SOME REMARKS ON THE SINGULARITIES OF THE COSMOLOGICAL SOLUTIONS OF THE GRAVITATIONAL EQUATIONS

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It is shown that for a space filled with dust matter the solution of the gravitational equation which depends on the maximal number (eight) of physically arbitrary functions of three variables possesses a true singularity.

IN the investigation of the problem of the singularities in the cosmological solutions of the gravitational equations it is of interest to consider the case of a space filled with dust matter (equation of state $p = 0$). We shall be interested in the solution of the gravitational equations which depends on the maximal number of physically arbitrary functions of three variables. It is known that for a free space of type I (in the classification of Petrov) the physical arbitrariness consists of four functions of three variables and a certain number of functions of two variables.[1] For a space filled with matter this arbitrariness is increased by four functions of three variables.[2,4] Thus we shall consider a solution which depends on eight physically arbitrary functions of three variables.

As a criterion for the existence of a physical (true) singularity we shall take the going to infinity of the energy density ϵ on some hypersurface (or on a geometric locus of lower dimensionality). It will be shown below that the solution of interest to us does indeed contain a physical singularity. It follows from the analysis of this solution that the world lines of the dust particles have an enveloping hypersurface. The intersection of the world lines in the neighborhood of their points of tangency with the enveloping hypersurface leads to a uniquely peculiar cumulation of the energy.

1. Let us assume that the physical singularity occurs on some hypersurface $f(x^0, x^1, x^2, x^3) = 0$. In solved form its equation is $x^0 = \varphi(x^1, x^2, x^3)$, where φ is an arbitrary function of three variables. Let us reduce the energy-momentum tensor of matter $T_{\mu\nu}$ to the form $T_{\mu\nu} = \epsilon u_\mu u_\nu$.[1)] The components of the four-dimensional velocity are related by the obvious identity $u^\mu u^\nu g_{\mu\nu} = 1$, by which e.g., the u^0 component can be expressed through the three other components. We shall use a synchronous (semi-geodesical) system of reference, i.e., a system subject to the condition $g_{00} = 1$, $g_{0i} = 0$.

Let us show that the solution of the gravitational equations having near the singularity the form

$$g_{ik} = a_{ik} + b_{ik}z + c_{ik}z^2 + d_{ik}z^3 + e_{ik}z^4 + \dots,$$
$$\varepsilon = \varepsilon^{(-1)}z^{-1} + \varepsilon^{(0)} + \varepsilon^{(1)}z + \dots,$$
$$u_\mu = u_\mu^{(0)} + u_\mu^{(1)}z + u_\mu^{(2)}z^2 + \dots \quad (1.1)$$

($z = f^{1/2}$, where the coefficients are functions of three variables) contains eight physically arbitrary functions of three variables.

Our task consists in the following. Substituting the expansions (1.1) in the Einstein equations and equating the terms with the same powers of z on both sides, we obtain certain relations between the coefficients of the expansions (1.1). These relations allow us in principle, to express all coefficients of the expansions (1.1) in terms of a certain number of coefficients which thus remain arbitrary functions. Since the form of the solution (1.1) does not change in going over to a new synchronous system of reference, this means that it contains four arbitrary functions of three variables allowing for the possibility of such a transformation. Taking account of this circumstance and reducing the number of arbitrary functions by four we find the number of physically arbitrary functions contained in the solution.

Let us write the Einstein equations in the form of two groups of equations:

$$R_{0\nu} = -(T_{0\nu} - {}^1/_2 g_{0\nu}T), \quad (1.2)$$

$$R_{ik} = -(T_{ik} - {}^1/_2 g_{ik}T). \quad (1.3)$$

Let us write down Eqs. (1.2) and (1.3) in the

1)Greek indices run through 0, 1, 2, 3 and Latin indices through 1, 2, 3. Moreover we set the velocity of light and the Einstein gravitational constant equal to unity.

first orders in z. In the approximation z^{-3} we have

$$R_{00} \equiv -{}^1/_8 b(f_{,0})^2 = 0, \quad (1.4)$$

$$R_{0i} \equiv {}^1/_8 f_{,0}(b_i^k f_{,k} + b f_{,i}) = 0, \quad (1.5)$$

$$R_{ik} \equiv -{}^1/_8(N b_{ik} + f_{,i} f_{,k} b - f_{,}^l f_{,i} b_{kl} - f_{,}^l f_{,k} b_{il}) = 0. \quad (1.6)$$

