ANALYTIC PROPERTIES OF THE SCATTERING AMPLITUDE IN A THEORY WITH THE LAGRANGIAN $g\varphi^4$

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The analytic properties of the scattering amplitude as a function of the coupling constant g and the angular momentum λ are considered. It is found that for a potential of the form $b/x^2 \ln x$ in a theory with the Lagrangian $g\varphi^4$, the scattering amplitude is analytic in g at the point $g = 0$ and has a logarithmic branch point at $\lambda = 0$ for $k^2 = 0$.

1. IN recent times a number of authors[1-7] have investigated the analytic properties of the scattering amplitude in the coupling constant g and the angular momentum λ by considering a theory with the Lagrangian $g\varphi^4$ in the ladder approximation. Thus it was shown in [1, 2] that for a potential

$$V(r) = -a\frac{g^2}{r^2} - b\frac{g^3}{r^2}\ln 2mr, \quad a > 0,\ b > 0,$$

corresponding to the graph of Fig. 1, the scattering amplitude has fixed branch points $\lambda = \pm\sqrt{ag^2} + O(g^2)$ in the λ plane and an essential singularity at $g = 0$ in the plane of the coupling constant. An analogous result was obtained in [3], where the analytic properties of the scattering length as a function of g were investigated for the potential

$$V(r) = \frac{g}{r^2}\ln\frac{R}{r}, \quad r < r_0,$$
$$V(r) = 0, \quad r > r_0.$$

For this potential the scattering length has, at $g = 0$, a branch point and an essential singularity of the type corresponding to a condensation of poles. A branch point of the root type in the λ plane has also been obtained in [4] by summing the most singular terms in an iterated series. The graphs of Fig. 2 were considered in [5, 6, 7]. For example, Charap and Dombey[7] showed that the potential corresponding to the graph of Fig. 2b, leads to an essential singularity of the wave function in the angular momentum λ at the point $\lambda = 0$. The question whether this essential singularity also appears in the scattering amplitude has not yet been resolved.

In the present note we consider the analytic properties of the scattering amplitude in g ($g \to 0$) and λ ($\lambda \to 0$) in a theory with the interaction $g\varphi^4$, using the ladder approximation; the graph of lowest order will in this case be the graph of Fig. 2b. We use the method proposed by Filippov,[1] who obtained a differential equation—the relativistic analog of the Schrödinger equation—from the quasi-potential equation for the partial scattering amplitude and found the corresponding general expression for the local potential.

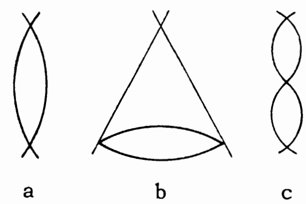

FIG. 1

2. The potential corresponding to the graph of Fig. 2b has the form

$$V(x) = a\frac{g}{x^2\ln x}, \quad x < x_0,$$
$$V(x) = 0, \quad x > x_0, \tag{1}$$

where

$$a = -(2\pi)^2\gamma(\lambda), \quad g > 0, \quad x_0 < 1,$$
$$\gamma(\lambda) = \frac{[\Gamma(\lambda+1)]^2}{(\lambda+{}^1/_2)[\Gamma(\lambda+{}^1/_2)]^2}.$$

The differential equation for the "wave function" $\varphi_\lambda(x)$ introduced in [1] is written in the form

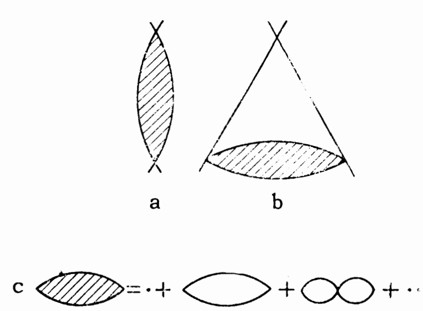

FIG. 2

$$\left[\frac{d^2}{dx^2}-\frac{\lambda^2-1/4}{x^2}+k^2\right]\varphi_\lambda(g,k,x)=V(x)\varphi_\lambda(g,k,x), \quad (2)$$

where $\lambda = l$ is the orbital angular momentum,[1)] $k^2 = k'^2/4m^2$, k' is the wave vector, and m is the mass of the intermediate particle.

