreducible representation of the permutation group of three particles, for an arbitrary total orbital angular momentum. Analogous questions have been discussed by many authors.[1-4] Here we indicate a simple algorithm for the construction of a complete set of independent functions. The expression for the polynomials with arbitrary L is written out explicitly. For L = 1 and 2 the polynomials have a very simple form.

In the work of Zickendraht[2] use was made of the method of separation of variables in the Laplace equation to construct a complete set of independent functions for the three particle system. Zickendraht,[5] as well as a number of other authors (see, for example [6]) make use of internal and external coordinates in the center of mass system. The internal coordinates describe the shape of the triangle formed by the particles; the external coordinates describe the orientation of the triangle in space.

As it turns out, the variables A and λ, introduced in [1], in essence coincide with the internal coordinates used by Zickendraht. Nevertheless, the final results obtained in this paper and the results of Zickendraht are substantially different. The problem has to do with the fact that the complete set of independent functions obtained by Zickendraht does not make it possibly to fully classify these functions according to their symmetry with respect to permutation of particles. None of his functions possesses definite symmetry under permutation of all three particles. The functions obtained in this paper constitute a complete set and, in addition, have extremely simple properties with respect to permutations of all three particles. Each of the functions is either symmetric or antisymmetric, or transforms according to the two-dimensional representation of the permutation group. These properties make the functions obtained below particularly useful for practical applications.

One more method of construction has been proposed by Dragt,[3] however the method is very complicated for the construction of polynomials with large K (Dragt has found an explicit form for the polynomials with $K < 3$).

In this paper we shall make extensive use of the previous article,[1] whose results will be assumed to be known; references to equations given there will be denoted by (I, ...).

2. QUANTUM NUMBERS

The quantum numbers enumerating the sought for polynomials are taken as the eigenvalues of operators that are conserved for free motion of three particles.

For some time now it has been understood[3, 7] that to take into account of the identity of the particles it is necessary to use the irreducible representations of the group U_3 as a complete set of states for the three particles. The method for the construction of the polynomials outlined in [1] automatically constitutes representations of the group U_3, however for L = 0 and 1 the polynomials were constructed also by the method of separation of variables in the Laplace equation, which gives rise to the same result.[4]

In the general case we should classify the states according to the subgroup O_3 of the group U_3, which means that in addition to the quantum numbers K and ν present in the case $L = 0$[1] there appear the quantum numbers L^2 and M—the square of the total angular momentum and its projection. Finally, there appears a fifth operator Ω, which has been discussed by a number of authors.[8, 9]

We construct the complete set of wave functions in the form of harmonic polynomials in six-dimensional space, which in the case of an arbitrary L will be constructed by forming tensor and scalar combinations of two complex vectors z_i and z_i^* (I.9). It follows from the general theory of harmonic polynomials (see for example, [10]) that such polynomials may be characterized by five quantum numbers. We shall use for these quantum numbers the following:

1) K—the degree of the polynomial; the corresponding operator for the square of the six-dimensional global angular momentum K^2 is the angular part of the six-dimensional Laplacian

$$\Delta = \frac{\partial^2}{\partial \mathbf{z}\, \partial \mathbf{z}^*} \tag{1}$$

and has the form

$$\hat{K}^2 = \left(\mathbf{z}\frac{\partial}{\partial \mathbf{z}} + \mathbf{z}^*\frac{\partial}{\partial \mathbf{z}^*}\right)\left(\mathbf{z}\frac{\partial}{\partial \mathbf{z}} + \mathbf{z}^*\frac{\partial}{\partial \mathbf{z}^*} + 4\right) - (\mathbf{z}\mathbf{z}^*)\Delta; \tag{2}$$

2) L^2—the square of the total orbital angular momentum. The operator in terms of the variables z_i and z_i^* has the form

$$\mathbf{L} = -i\hbar\left(\left[\mathbf{z}, \frac{\partial}{\partial \mathbf{z}}\right] + \left[\mathbf{z}^*, \frac{\partial}{\partial \mathbf{z}^*}\right]\right); \tag{3}$$

3) M—the projection of the total orbital angular momentum on the axis z;

4) ν—the operator characterizing the behavior of the polynomials under permutations. If p is the number of the variables z and q the number of the variables z* in the polynomial, then

$$\nu = (p - q) / 2. \tag{4}$$

It is easy to write the corresponding operator:

$$2\hat{\nu} = \mathbf{z}\frac{\partial}{\partial \mathbf{z}} - \mathbf{z}^*\frac{\partial}{\partial \mathbf{z}^*}. \tag{5}$$

In what follows it will be convenient to break up ν into two parts: τ and κ ($\nu = \tau + \kappa$), where τ characterizes the scalar part of the polynomial and κ the tensor part;

5) for the fifth quantum number we shall make use of the number ω, whose meaning will become clarified below.

The four indicated operators K^2, L^2, M, and ν commute with each other. The functions obtained by us are orthogonal with respect to the corresponding indices. We were not able to show that our functions are also orthogonal with respect to ω, however the independence of functions with different ω has been shown. Moreover the total number of functions for a given K, calculated with respect to the subscripts L, M, ν, ω, is equal to

$$n_K = (K + 3)!(K + 2) / 12K! \tag{6}$$

as it should (see Appendix III). Further, for $L = 0$, 1 and 2 the set of functions obtained by us is orthogonal with respect to all five indices.

The fifth operator which commutes with all the operators K, L^2, M and ν is the operator

$$\Omega = \sum_{ij} L_i \left(z_i \frac{\partial}{\partial z_j} - z_j^* \frac{\partial}{\partial z_i^*}\right) L_j,$$

whose physical meaning is not quite clear to us. For practical purposes it is more convenient to make use of the quantum number ω. The number of eigenvalues of ω and Ω for given K, L^2, M and ν is the same.

3. METHOD OF CONSTRUCTION OF THE POLYNOMIALS

The polynomials satisfying the required symmetry properties will be found as follows. Let us denote the differentiation operators by:

$$\frac{\partial}{\partial z_i} \equiv \partial_i, \qquad \frac{\partial}{\partial z_i^*} \equiv \partial_i^*. \tag{7}$$

1. The application of the differentiation operators to the scalar polynomial ($L = 0$) L times gives us a tensor of rank L. It is clear that this yields a harmonic polynomial, since the differentiation operators (7) commute with the Laplacian $\partial_i \partial_i^*$(1).