Here $f_{,\nu} \equiv \partial f/\partial x^\nu \neq 0$, $N \equiv (f_{,0})^2 + f_{,i} f_{,}^i \neq 0$. Assuming that $N \neq 0$ we exclude from our consideration the case where the hypersurface is isotropic.[1] The raising and lowering of indices is carried out in the three-dimensional space with the metric a_{ik}. We find from (1.4) to (1.6) that all b_{ik} are equal to zero. The next order of the equations of gravitation is z^{-1}. For ease of calculation we use instead of (1.2) the equations

$$(R^{\mu\nu} - {}^1/_2 g^{\mu\nu} R) f_{,\nu} = -T^{\mu\nu} f_{,\nu}. \quad (1.2')$$

The equations (1.3) are left in the previous form. Then we obtain in the approximation z^{-1}

$$(R^{0\nu} - {}^1/_2 g^{0\nu} R) f_{,\nu} \equiv 0 = -\varepsilon^{(-1)} u^{0(0)} u^{\nu(0)} f_{,\nu}, \quad (1.7)$$

$$(R^{i\nu} - {}^1/_2 g^{i\nu} R) f_{,\nu} \equiv 0 = -\varepsilon^{(-1)} u^{i(0)} u^{\nu(0)} f_{,\nu} \quad (1.8)$$

$$R_{ik} \equiv {}^3/_8 (N d_{ik} + f_{,i} f_{,k} d - f_{,}^l f_{,i} d_{kl} - f_{,}^l f_{,k} d_{il})$$
$$= -\varepsilon^{(-1)} (u_i^{(0)} u_k^{(0)} - {}^1/_2 a_{ik}). \quad (1.9)$$

With the help of (1.9) we can express the quantities d_{ik} in terms of $f_{,\mu}$, a_{ik}, $\epsilon^{(-1)}$, and $u_i^{(0)}$:

$$d_{ik} = \frac{8}{3N}\varepsilon^{(-1)} \left[\frac{u_0^{(0)}}{f_{,0}} (f_{,i} u_k^{(0)} + f_{,k} u_i^{(0)}) - \frac{f_{,i} f_{,k}}{(f_{,0})^2}\left(u_0^{(0)2} - \frac{1}{2}\right) - u_i^{(0)} u_k^{(0)} + \frac{1}{2} a_{ik} \right]. \quad (1.10)$$

It follows from (1.7) and (1.8) that

$$u^{\nu(0)} f_{,\nu} = 0. \quad (1.11)$$

Let us go over to the next order in z ($\sim z^0$) of the gravitational equations. The calculation of the quantities $R_{\mu\nu}$ shows that in this approximation

$$R_{00} \equiv e (f_{,0})^2 + r_{00}, \quad (1.12)$$

$$R_{0i} \equiv f_{,0}(e f_{,i} - e_i^k f_{,k}) + r_{0i}, \quad (1.13)$$

$$R_{ik} \equiv N e_{ik} + f_{,i} f_{,k} e - f_{,}^l f_{,i} e_{kl} - f_{,}^l f_{,k} e_{il} + r_{ik}, \quad (1.14)$$

where $r_{\mu\nu}$ contains terms constructed out of $f_{,\mu}$, a_{ik}, c_{ik}, and their derivatives ($r_{00} + r_{ik} a^{ik} \equiv r$). The quantities d_{ik} do not enter in (1.12) to (1.14) at all. Substituting (1.12) to (1.14) in (1.2′) and (1.3), we find that on the left-hand sides of the equations (1.2′) the terms containing e_{ik} cancel one another; we have then in the approximation z^0

$$(R^{0\nu} - {}^1/_2 g^{0\nu} R) f_{,\nu} \equiv (r^{0\nu} - {}^1/_2 g^{0\nu} r) f_{,\nu}$$
$$= -\varepsilon^{(-1)} u^{0(0)} u^{\nu(1)} f_{,\nu}, \quad (1.15)$$

$$(R^{i\nu} - {}^1/_2 g^{i\nu} R) f_{,\nu} \equiv (r^{i\nu} - {}^1/_2 g^{i\nu} r) f_{,\nu}$$
$$= -\varepsilon^{(-1)} u^{i(0)} u^{\nu(1)} f_{,\nu}, \quad (1.16)$$

$$R_{ik} \equiv N e_{ik} + f_{,i} f_{,k} e - f_{,}^l f_{,k} e_{il} - f_{,}^l f_{,i} e_{kl} + r_{ik}$$
$$= -\varepsilon^{(-1)} (u_i^{(0)} u_k^{(1)} + u_i^{(1)} u_k^{(0)}) - \varepsilon^{(0)} (u_i^{(0)} u_k^{(0)} - {}^1/_2 a_{ik}). \quad (1.17)$$