For $x = 0$ we have the following boundary condition:

$$\varphi_\lambda(g,\ k,\ x)\,|_{x\to 0}=0.$$

A particular solution of (2) satisfying this condition can be found from the integral equation ($x < x_0$):

$$\varphi_\lambda'(g,k,x)=\Psi_\lambda^{(1)}(g,0,x)+\frac{k^2\Gamma(1-ag/2\lambda)}{2\lambda}$$
$$\times\int_0^x[\Psi_\lambda^{(1)}(g,0,x)\Psi_\lambda^{(2)}(g,0,x')$$
$$-\Psi_\lambda^{(1)}(g,0,x')\Psi_\lambda^{(2)}(g,0,x)]\,\varphi_\lambda'(g,k,x')\,dx', \quad (3)$$

where

$$\Psi_\lambda^{(1)}(g,0,x)=x^{\lambda+1/2}\,\Psi\left(-\frac{ag}{2\lambda},0,-2\lambda\ln x\right), \quad (4)$$

$$\Psi_\lambda^{(2)}(g,0,x)=x^{\lambda+1/2}(-2\lambda\ln x)$$
$$\times\Phi\left(1-\frac{ag}{2\lambda},2,-2\lambda\ln x\right), \quad (5)$$

and Ψ and Φ are confluent hypergeometric functions of the second and first kind.[8] For $x \to 0$ we have

$$\Psi_\lambda^{(1)}(g,\ 0,\ x)\ \sim\ x^{1/2+\lambda}(-2\lambda\ln x)^{ag/2\lambda},$$
$$\Psi_\lambda^{(2)}(g,\ 0,\ x)\ \sim\ x^{1/2-\lambda}(-2\lambda\ln x)^{-ag/2\lambda}.$$

The other independent particular solution which behaves like $\Psi_\lambda^{(2)}(g, 0, x)$ at the origin becomes infinite at $x = 0$ for integer λ. Therefore we consider the following solution of (2) in the region $x < x_0$:

$$\varphi_\lambda(g,\ k,\ x)\ =\ \mathrm{const}\,\varphi_\lambda'(g,\ k,\ x).$$

The boundary condition at infinity has the form

$$\varphi_\lambda(g,k,x)\,|_{x\to\infty}\sim\frac{\sin(kx-l'\pi/2)}{k}$$
$$-\frac{\pi}{2}\frac{1}{k^2}\frac{1}{\sqrt{k^2+m^2}}A_\lambda(g,k,x_0)\,e^{i(kx-l'\pi/2)}, \quad (6)$$

where $l' = \lambda - 1/2$ and $A_\lambda(g, k, x_0)$ is the relativistic scattering amplitude[2)] which coincides with $A_\lambda(g, p, p', x_0)$ for $p = p' = k$; the latter satisfies the relativistic Lippmann-Schwinger equation.[1] The condition (6) can be obtained easily by taking account of the relation between the p representation of $\varphi_\lambda(x)$ and $A_\lambda(g, p, k, x_0)$.[1]

The Jost solution in the region $x > x_0$ is taken in the form

$$g_\lambda(g,\ k,\ x)\ =\ (1/2\pi k^{-1}x)^{1/2}J_\lambda(kx),$$
$$f_\lambda(g,\ -k,\ x)\ =\ (1/2\pi kx)^{1/2}iH_\lambda^{(1)}(kx),$$

where J_λ and $H_\lambda^{(1)}$ are the Bessel and Hankel functions. From this we obtain in the usual way

$$A_\lambda(g,k,x_0)=\frac{2k}{\pi i}\sqrt{k^2+m^2}$$
$$\times\frac{kJ_\lambda'(kx_0)+B_\lambda(g,k,x_0)J_\lambda(kx_0)}{kH_\lambda^{(1)\prime}(kx_0)+B_\lambda(g,k,x_0)H_\lambda^{(1)}(kx_0)}, \quad (7)$$

and

$$B_\lambda(g,k,x_0)=\frac{1}{2x_0}-\frac{\varphi_\lambda'(g,k,x)_x'}{\varphi_\lambda'(g,k,x)}\Big|_{x=x_0}.$$