2. By symmetrizing the resultant tensors and making them traceless with respect to each pair of indices we arrive at polynomials which are eigenfunctions of the square of the total orbital angular momentum.

3. By utilizing "spherical" coordinates we automatically obtain classification of the polynomials with respect to the projection M of the total orbital angular momentum.

4. The construction of functions which are symmetric, antisymmetric or transform according to the two-dimensional representation with respect to permutations is easily carried out if it is remem-

bered how z and z* transform under permutations (see (I.10) and (I.11)).

5. For $L \geq 3$ the resultant system should be orthogonalized with respect to the subscript ω.

4. MATHEMATICAL FORMALISM

We shall write the tensor of rank L with the tensor indices $m_1 m_2 \ldots m_L$ in the form

$$T^L_{m_1 m_2 \ldots m_L}\,(T^L_{12\ldots L}). \tag{8}$$

In order to make an arbitrary tensor symmetric and traceless with respect to every pair of tensor indices, we make use of the symmetrization operator $\mathcal{P}^L_s$ and the tracelessness operator $\mathcal{P}^L_t$, following the work of Zemach.[11] Afterwards, like Zemach, we introduce spherical coordinates.

A. The operators $\mathcal{P}^L_s$ and $\mathcal{P}^L_t$. The symmetrization operator $\mathcal{P}^L_s$ is defined by the equality

$$\mathcal{P}_s{}^L T^L_{12\ldots L} = (L!)^{-1} \Sigma_P T^L_{12\ldots L}, \tag{9}$$

where the symbol Σ_p denotes summation over all essentially different permutations. If the tensor T^L is symmetric then

$$\mathcal{P}_s{}^L T^L = T^L. \tag{10}$$

Let $\mathcal{P}^L_t$ be the operator which ensures that an arbitrary tensor is traceless. For small L the action of $\mathcal{P}_t$ on tensors which are already symmetric can be written very simply:

$$\begin{aligned}
&\mathcal{P}_t T_{12} = T_{12} - {}^1/_3 \delta_{12} T_{..},\\
&\mathcal{P}_t T_{123} = T_{123} - {}^1/_5 (\delta_{12} T_{..3} + \delta_{13} T_{..2} + \delta_{23} T_{..1}),\\
&\mathcal{P}_t T_{1234} = T_{1234} - {}^1/_7 \Sigma_P \delta_{12} T_{nn34} +\\
&\quad + {}^1/_{35} (\delta_{12}\delta_{34} + \delta_{13}\delta_{24} + \delta_{14}\delta_{23}) T_{nnkk}.
\end{aligned} \tag{11}$$

The general formula for a symmetric T^L is

$$\begin{aligned}
\mathcal{P}_t{}^L T^L_{12\ldots L} &= T_{12\ldots L} - (2L-1)^{-1} \Sigma_P \delta_{12} T_{nn3\ldots L}\\
&+ (2L-1)^{-1} (2L-3)^{-1} \Sigma_P \delta_{12}\delta_{34} T^L_{nnkk5\ldots L} - \ldots,
\end{aligned} \tag{12}$$

which is easily verified by calculating the trace of the expression (12) and cancelling similar terms.

The operators $\mathcal{P}_s$ and $\mathcal{P}_t$ commute with each other. The complete operator which reduces an arbitrary tensor to a symmetric and traceless form is therefore

$$\mathcal{P}^L = \mathcal{P}_s{}^L \mathcal{P}_t{}^L. \tag{13}$$

B. Tensors constructed out of two vectors. We shall construct tensors out of two complex vectors z_i and z_i^*. The tensor of order L is characterized by the additional index $\kappa = (p_L - q_L)/2$ (see Eq. (4)), which runs over the values from $-L/2$ to $+L/2$ at unit intervals, taking on (depending on the nature of L) integer or half integer values, altogether $L + 1$ values. Let us denote an arbitrary tensor formed out of two vectors by

$$R^{L\kappa}_{12\ldots L}(z, z^*). \tag{14}$$

Acting on an arbitrary tensor with the operator $\mathcal{P}^L$, Eq. (13), we obtain a symmetric and traceless tensor constructed out of two vectors:

$$T^{L\kappa}_{12\ldots L}(z, z^*) = \mathcal{P}^L R^{L\kappa}_{12\ldots L}(z, z^*). \tag{15}$$

For example, for $L = 2$

$$T^{21}_{ij} = z_i z_j - {}^1/_3 \delta_{ij} z^2, \tag{16}$$

$$T^{20}_{ij} = {}^1/_2 (z_i z_j^* + z_i^* z_j) - {}^1/_3 \delta_{ij} (\mathbf{zz}^*), \tag{17}$$

$$T^{2-1}_{ij} = z_i^* z_j^* - {}^1/_3 \delta_{ij} z^{*2}. \tag{18}$$

In addition to ordinary tensors it is possible to construct pseudotensors out of two vectors. Let us introduce the vector A_i which is the vector product of the vectors z_i and z_i^*:

$$A_i = \varepsilon_{ikl} z_k z_l^*, \tag{19}$$

where ϵ_{ikl} is the completely antisymmetric tensor. The product of two components of the vector A_i may again be expressed in terms of z_i and z_i^*:

$$A_i A_j = {}^1/_3 \delta_{ij} A^2 - z^{*2} T^{21}_{ij} - z^2 T^{2-1}_{ij} + 2(\mathbf{zz}^*) T^{20}_{ij}, \tag{20}$$

it is therefore sufficient to consider only such pseudotensors in which A_i enters only once. In such a tensor κ runs over the values from $-(L-1)/2$ to $(L-1)/2$ at unit intervals and takes on L values. Symmetrized and traceless pseudotensors will be denoted by

$$A^{L\kappa}_{12\ldots L}(A, z, z^*). \tag{21}$$

For $L = 2$ we have

$$A^{2\,1/2}_{ij} = {}^1/_2 (A_i z_j + A_j z_i), \tag{22}$$

$$A^{2-1/2}_{ij} = {}^1/_2 (A_i z_j^* + A_j z_i^*). \tag{23}$$

C. Spherical coordinates. The spherical components of a vector are expressed in terms of its Cartesian components by

$$p_+ = \frac{1}{\sqrt{2}}(p_1 + ip_2), \quad p_0 = p_3, \quad p_- = -\frac{1}{\sqrt{2}}(p_1 - ip_2). \tag{24}$$