The equations (1.17) [in analogy to (1.9)] serve for the determination of the quantities e_{ik}, which are expressed in terms of $f_{,\mu}$, a_{ik}, c_{ik}, $\epsilon^{(-1)}$, $\epsilon^{(0)}$, $u_i^{(0)}$, $u_i^{(1)}$. In the following we shall not consider (1.3) any longer since the structure of the resulting relations is clear. From (1.15) we can determine the quantity $u^{\nu(1)} f_{,\nu}$. It will be expressed in terms of $f_{,\mu}$, a_{ik}, c_{ik}, $\epsilon^{(-1)}$, $u_i^{(0)}$. Substituting $u^{\nu(1)} f_{,\nu}$ in (1.16), we obtain three relations between the quantities $f_{,\mu}$, a_{ik}, c_{ik}, and $u_i^{(0)}$:

$$u^{i(0)} (r^{\mu\nu} - {}^1/_2 g^{\mu\nu} r) f_{,\nu} u_\mu^{(0)} = (r^{i\nu} - {}^1/_2 g^{i\nu} r) f_{,\nu}. \quad (1.18)$$

In the next order in z we consider only the equations (1.2′). Clearly they have a form analogous to (1.15) and (1.16):

$$(R^{0\nu} - {}^1/_2 g^{0\nu} R) f_{,\nu} \equiv (s^{0\nu} - {}^1/_2 g^{0\nu} s) f_{,\nu} = -\varepsilon^{(-1)} (u^{0(0)} u^{\nu(2)} f_{,\nu} + u^{0(1)} u^{\nu(1)} f_{,\nu}) - \varepsilon^{(0)} u^{0(0)} u^{\nu(1)} f_{,\nu}, \quad (1.19)$$

$$(R^{i\nu} - {}^1/_2 g^{i\nu} R) f_{,\nu} \equiv (s^{i\nu} - {}^1/_2 g^{i\nu} s) f_{,\nu} = -\varepsilon^{(-1)} (u^{i(0)} u^{\nu(2)} f_{,\nu} + u^{i(1)} u^{\nu(1)} f_{,\nu}) - \varepsilon^{(0)} u^{i(0)} u^{\nu(1)} f_{,\nu}. \quad (1.20)$$

Here $s_{\mu\nu}$ includes terms which contain $f_{,\mu}$, a_{ik}, c_{ik}, d_{ik} and their derivatives ($s_{00} + s_{ik} a^{ik} \equiv s$). The relations (1.19) and (1.20) will be treated in the following way. We determine the quantity $u^{\nu(2)} f_{,\nu}$ from (1.19). Substituting it in (1.20) we obtain three equations for the determination of $u^{i(1)}$. Then $u^{\nu(1)}$ can be eliminated from (1.15) and the latter will then relate the functions $f_{,\mu}$, a_{ik}, c_{ik}, d_{ik}, $\epsilon^{(-1)}$, and $u_0^{i( )}$. The equations (1.20) and (1.15) transformed in this way have the form

$$u^{i(1)} = [(s^{i\nu} - {}^1/_2 g^{i\nu} s) f_{,\nu} - u^{i(0)} (s^{\mu\nu} - {}^1/_2 g^{\mu\nu} s) u_\mu^{(0)} f_{,\nu}] \times [r^{\mu\nu} - {}^1/_2 g^{\mu\nu} r) f_{,\nu} u_\mu^{(0)}]^{-1}, \quad (1.20')$$

$$[(r^{\mu\nu} - {}^1/_2 g^{\mu\nu} r) f_{,\nu} u_\mu^{(0)}]^2 + \varepsilon^{(-1)} (s^{\mu\nu} - {}^1/_2 g^{\mu\nu} s) f_{,\mu} f_{,\nu} = 0. \quad (1.15')$$

If we substitute the expression (1.10) for d_{ik} in (1.15′), then the latter will relate the functions $f_{,\mu}$, a_{ik}, c_{ik}, $\epsilon^{(-1)}$, and $u_i^{(0)}$. With the help of (1.10) and (1.20′) the quantities d_{ik} and $u_i^{(1)}$ can thus be expressed in terms of $f_{,\mu}$, a_{ik}, c_{ik}, $\epsilon^{(-1)}$ and $u_i^{(0)}$. Let us show finally that $\epsilon^{(0)}$ can also be expressed in terms of these functions.

