THEORY OF AMPLIFICATION OF LONGITUDINAL WAVES BY A CHARGED PARTICLE BEAM IN A NONLINEAR PLASMA

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The one-dimensional problem of amplification of a monochromatic longitudinal wave by a charged-particle beam in a nonlinear plasma, which is described by a dielectric constant $\epsilon \equiv 1 - (\omega_p^2/\omega^2)\exp(-E^2/E_0^2)$ (ω_p is the plasma frequency, ω the operating frequency, E the amplitude of the excited field; $\omega < \omega_p$) is considered. It is shown that for a sufficiently high beam density, the reaction of the excited oscillations on the motion of the beam particles can be neglected, at least in the vicinity of plasma resonance ($|\omega - \omega_p| \ll \omega_p$). The maximal amplitude of the amplified wave, and also the dependence of the amplitude on the coordinate, are found. The possibility of appearance of accelerated particles at the amplifier output is noted.

As is well known, upon passage of a beam of charged particles through a plasma, amplification of longitudinal oscillations takes place at frequencies $\omega < \omega_p$ for which the dielectric constant of the plasma is negative. In the linear theory, the amplitude of the oscillations here increases exponentially with the coordinate. However, at large amplitudes, the linear theory is inapplicable, and it appears to be necessary to take into account the nonlinear effects of the interaction of the beam with the plasma.

One of these effects is the reaction of the excited oscillations on the motion of the beam of charged particles. This effect is the decisive one if the beam density is small while the frequencies of the oscillations excited by it are close to the resonant frequencies of the particles of the beam. Under the condition that there is no correlation of the phases of the excited oscillations, this effect can be regarded in a quasilinear approximation; [1,2] the case of oscillations with fixed phase was considered in [3-5].

In the other limiting case, when the beam density is not small, and the frequencies of the amplified oscillations are close to the resonant frequency of the plasma, the most significant situation is the nonlinear dependence of the parameters of the plasma on the amplitude of the excited oscillations, which leads to a dependence of the growth increment (the amplification coefficient) on the amplitude of oscillations.[6] We shall consider below the effect of this on the amplification of the longitudinal oscillations in a plasma by a monoenergetic beam of charged particles.

The initial system of equations consists of the equations of motion and continuity for a beam of particles and Poisson's equation:

$$\frac{\partial v}{dt} + v\frac{\partial v}{\partial z} = \frac{e}{m}E, \quad \frac{\partial}{\partial t}n + \frac{\partial}{\partial z}nv = 0,$$

$$\frac{\partial D}{\partial z} = 4\pi e(n - n_b), \quad D = \varepsilon(E)E, \tag{1}$$

where n is the density of the plasma beam and v its velocity (n_b is the equilibrium beam density). The nonlinear properties of the plasma here are taken into account by the dielectric constant ϵ (E), the dependence of which on the amplitude of the field is given in a number of researches.[7-9] We have

$$\varepsilon(E) = 1 - \frac{\omega_p^2}{\omega^2}\exp\left(-\frac{|E|^2}{E_0^2}\right),$$

$$\omega_p^2 \equiv \frac{4\pi e^2}{m}n_p, \quad E_0^2 = \frac{8\omega^2 mkT}{e^2}; \tag{1a}$$

Here n_p is the density of the plasma at the point where the amplitude of the field is equal to zero.[1)]

Neglecting nonlinear effects in the equations of motion of the beam particles (1) (an estimate of

[1)]Equation (1a) was obtained in [7-9] for a transverse field with amplitude depending on the longitudinal coordinate. For the case of a longitudinal field, a similar expression can be obtained by expansion of the hydrodynamic equation of motion of the particles of the plasma in the amplitude of the field, with subsequent averaging over the period of oscillation.

the conditions of applicability of this approximation will be given below), we obtain the following equation for the field E:

$$\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial z}\right)^2 D+\omega_b{}^2 E=0. \quad (2)$$

We shall seek a solution of this equation in the form of a wave traveling with a velocity equal to the beam velocity v_0, and a z-dependent amplitude:

$$E=E_0 y(z)\cos[\omega(t-z/v_0)]. \quad (3)$$

In this case,

$$\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial z}\right)\cos[\omega(t-z/v_0)]=0,$$

and from Eq. (2) we get for the dimensionless amplitude of the field y(z) the ordinary differential equation:

$$\frac{d^2}{dx^2}[y\varepsilon(y)]+\nu^2 y=0;$$
$$\nu^2\equiv n_b/n_p,\quad x\equiv\omega_p z/v_0. \quad (4)$$

In the linear approximation ($y \ll 1$), $\epsilon(y) = \epsilon(0) < 0$, and the exponential increase in the amplitude of y along the x coordinate follows from Eq. (4), with the amplification coefficient $\alpha \equiv \nu|\epsilon(0)|^{-1/2}$.