The scalar product

$$\begin{aligned}
\mathbf{pq} &= p_0 q_0 + (p_+)^* q_+ + (p_-)^* q_-\\
&= p_0 q_0 - p_+ q_- - p_- q_+ \equiv g^{AB} p_A q_B,
\end{aligned} \tag{25}$$

where

$$g^{00} = 1, \quad g^{+-} = g^{-+} = -1, \quad \text{all other } g^{AB} = 0. \tag{26}$$

Let us consider tensors formed out of one vector. All vector indices of such a tensor may be expressed in spherical coordinates. The formulae previously obtained remain valid if the scalar product is taken as described above and the Kronecker symbol replaced by g^{AB}.[11] In this scheme the indices +, 0, − play an important role. Let the number of the corresponding components be m, l, n. Then the symmetric and traceless tensor can be written in the form

$$T^L_{12\ldots L} = T^L_{lmn}. \quad (27)$$

Such a tensor component has a definite quantum number

$$M = m - n. \quad (28)$$

Two arbitrary tensor components T_{+-} and T_{00}, with the same value of M, should be proportional. From the symmetry of the tensor (27) and from the fact that its trace equals zero ($g^{AB}T^L_{AB\ldots} = 0$), we get

$$T_{+-\ldots} = T_{-+\ldots} = {}^1/_2 T_{00\ldots}. \quad (29)$$

Therefore the tensor T^L_{lmn} may be characterized by the quantum number M alone:

$$T^L_{lmn} = T^{LM}. \quad (30)$$

Let us consider now the tensor (15) formed out of two vectors. We introduce spherical components of the complex vectors z_i and z_i^*:

$$z_+ = \frac{1}{\sqrt{2}}(z_1 + iz_2), \quad z_0 = z_3, \quad z_- = -\frac{1}{\sqrt{2}}(z_1 - iz_2). \quad (31)$$

Correspondingly

$$z_+^* = \frac{1}{\sqrt{2}}(z_1^* + iz_2^*), \quad z_0^* = z_3^*,$$

$$z_-^* = -\frac{1}{\sqrt{2}}(z_1^* - iz_2^*). \quad (32)$$

Analogous to the expression (27), the tensor (15) constructed out of two vectors will be written in the form

$$T^{L\varkappa}_{12\ldots L} = T^{LM\varkappa}_{lmnrst}, \quad (33)$$

where l, m, n, r, s, t —is the number of the corresponding components of z_A and z_A^*. The total orbital angular momentum (3) consists of the sum of the two moments with respect to z_i and z_i^* and therefore the tensor (33) will be characterized by a definite projection M of the total orbital angular momentum, equal to the sum of the projections of the constituent momenta. According to (28),

$$M = (m - n) + (s - t). \quad (34)$$

For later reference it is necessary to establish how the tensor (33) transforms under the action of the operator P_{12}, which interchanges the first and second particles. The operation P_{12} takes z into z* (I, 10), and z* into z (I, 11). It follows from the definition of κ, Eq. (4), that κ will go over into $-\kappa$, and M will remain unchanged since in Eq. (34) the two terms interchange places. It follows from here that

$$P_{12}T^{LM\varkappa} = T^{LM-\varkappa}. \quad (35)$$

Let us also establish what happens to the tensor (33) under complex conjugation. The operation of complex conjugation on spherical components of the vectors z_A and z_A^* may be written with the help of the metric tensor (26) in the form

$$(z_A)^* = g^{AB}z_B^*, \quad (z_A^*)^* = g^{AB}z_B. \quad (36)$$

It follows from here that κ, Eq. (4), goes over into $-\kappa$, and M, Eq. (34), will go over into −M, and there will appear the numerical factor $(-1)^M$. Thus

$$(T^{LM\varkappa})^* = (-1)^M T^{L-M-\varkappa}. \quad (37)$$

5. GENERAL METHOD OF CONSTRUCTION OF A COMPLETE ORTHOGONAL SYSTEM

The first step in the method is to introduce differential tensors with the help of which one finds eigenfunctions of the total orbital angular momentum and its projection on the axis z. The second step is to classify the resultant functions with respect to the symmetry under permutation of particles.

The differential tensors are constructed out of derivatives with respect to spherical components of the vectors z_A and z_A^*:

$$\frac{\partial}{\partial z_A} \equiv \partial_A, \quad \frac{\partial}{\partial z_A^*} \equiv \partial_A^*. \quad (38)$$

A given differential tensor will be characterized by its order L, projection N and the number ω which determines the relative number of components ∂ and ∂^* in accordance with Eq. (4). In its general form

$$D^{LN\omega}(\partial, \partial^*) = \mathcal{P}^L \partial_A \partial_B \ldots \partial_C^* \partial_D^* \ldots, \quad (39)$$

where the operator $\mathcal{P}^L$ is determined by Eq. (13).

In general the derivatives ∂_A and ∂_A^*, as well as the vectors z_A and z_A^*, are characterized by six variables. Three internal coordinates describe the configuration of the triangle formed by the three particles, and three coordinates describe the orientation of the triangle in space. In the given situation the tensor (39) acts on the scalar polynomial (I, 33), which depends only on the internal variables ρ, A, λ:

$$\rho^2 = (\mathbf{zz}^*), \quad A^2 = \rho^{-4} z^2 z^{*2}, \quad e^{-i\lambda} \equiv \sigma = -\rho^{-2} A^{-1} z^2 \tag{40}$$

(see Eqs. (I, 3), (I, 29) and (I, 31)), and therefore the tensor (39) may be reduced to such a form that the differential operators will contain only internal coordinates. First of all we shall write for the differential vector:

$$\partial_B = (\partial_B \rho)\frac{\partial}{\partial \rho} + (\partial_B A)\frac{\partial}{\partial A} + (\partial_B \sigma)\frac{\partial}{\partial \sigma}. \tag{41}$$

Making use of the expressions for ρ, A, and σ in terms of z, z* we may express the derivative ∂_A with the help of the metric tensor (26) in the form

$$\partial_A = g^{AB}(z_B \Lambda_1 + z_B^* \Lambda_2), \tag{42}$$

where Λ_1 and Λ_2 are differentiation operators depending only on ρ, A and σ:

$$\Lambda_1 = -\frac{1}{\rho^2 \sigma}\frac{\partial}{\partial A} - \frac{1}{\rho^2 A}\frac{\partial}{\partial \sigma}, \tag{43}$$

$$\Lambda_2 = \frac{1}{2\rho}\frac{\partial}{\partial \rho} - \frac{A}{\rho^2}\frac{\partial}{\partial A}. \tag{44}$$

Analogously for ∂_A^* we find

$$\partial_A^* = g^{AB}(z_B^* \Lambda_1^* + z_B \Lambda_2), \tag{45}$$

where

$$\Lambda_1^* = -\frac{\sigma}{\rho^2}\frac{\partial}{\partial A} + \frac{\sigma^2}{\rho^2 A}\frac{\partial}{\partial \sigma} \tag{46}$$

It is easily verified that all the differential operators Λ_1, Λ_2, and Λ_1^* commute, and it is therefore not necessary to worry about their order in the expressions obtained below.