3. Of paramount interest in field theory is the behavior of the scattering amplitude for large momenta, which corresponds to small distances. At small distances the analytic properties of the scattering amplitude are determined by the most singular part of the potential which in our case has the form (1). Therefore we consider in the following the case $x_0 \ll 1$. The function $\varphi_\lambda'(g, k, x)$ is analytic in g in the neighborhood of $g = 0$, since it is determined by the Volterra equation (3) and $\Psi(-ag/2\lambda, \beta, -2\lambda \ln x)$ is analytic in g. For small g we find

$$B_\lambda(g,k,x_0)=-\frac{1}{x_0}\left[\lambda+\frac{ag}{2\lambda\ln x_0}+O(g^2)\right],$$

where $O(g^2)$ contains only integer powers of g. Thus $A_\lambda(g, k, x_0)$ has no singularity in g near $g = 0$. For $k^2 = 0$ this also follows from (8) (see below). We note that the potential under consideration does not satisfy the condition

$$\int_0^\eta x\,|V(x)|\,dx<\infty,$$

and the trace of the kernel of the corresponding integral equation diverges.

Let us consider the quantity $f_\lambda(g, x_0)$:

$$f_\lambda(g,x_0)=\lim_{k^2\to 0}\left\{\frac{A_\lambda(g,k,x_0)}{k^{2\lambda+2}}\right\}=2\left(\frac{x_0}{2}\right)^{2\lambda}\frac{1}{\lambda[\Gamma(\lambda)]^2}$$
$$\times\left[1-\frac{\Psi(-ag/2\lambda,0,-2\lambda\ln x_0)}{\Psi(-ag/2\lambda,1,-2\lambda\ln x_0)}\right]. \quad (8)$$

In the region $|\lambda| \ll 1$ we use the expansion of the degenerate hypergeometric function $\Psi(\alpha, \beta, z)$ for integer β:

[1)]We note that the "centrifugal" term is here $(l^2 - 1/4)/x^2$ and not $[(l + 1/2)^2 - 1/4]/x^2$, as in the usual Schrödinger equation, and $l \neq 0$.

[2)]$A_\lambda = -2k\sqrt{k^2+m^2}\,\pi^{-1}e^{i\delta_l}\sin\delta_l$.

$$\Psi(\alpha, \beta, z) = \frac{e^{z/2}}{\pi}(-1)^{\beta} z^{(1-\beta)/2} \rho^{1/2} \Gamma(\rho) \sin \alpha\pi [\mathrm{const}(g, \beta, x_0) + O(\lambda) + O(\lambda \ln \lambda)], \quad (9)$$

where

$$\rho = {}^{1}/_{2}\beta - \alpha, \quad \alpha = -ag/2\lambda, \quad z = 2\lambda |\ln x_0|, \quad |\arg z| < \pi.$$

Formula (9) is easily obtained from the representation of $\Psi(\alpha, \beta, z)$ in the form of a series for integer β.[8] Then $f_\lambda(g, x_0)$ is written in the following form:

$$f_\lambda(g, x_0) = 2(x_0/2)^{2\lambda}\lambda[1 + \lambda A], \quad (10)$$

where

$$A = \mathrm{const}(g, x_0)\{1 + O(\lambda) + O(\lambda \ln \lambda)\}.$$

The analytic properties of the scattering amplitude for $k^2 = 0$ in the region $|\lambda| \ll 1$ follow immediately from (10) and (8) with account of the properties of the function $\Psi(\alpha, \beta, z)$.[8] In this region the scattering amplitude has a logarithmic branch point at $\lambda = 0$. The wave function contains a logarithmic branch point and an essential singularity at $\lambda = 0$, as follows from (3), (4), and (9).

4. The example considered above shows that a potential of the type (1) leads to a non-analytic dependence of the relativistic scattering amplitude on the complex angular momentum λ. The replacement of the kernel of Fig. 1b by the kernel of Fig. 2b in the corresponding integral equation alters essentially the analytic properties of the scattering amplitude in the coupling constant. The presence of an essential singularity in λ in the wave function of an equation of the Schrödinger type does not yet mean that this singularity also appears in the scattering amplitude. With obvious modifications the results obtained above are also valid for the usual nonrelativistic Schrödinger equation with a potential of the type (1).

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