It is most convenient to study the nonlinear solutions of Eq. (4) by means of the first integral of this equation, which we shall write down for the case of small field amplitude y, taking into account the smallness of $|\epsilon(0)|$:

$$y'^2[|\varepsilon(0)|-3y^2]^2+\nu^2y^2[\varepsilon(0)+{}^3/_2y^2]=C. \quad (5)$$

First of all, it is necessary to note that, in accord with (5), there is a maximal amplitude, determined by the condition $y'(y_m) = 0$. The value of this amplitude, and also the change in the field amplitude with the coordinate, are determined by (5) and depend materially on the value of the constant C. When C = 0, the solution of (5) has the form of a unit pulse with maximum amplitude $y_m = [{}^2/_3|\epsilon(0)|]^{1/2}$. Thus, for frequencies close to the plasma frequency, the maximum amplitude is actually small. The characteristic dimensions of the pulse L are determined by the amplification coefficient of linear theory: $L \sim \alpha^{-1}$.

It is easy to see that for $C \neq 0$, the solution of Eq. (5) varies periodically with the coordinate; for $C > 0$, we have $y_{min} = -y_{max}$, and for $C < 0$, we have $1 > y_{min}/y_{max} > 0$. For the form of the dependence of the field on the coordinate and time chosen by us (3), the function y(x) determines the field in the coordinate system in which the beam is at rest. It then follows that in the given system, for $C \leq 0$, each particle moves in a field that does not change sign, and can be retarded or accelerated, depending on the phase of the modulating field. Inasmuch as we have not taken into account the effect of the amplified field on the motion of the beam particles, assuming this motion to be given, the condition for applicability of the considerations given above amounts to the requirement of the smallness of the displacement of the particle $\Delta z = eE_0y_mL^2/mv_0^2$ under the action of the field at distances of the order of a period of the stationary structure L in comparison with the length of the amplified wave v_0/ω_p ($\alpha \lesssim 1$). This condition is essentially identical with the condition of the possible neglect of terms $\tilde{v}\nabla\tilde{v}$ in the equations of motion of the beam, and leads to the inequality

$$|\varepsilon(0)|^{3/2}n_p v_T/n_b v_0 \ll 1,$$

which we assume to be satisfied. Thus, our consideration is actually valid for not too small a beam density and in the vicinity of the plasma resonance.

It must be noted that the considerations given above are not valid close to the points $y_c \equiv \pm y_m/\sqrt{2}$, where the nonlinear dielectric constant of the plasma, which is defined as the derivative of the induction with respect to the amplitude of the electric field, vanishes. Near these points, the derivative with respect to the field amplitude increases; therefore, one must consider spatial dispersion of the dielectric constant of the plasma (high-frequency contribution to the pressure gradient of the plasma), which is not taken into account in the dielectric constant (1a). Far from the points, where $y^2 = y_c^2$, account of these effects leads to corrections of the order of v_T^2/v_0^2, which we assume to be small.

Account of spatial dispersion in the vicinity of the critical points leads to the following equation for the field amplitude y(x) (under the assumption that the dimensions of this region are small in comparison with the wavelength v_0/ω_p):

$$\mu^2\frac{d^4}{dx^4}y+\frac{d^2}{dx^2}[y\varepsilon(y)]+\nu^2y=0,\quad \mu=\frac{v_T}{v_0}. \quad (6)$$

We must study the solution of this equation in the vicinity of the singular points, where the difference $w \equiv y - y_c$ is small. In this case, the order of Eq. (6) can be reduced. By integrating twice with respect to $\xi \equiv x - x_c$, we get

$$\mu^2\frac{d^2w}{dx^2}+3y_cw^2=C_1\xi, \quad (7)$$

where C_1 is a constant of integration (which can be expressed in terms of the constant C of Eq. (5)). The theory of the last equation is set forth in[10], in which it was shown that its solution is not expressed in terms of elementary functions. We are interested in the behavior of the solution of Eq. (6) in the vicinity of the point $w = 0$, $\xi = 0$. In this case, the solutions can be sought in the form of a series in powers of ξ:

$$w = C_2\xi + \frac{C_1}{6\mu^2}\xi^3 + O(\xi^4). \quad (8)$$

Thus the account of spatial dispersion leads to the result that the derivative w' remains finite as $\xi \to 0$.

This statement is valid, according to (8) only if $\xi \ll \mu$. For $\xi \gg \mu$, when w is not too small, we can neglect the term in Eq. (7) with the second derivative, as a result of which we obtain $w \sim (|\xi|)^{1/2}$ and $w' \sim (|\xi|)^{-1/2}$, that is, growth of the derivative with decrease in $|\xi|$. Substituting this solution in (7), we can establish the fact that it is valid for $\xi \gg \mu^{4/5}$, i.e., at distances greater than the Debye radius. Thus, far from the critical points, we can solve Eq. (5) and match the solutions on both sides of the cut, starting from the condition of continuity of the field and its derivative.

It must also be emphasized that the particles of the beam at the output of the plasma layer can have an energy greater than the energy of inertia. Such an effect can obviously lead to the appearance of fast particles in the experiments.[11]

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51