The transformation of tensors (39) for $L \geq 2$ is performed in succession from tensors of rank L to tensors of rank L + 1. In order to demonstrate the method we consider in detail the case L = 2, for example for the tensor $\partial_A \partial_B$:

$$\begin{aligned}\partial_A \partial_B &= \partial_A g^{BD}(z_D \Lambda_1 + z_D^* \Lambda_2)\\ &= g^{BD}(\delta_{AD}\Lambda_1 + z_D \partial_A \Lambda_1 + z_D^* \partial_A \Lambda_2) =\\ &= g^{AB}\Lambda_1 + g^{AC}g^{BD}\{z_C z_D \Lambda_1^2 + (z_C z_D^* + z_C^* z_D)\Lambda_1\Lambda_2\\ &+ z_C^* z_D^* \Lambda_2^2\}.\end{aligned}$$

Applying to this tensor the operator $\mathcal{P}^2$, Eq. (13), we obtain the symmetric traceless tensor

$$\begin{aligned}\mathcal{P}^2 \partial_A \partial_B = g^{AC}g^{BD}\{&(z_C z_D - {}^1\!/_3 g^{CD} z^2)\Lambda_1^2\\ &+ (z_C z_D^* + z_C^* z_D - {}^2\!/_3 g^{CD}(\mathbf{zz}^*))\Lambda_1\Lambda_2\\ &+ (z_C^* z_D^* - {}^1\!/_3 g^{CD} z^{*2})\Lambda_2^2\}.\end{aligned} \tag{47}$$

Analogous expressions are obtained for two other tensors of rank 2:

$$\begin{aligned}\mathcal{P}^2 \partial_A \partial_B^* = g^{AC}g^{BD}\{&(z_C z_D - {}^1\!/_3 g^{CD} z^2)\Lambda_1\Lambda_2\\ &+ {}^1\!/_2(z_C z_D^* + z_C^* z_D - {}^2\!/_3 g^{CD}(\mathbf{zz}^*))(\Lambda_1\Lambda_1^* + \Lambda_2^2)\\ &+ (z_C^* z_D^* - {}^1\!/_3 g^{CD} z^{*2})\Lambda_1^*\Lambda_2\};\end{aligned} \tag{48}$$

$$\begin{aligned}\mathcal{P}^2 \partial_A^* \partial_B^* = g^{AC}g^{BD}\{&(z_C z_D - {}^1\!/_3 g^{CD} z^2)\Lambda_2^2\\ &+ (z_C z_D^* + z_C^* z_D - {}^2\!/_3 g^{CD}(\mathbf{zz}^*))\Lambda_1^*\Lambda_2\\ &+ (z_C^* z_D^* - {}^1\!/_3 g^{CD} z^{*2})\Lambda_1^{*2}\}.\end{aligned} \tag{49}$$

The result is easily written for the general case if it is noted that the action of the operator $\mathcal{P}^L$ is equivalent to having the operators Λ_1, Λ_2, Λ_1^* commute with the vectors z_A and z_A^*. Therefore the differential tensor, Eq. (39), may be written in the form

$$D^{LN\omega} = \mathcal{P}^L \partial_A \partial_B \ldots \partial_C^* \partial_D^* \ldots = \mathcal{P}^L [g^{AB}(z_B \Lambda_1 + z_B^* \Lambda_2)]^p \times [g^{CD}(z_D^* \Lambda_1^* + z_D \Lambda_2)]^q, \tag{50}$$

where p and q are the number of the derivatives ∂_A and ∂_A^*. In this expression one must regard the operators Λ and the vectors z formally as numbers with respect to each other. The tensor indices are easily written down. Now the expression (50) will be rewritten in the form

$$D^{LN\omega} \sim \sum_{m=0}^{(L+2\omega)/2} \sum_{n=0}^{(L-2\omega)/2} \binom{(L+2\omega)/2}{m}\binom{(L-2\omega)/2}{n} \times \mathcal{P}^L (z^{L/2+(\omega-m+n)} z^{*L/2-(\omega-m+n)}) \Lambda_1^{L/2+\omega-m} \Lambda_1^{*L/2-\omega-n} \Lambda_2^{m+n},$$

where $\binom{s}{m}$ are the binomial coefficients. We have expressed here p and q in terms of L and ω. Noting that $\kappa = \omega - m + n$, we go over from summation over m and n to summation over κ and m. We obtain

$$D^{LN\omega} = g^{AB}g^{CD}\ldots g^{FG} \sum_{\varkappa=-L/2}^{L/2} T^{L\varkappa}_{BD\ldots G} \sum_{m=0}^{(L+2\omega)/2} \binom{(L+2\omega)/2}{m} \times \binom{(L-2\omega)/2}{\varkappa - \omega + m} \Lambda_1^{L/2+\omega-m} \Lambda_1^{*L/2-\varkappa-m} \Lambda_2^{2m+\varkappa-\omega} \tag{51}$$

For L = 2 we arrive at the expressions (47), (48), (49) as we should.

The scalar polynomial, on which the differential tensor (51) acts, is of the form (see (I.33)).

$$P_{K^\nu} = \rho^K \sigma^\nu A^{|\nu|} P^{(|\nu|,0)}_{(\mu-\nu)/2}(1 - 2A^2), \tag{52}$$

where μ = K/2. All the operators Λ, Eqs. (43), (44) and (46), lower the order of ρ by two units. The operator Λ_1 lowers, in addition, by one unit the orders of σ and A, and the operator Λ_1^* lowers by unity the order of A and increases by unity the order of σ. The action of the operators Λ on the scalar polynomial, Eq. (52), may be expressed with the help of the well known properties of Jacobi polynomials,[12] in the form

$$\Lambda_1 P_K^\nu = -(\mu+\nu)\rho^{K-2}\sigma^{\nu-1}A^{\nu-1}P^{(\nu-1,1)}_{(\mu-\nu)/2}(1-2A^2), \quad (53)$$

$$\Lambda_2 P_K^\nu = (\mu+\nu)\rho^{K-2}\sigma^{\nu}A^{\nu}P^{(\nu,1)}_{(\mu-\nu)/2-1}(1-2A^2), \quad (54)$$

$$\Lambda_1^* P_K^\nu = (\mu+\nu+2)\rho^{K-2}\sigma^{\nu+1}A^{\nu+1}P^{(\nu+1,1)}_{(\mu-\nu)/2-1}(1-2A^2). \quad (55)$$

The formulae for successive application of the operators Λ are given in the Appendix I.

Let us establish the intervals of values which the numbers K' and ν' of the scalar polynomial, Eq. (52), may take on. We assume that the quantum numbers K, L, ν of the resultant polynomial are given. It is clear that K' can take on only one value, $K' = K + L$. The interval of values for ν' is determined with the help of the equality $\nu = \nu' - \omega$ by two relations

$$-K/2 \leqslant \nu' - \omega \leqslant K/2, \qquad -L/2 \leqslant \omega \leqslant L/2. \quad (56)$$

ν' takes on only integer even or odd values at intervals of two units, depending on whether $K'/2$ is even or odd. If we consider pseudopolynomials with the quantum numbers K, L, ν we obtain in place of the relations (56)

$$-(K-1)/2 \leqslant \nu' - \omega \leqslant (K-1)/2,$$

$$-(L-1)/2 \leqslant \omega \leqslant (L-1)/2. \quad (57)$$

The inequalities corresponding to a given set of numbers K, L, ω will be given in Appendix III in the process of enumerating the polynomials.

Let us consider the properties of the resultant polynomials.

1. The polynomials represent an expansion in terms of tensors constructed out of the two vectors z_i and z_i^*, with κ running over all $L + 1$ values. The scalar part of the polynomials is such that each term of the superposition has a definite quantum number $\nu = \tau + \kappa$.

2. It is clear from the expressions (42) and (45) for the derivatives ∂_A and ∂_B^* that the vectors corresponding to them are $g^{AB}z_B$ and $g^{AB}z_B^*$. From here we conclude that the polynomials obtained with the help of the differential tensor (51) has a definite projection M of the total orbital angular momentum on the z axis, equal to $-N$.

3. Let us construct now polynomials which possess definite symmetry with respect to particle permutations. We note that under permutation the vectors z and z^* go over into each other with certain phase factors (see (I.10) and (I.11)). In particular, no phase factor appears under the permutation P_{12}. In general the scalar part of the polynomials (by scalar part we mean henceforth the function which depends only on A) are invariant with respect to permutations as a result of invariance of A (I.29). It follows hence that the symmetric and antisymmetric functions with respect to, for example, the permutation P_{12} should be constructed out of tensor functions with identical scalar parts for the conjugate tensors. Such functions may be found making use of the properties of the polynomial P_K^ν, Eq. (52), with $L = 0$ and the differential tensor $D^{LN\omega}$, Eq. (51). First of all we note that all the polynomials can be obtained with the help of the differential tensor (51) with $\omega \geq 0$ and the permutation operator P_{12}. Indeed the polynomial in its general form can be written as follows:

$$u^{\nu,\omega} = D^{LN\omega}P_K^{\nu+\omega}. \quad (58)$$

In the symbol on the left we omit all indices inessential for the following discussion. Let us apply to (58) the operation P_{12}. Making use of (35), which is also valid for the tensor (51), we obtain

$$P_{12}u^{\nu,\omega} = P_{12}D^{LN\omega}P_K^{\nu+\omega} = D^{LN-\omega}P_K^{\ -\omega} = u^{-\nu,-\omega}. \quad (59)$$

It is clear that the functions $u^{\nu,\omega}$ and $u^{-\nu,-\omega}$ possess the desired property, i.e., have identical scalar parts for the conjugate tensors, since the permutation P_{12} transforms the vectors z and z^* into each other without affecting the scalar part. According to (35), the projection M of the total orbital angular momentum is unchanged here. The permutation P_{13} and P_{23} give respectively:

$$P_{13}u^{\nu,\omega} = u^{-\nu,-\omega}e^{-4\pi i\nu/3}, \qquad P_{23}u^{\nu,\omega} = u^{-\nu,-\omega}e^{4\pi i\nu/3}. \quad (59')$$

It is now a simple matter to construct polynomials with definite symmetry with respect to permutation of particles. Taking (35) and (59) into account we find

$$u_1 = \frac{1}{\sqrt{2}}(1+P_{12})D^{LN\omega}P_{K+L}^{\nu+\omega}, \quad (60)$$

$$u_2 = \frac{1}{\sqrt{2}i}(1-P_{12})D^{LN\omega}P_{K+L}^{\nu+\omega}\begin{cases}1, & \nu = \frac{3}{2}n\\ \frac{2}{\sqrt{3}}\sin\frac{4\pi}{3}\nu, & \nu \neq \frac{3}{2}n\end{cases}, \quad (61)$$

where n is an integer.

If the starting polynomial (58) has ν equal to a multiple of $^3/_2$ (I.19), then the functions u_1 are symmetric and u_2 antisymmetric. For other values of ν the functions u_1 and u_2 with the same quantum numbers form in pairs the two-dimensional representation of the permutation group on three particles. In the starting polynomial (58) ω runs over integer (for L even) or half integer (for L odd) positive values:

$$0 \leqslant \omega \leqslant L/2. \quad (62)$$

The lower limit is reached only for even L. The index ν for $\omega \neq 0$ runs over both positive and negative values. ν is positive ($\nu \geq 0$) for $\omega = 0$.

4. The meaning of the quantum number ω and the independence of the resultant polynomials in ω are studied in Appendix II. Here we shall only say that from L + 1 independent tensors constructed out of two vectors z and z* it is possible to construct L + 1 independent combinations satisfying the Laplace equation (I.20), with coefficients depending on the scalars constructed out of z and z*.

The pseudotensor functions are obtained in an analogous manner. In spherical coordinates the pseudovector (19) has the form

$$A_+ = -i(z_0 z_+^* - z_+ z_0^*), \quad A_0 = i(z_+ z_-^* - z_- z_+^*), \quad A_- = i(z_0 z_-^* - z_- z_0^*). \tag{63}$$

Application of the permutation operator P_{12} gives

$$P_{12} A_B = -A_B. \tag{64}$$

The operation of complex conjugation, with (36) taken into account, gives rise to the expression

$$A_B^* = g^{BC} A_C. \tag{65}$$

We set in correspondence to the pseudovector A a differential operator ∂^A in just the same way as we set in correspondence to the differential operator ∂ and ∂^* to the vectors z and z*. Transformations analogous to those which gave rise to the expression (47) now give rise to

$$\partial_B{}^A = -g^{BC} A_C (\Lambda_1 \Lambda_1^* - \Lambda_2^2). \tag{66}$$

The pseudodifferential tensor will differ from the differential tensor (50) only in that it will contain once the factor ∂^A, Eq. (66).

The pseudopolynomials with definite symmetry with respect to permutations are now constructed very simply. They differ from (60) and (61) only by the sign at the operator P_{12}, as follows from (64). Thus

$$u_1{}^A = \frac{1}{\sqrt{2}}(1 - P_{12}) D_A{}^{LN\omega} P_{K+L+1}^{\nu+\omega}, \tag{67}$$

$$u_2{}^A = \frac{1}{i\sqrt{2}}(1 + P_{12}) D_A{}^{LN\omega} P_{K+L+1}^{\nu+\omega} \begin{cases} 1, & \nu = \frac{3}{2} n \\ \frac{2}{\sqrt{3}} \sin \frac{4\pi}{3} \nu, & \nu \neq \frac{3}{2} n \end{cases}. \tag{68}$$

6. COMPLETE ORTHOGONAL SET FOR L = 0, 1, 2

To save space we shall only give the basis functions out of which the functions with definite symmetry with respect to permutations of particles are obtained trivially in accordance with (60) and (61) for the tensors and (67) and (68) for the pseudotensors. The functions for L = 0 are given for completeness and are taken from [1], Eq. (I.33). All functions are normalized except for the tensor version L = 2. The full ν, Eq. (4), is indicated in the notation for the polynomials on the left; for convenience in writing the scalar part of the polynomials, the part ϵ is extracted out of ν.

Tensor Version

Case L = 0:

$$\nu = -K/2, \; -K/2 + 2, \ldots, K/2;$$
$$u_K{}^\nu = ((K+2)/2\pi^3)^{1/2} H_n^{(\nu, 0)}.$$

Case L = 1:

$$\varepsilon = -\frac{K-1}{2}, \; -\frac{K-1}{2} + 2, \; \ldots, \; \frac{K-1}{2};$$
$$u_K{}^{\varepsilon+1/2} = \left[\frac{3}{4} \frac{K+2\varepsilon+3}{\pi^3}\right]^{1/2} \left\{ \frac{z_A}{\rho} H_n{}^{(\varepsilon, 1)} - \frac{z_A^*}{\rho} H_{n-1}^{(\varepsilon+1, 1)} \right\}.$$

Case L = 2:

$$\omega = 1; \quad \varepsilon = -\frac{K-2}{2}, \; -\frac{K-2}{2} + 2, \; \ldots, \; \frac{K-2}{2};$$

$$\begin{aligned} {}_1u_K{}^{\varepsilon+1} \sim & \left(\frac{K-2}{2} + \varepsilon + 2\right) T^{2M1} H_n{}^{(\varepsilon, 2)} \\ & - 2\left(\frac{K-2}{2} + \varepsilon + 2\right) T^{2M0} H_{n-1}^{(\varepsilon+1, 1)} \\ & + T^{2M-1}\left[H_{n-1}^{(\varepsilon+2, 1)} + \left(\frac{K-2}{2} + \varepsilon + 2\right) H_{n-2}^{(\varepsilon+2, 2)}\right]; \end{aligned}$$

$$\omega = 0; \quad \varepsilon = 0, 2, \ldots, \frac{K-2}{2}, \quad \text{if } \frac{K-2}{2} \text{ is even};$$
$$\varepsilon = 1, 3, \ldots, \frac{K-2}{2}, \quad \text{if } \frac{K-2}{2} \text{ is odd};$$

$$\begin{aligned} {}_0u_K{}^\varepsilon \sim & -\left(\frac{K-2}{2} + \varepsilon\right) T^{2M1} H_n^{(\varepsilon-1, 2)} \\ & T^{2M0}\left[H_n^{(\varepsilon, 1)} + \left(\frac{K-2}{2} + \varepsilon\right) H_{n-1}^{(\varepsilon, 2)} \right. \\ & \left. - \left(\frac{K-2}{2} + \varepsilon + 4\right) H_n^{(\varepsilon, 2)}\right] \\ & \times \left(\frac{K-2}{2} + \varepsilon + 4\right) T^{2M-1} H_{n-1}^{(\varepsilon+1, 2)}. \end{aligned}$$

Pseudotensor Version

Case L = 1:

$$\varepsilon = -\frac{K-2}{2}, \; -\frac{K-2}{2} + 2, \; \ldots, \; \frac{K-2}{2};$$

$$\tilde{u}_K{}^{\varepsilon} = \left[3\left(\frac{K-2}{2}+\varepsilon\right)\left(\frac{K-2}{2}+\varepsilon+2\right)\right]^{1/2}$$
$$\times\left[\left(\frac{K-2}{2}-\varepsilon+2\right)\pi^3\right]^{-1/2}\left[\frac{z}{\rho},\frac{z^*}{\rho}\right]H_{\tilde{n}}^{(\varepsilon,1)}.$$

Case L = 2:

$$\varepsilon = -\frac{K-3}{2},\ -\frac{K-3}{2}+2,\ \ldots,\ \frac{K-3}{2};$$

$$\tilde{u}_K{}^{\varepsilon+1/2} = -\frac{3}{2}\left[\left(\frac{K-3}{2}+\varepsilon+2\right)\left(\frac{K-3}{2}+\varepsilon+4\right)\right]^{1/2}$$
$$\times\left[\left(\frac{K-3}{2}-\varepsilon+2\right)\pi^3\right]^{-1/2}$$
$$\times\{\tilde{T}^{2M1/2}H_{\tilde{n}}^{(\varepsilon,2)}-\tilde{T}^{2M-1/2}H_{\tilde{n}-1}^{(\varepsilon+1,2)}\}.$$

In writing the polynomials above we have made use of the following notation:

$$n=\left(\frac{K-L}{2}-\varepsilon\right)\Big/2,\qquad \tilde{n}=\left(\frac{K-L-1}{2}-\varepsilon\right)\Big/2,$$

$$H_{n-s}^{(\alpha,\beta)} = \sigma^{\alpha}A^{\alpha}P_{n-s}^{(\alpha,\beta)}(1-2A^2); \tag{69}$$

$H \equiv 0$ for $n - s < 0$;

$$T^{2M1} = \rho^{-2}(z_A z_B - {}^1\!/_3 g^{AB}z^2),$$
$$T^{2M0} = \rho^{-2}\left(\frac{z_A z_B^* + z_A^* z_B}{2} - \frac{1}{3}g^{AB}(zz^*)\right),$$
$$T^{2M-1} = P_{12}T^{2M1},$$
$$\tilde{T}^{2M1/2} = \frac{z_A}{\rho}\left[\frac{z}{\rho},\frac{z^*}{\rho}\right]_B + \frac{z_B}{\rho}\left[\frac{z}{\rho},\frac{z^*}{\rho}\right]_A,$$
$$\tilde{T}^{2M-1/2} = -P_{12}\tilde{T}^{2M1/2}.$$

7. CONCLUSIONS

The polynomials obtained by us constitute a convenient basis for the expansion in terms of them of the wave functions of three nucleons. The symmetry properties are taken into account here very simply, and the Schrödinger equation with spin and isospin taken into account goes over into a system of equations for the partial waves. In this manner we can, for example, take into account the contribution of D-waves to the wave functions of T and He^3, as well as higher partial waves in the problem for the continuum of three nucleons. In addition, the resultant functions constitute the basis for expansion in terms of them of the decay amplitude of a particle of arbitrary spin into three particles and give the distribution of events on a Dalitz graph, which will be discussed in a different paper.

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APPENDIX I

With the help of differentiation formulae and identities for the Jacobi polynomials[12] we express the action of the operators Λ, Eq. (43), (44) and (46), in the following form:

$$\Lambda_1{}^s P_K{}^\nu = (-1)^s(\mu+\nu)(\mu+\nu-2)\ldots(\mu+\nu-2(s-1))$$
$$\times\rho^{K-2s}H_n^{(\nu-s,s)}; \tag{A.1}$$

$$\Lambda_1{}^{*s}P_K{}^\nu = (\mu+\nu+2)(\mu+\nu+4)\ldots(\mu+\nu+2s)$$
$$\times\rho^{K-2s}H_{n-s}^{(\nu+s,s)}. \tag{A.2}$$

Here and below μ = K/2 and $(\mu - \nu)/2$. The function H is defined by the expression (69).

In the case of Λ_2 different expressions arise for even and odd powers of the operator:

$$\Lambda_2{}^s P_K{}^\nu = \rho^{K-2s}(\mu+\nu)(\mu+\nu-2)\ldots\left(\mu+\nu-s+\begin{Bmatrix}2\\1\end{Bmatrix}\right)$$
$$\times\sum_{p=0}^{\begin{Bmatrix}s/2\\(s-1)/2\end{Bmatrix}}\left(\mu+\nu-s-\begin{Bmatrix}0\\1\end{Bmatrix}\right)\left(\mu+\nu-s-\begin{Bmatrix}2\\3\end{Bmatrix}\right)\ldots$$
$$\ldots(\mu+\nu-2s+2p+2)\,Q_{s-p}^{s}\,H_{n-s+p}^{(\nu,\,s-p)}, \tag{A.3}$$

where the upper and lower lines refer to even and odd values of s. The coefficients Q_{s-p}^{s} are determined by the equations

$$Q_{s-p}^{s} = (s-2p+1)\,Q_{s-p}^{s-1} + Q_{s-p-1}^{s-1}. \tag{A.4}$$

For $Q_s^s = 1$ we have $Q_{s-p}^s = 0$ if p is negative; $Q_{s-p}^s = 0$ if p > s/2 for even s and if p > (s - 1)/2 for odd s.

To calculate these coefficients it is convenient to make use of the accompanying table. The lines refer to the upper index of the coefficients Q_{s-p}^s, and the columns to the lower. It follows from (A.4) that we have ones on the main diagonal and zeroes above the main diagonal. The zeroes below the main diagonal are also easily obtained. The sequence in which the remaining "cells" of the table are filled is as follows: The first diagonal is filled from above downwards, then the second, then the third, etc. The diagonals are numbered from the main one downwards. In fact, the number of the diagonal coincides with the value of the index p. The table demonstrates this method for obtaining the coefficients Q_{s-p}^s.

The result of the action of the product of the operators Λ in arbitrary degree can be written in the following form:

$$\Lambda_1{}^t\Lambda_1{}^{*s}P_K{}^\nu = (-1)^t(\mu+\nu+2)(\mu+\nu+4)\ldots(\mu+\nu+2s)$$
$$\times(\mu+\nu)(\mu+\nu-2)\ldots(\mu+\nu-2(t-1))$$
$$\times\rho^{K-2(s+t)}H_{n-s}^{(\nu+s-t,\,s+t)}; \tag{A.5}$$

$$\Lambda_2{}^r\Lambda_1{}^t\Lambda_1{}^{*s}P_K{}^\nu = (-1)^t(\mu+\nu+2)\ldots(\mu+\nu+2s)(\mu+\nu)$$

$$\ldots(\mu+\nu-2(t-1))(\mu+\nu-2t)$$

$$\ldots\left(\mu+\nu-2t-r+\begin{Bmatrix}2\\1\end{Bmatrix}\right)\cdot$$

$$\rho^{K-2(s+t+r)}\sum_{p=0}^{\begin{Bmatrix}r/2\\(r-1)/2\end{Bmatrix}}\left(\mu+\nu-2t-r-\begin{Bmatrix}0\\1\end{Bmatrix}\right)$$

$$\ldots(\mu+\nu-2t-2r+2p+2)\,Q^r_{r-p}\,H^{(\nu+s-t,\ s+t+r-p)}_{n-s-r+p}. \quad (A.6)$$

The coefficients Q^r_{r-p} are the same as above.

It is now obvious how to write an expression for a polynomial with an arbitrary orbital angular momentum with the help of the differential tensor (51).

APPENDIX II

We prove the independence of the polynomials (58) with respect to the quantum number ω as follows. The harmonic polynomials in the variables z, z* satisfy the Laplace equation (I.20). In accord with the property 1 mentioned in Sec. 5, the polynomials have the form

$$P_K = \sum_{\varkappa=-L/2}^{L/2}\ \sum_{\alpha+\beta+\gamma=(K-L)/2} Q^{\varkappa}_{\alpha\beta\gamma}T^{L\varkappa}(z^2)^\alpha(z^{*2})^\beta(\mathbf{zz}^*)^\gamma \quad (A.7)$$

(compare with (I.22)). Since these polynomials satisfy Laplace's equation, the coefficients $Q^K_{\alpha\beta\gamma}$ are related by a set of equations. By investigating the general form it is seen that the system of equations is such that when $\gamma = 0$ the coefficients $Q^K_{\alpha\beta\gamma}$ may be taken arbitrarily. Thus, in order to prove the independence of the resultant polynomials with respect to the quantum number ω, it is sufficient to show that all L + 1 polynomials correspond to an independent set of coefficients $Q^K_{\alpha\beta 0}$. In order to separate the term corresponding to $\gamma = 0$ we note that it contains the maximal degree of A, namely $\alpha + \beta = (K - L)/2$, and that the operator Λ_2 does not change the degree of A, and the operators Λ_1 and Λ_1^* decrease the degree of A by unity. We therefore separate in (51) the terms with the maximum degree of Λ_2, namely the terms

$$\mathcal{P}(z^*\Lambda_2)^p(z\Lambda_2)^q = \mathcal{P}z^qz^{*p}\Lambda_2{}^{p+q}. \quad (A.8)$$

This is equivalent to specifying the coefficient $Q^{-\omega}_{\alpha\beta 0}$, and taking all others equal to zero.

APPENDIX III

Let us establish the total number of functions with a given K, enumerated according to the subscripts L, M, ν, ω. The interval of values assumed by the index ν' of the scalar polynomial (52) can be found by writing the Jacobi polynomial in the expression (II.6) in the form of a hypergeometric series.[12] This gives

$$-\frac{K+L}{2}+2(r+t-p)\leqslant\nu'\leqslant\frac{K+L}{2}-2(s+r-p). \quad (A.9)$$

Noting that

$$r+s+t=L,\qquad t-s=2\omega,$$

we obtain

$$-\frac{K-L}{2}+2\omega+(r-2p)\leqslant\nu'\leqslant\frac{K-L}{2}+2\omega-(r-2p). \quad (A.10)$$

The maximal interval of possible values of ν' is therefore

$$-\frac{K-L}{2}+2\omega\leqslant\nu'\leqslant\frac{K-L}{2}+2\omega; \quad (A.11)$$

Altogether we have (K − L)/2 + 1 values, disregarding the dependence on the index ω. Recognizing now that ω runs over L + 1 values, and the projection M runs over 2L + 1 values, the total number of functions with a given K can be ex-

s	s−p					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	$s-1=1$	1	0	0	0	0
3	0	$s-1+1=3$	1	0	0	0
4	0	$(s-3)\cdot 3+0=3$	$s-1+3=6$	1	0	0
5	0	0	$(s-3).6+3=15$	$s-1+6=10$	1	0
6	0	0	$(s-5)\cdot 15+0=15$	$(s-3).10+15=45$	$s-1+10=15$	1
7	0	0	0	$(s-5)\cdot 45+15=105$	$(s-3)\cdot 15+45=105$	$s-1+15=21$

pressed in the tensor version in the form of a sum over L:

$$n_K{}^V = \sum_L \left(\frac{K-L}{2}+1\right)(L+1)(2L+1). \quad \text{(A.12)}$$

In the pseudotensor version we arrive at an analogous expression

$$n_K{}^A = \sum_L \left(\frac{K-L-1}{2}+1\right)L(2L+1). \quad \text{(A.13)}$$

Since the degree of the scalar part of the polynomials is even, it follows that L and K are both even or both odd in the tensor version, and that in the pseudotensor version L is odd when K is even and vice versa. We find

$$\left.\begin{aligned} n_K^V &= {}^1/_{24}(K+2)(K^3+9K^2+23K+12)\\ n_K^A &= {}^1/_{24}(K+2)(K^3+3K^2-K)\end{aligned}\right\}, K, \text{ even;} \quad \text{(A.14)}$$

$$\left.\begin{aligned} n_K^V &= {}^1/_{24}(K+1)(K+3)(K^2+7K+10)\\ n_K^A &= {}^1/_{24}(K+1)(K-1)(K^2+5K+6)\end{aligned}\right\}, K, \text{ odd:} \quad \text{(A.15)}$$

Hence the total number of tensor and pseudotensor polynomials for arbitrary K equals

$$n_K = (K+3)!(K+2)/12K!, \quad \text{(A.16)}$$

as it should.

From (A.11) we obtain the range of variation of ν:

$$-\frac{K-L}{2}+\omega \leqslant \nu \leqslant \frac{K-L}{2}+\omega. \quad \text{(A.17)}$$

Together with the condition that ν be a multiple of $^3/_2$,

$$\nu = {}^3/_2 n, \quad \begin{aligned} &n = 0 \pm 1, \pm 2, \ldots && \text{for } \omega \neq 0,\\ &n = 0, 1, 2, \ldots && \text{for } \omega = 0. \end{aligned} \quad \text{(A.18)}$$

This interval determines the number of symmetric and antisymmetric functions.

In the Appendices it is important to know the K_{min} for which symmetric and antisymmetric functions appear for a given L. From the inequality (A.17) and the condition (A.18) we find that for the symmetric functions K_{min} coincides with L in the case of even L. For odd L we have $K_{min} = L$ if $L \geq 3$ and $K_{min} = 3$ for $L = 1$. For the antisymmetric functions, we have $K_{min} = L$ for odd L, if $L \geq 3$ and $K_{min} = 3$ for $L = 1$. In the case of even L we have $K_{min} = L$ if $L \geq 6$ and $K_{min} = 6$ for $L = 0, 2, 4$.

In the pseudotensor version K_{min} is larger by one unit.

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