Indeed, let us go over to the next order in (1.2′). In analogy to (1.19) and (1.20), one of the equations will determine $u^{\nu(3)} f_{,\nu}$ and the three quantities $u^{i(2)}$ can be found from the three remaining equations. Eliminating $u^{\nu(2)}$ from (1.19) we find an equation which relates the functions $f_{,\mu}$, a_{ik}, c_{ik}, d_{ik}, e_{ik}, $\epsilon^{(-1)}$, $\epsilon^{(0)}$, and $u^{(0)}$:

$$\varepsilon^{(0)}[(r^{\mu\nu} - {}^1/_2 g^{\mu\nu} r) f_{,\nu} u_\mu^{(0)}]^2 = 2\varepsilon^{(-1)}(s^{\mu\nu} - {}^1/_2 g^{\mu\nu} s) f_{,\nu} u_\mu^{(0)} (r^{\alpha\beta} - {}^1/_2 g^{\alpha\beta} r) f_{,\beta} u_\alpha^{(0)} + [\varepsilon^{(-1)}]^2 (t^{\mu\nu} - {}^1/_2 g^{\mu\nu} t) f_{,\mu} f_{,\nu}. \quad (1.21)$$

Here $t^{\mu\nu}$ includes terms which contain $f_{,\mu}$, a_{ik}, c_{ik}, d_{ik}, e_{ik} and their derivatives ($t_{00} + t_{ik}a^{ik} \equiv t$). If e_{ik} and d_{ik} in (1.21) are replaced by their expressions in terms of the functions $f_{,\mu}$, a_{ik}, c_{ik}, $\epsilon^{(-1)}$, $\epsilon^{(0)}$, and $u_i^{(0)}$, then (1.21) can be regarded as a relation between $\epsilon^{(0)}$ and the functions just mentioned.

It is clear that in the next order in z the equations analogous to (1.21) and (1.20′) determine $\epsilon^{(1)}$ and $u^{i(2)}$, respectively, and the functions e_{ik} can be found from (1.17), as we know.

In the solution (1.1) therefore, the coefficients d_{ik}, $\epsilon^{(0)}$, $u_\mu^{(1)}$ and all the ones following can be expressed in terms of 17 functions of three variables: a_{ik}, c_{ik}, $\epsilon^{(-1)}$, $u_i^{(0)}$, and φ. The function φ is contained in z; its meaning is obvious if the equation of the hypersurface $f = 0$ is formally solved for x_0. The 17 functions are connected by the five relations (1.11), (1.18), and (1.15′). Taking account of the fact that four functions are connected with the possibility of going to another reference system, we find that the solution (1.1) depends on eight physically arbitrary functions of three variables.

Let us now turn to Eq. (1.11), which permits a physical interpretation of the solution under consideration. Since the vector $f_{,\nu}$ is orthogonal to the hypersurface $f = 0$, it follows from (1.11) that the vector $u^{\nu(0)}$ is tangent to this hypersurface. Thus the hypersurface $f = 0$ is an envelope of the world lines (geodesics for $p = 0$) of the particles of the medium. The intersection of the world lines at their points of tangency with the hypersurface $f = 0$ causes the energy density ϵ to go to infinity on this hypersurface (at all points).

We note that the singularity has an essentially non-simultaneous character, since the hypersurface $f = 0$ contains line elements of time-like world lines and is therefore oriented in time.

2. In the solution considered above the components of the metric tensor as well as their first derivatives were regular for $f = 0$, and only the second derivatives became infinite on the hypersurface. This behavior of the metric on the hypersurface where a physical singularity occurs, is by no means inevitable. A synchronous reference system can be chosen such, for example, that the first and second derivatives remain finite and the metric determinant vanishes.

This problem is of interest since it is known that in a synchronous reference system the metric determinant must vanish after a finite time by virtue of one of the gravitational equations. In general, this means that in this reference system the metric has a singularity. In the paper of Lifshitz, Sudakov, and Khalatnikov[3] the geometric reasons for the occurrence of such a singularity were found, which are connected with the fact that in a synchronous reference system the time lines which form a family of geodesics intersect on the enveloping hypersurface (caustic), Lifshitz and Khalatnikov[2,3] have shown that the singularity arising in this way in the solution of the gravitational equations is fictitious in free space, and the solution contains the maximal number (in this case, four) of physically arbitrary functions. The authors also arrive at the conclusion that an analogous result must obtain in the case of space filled with matter. It was shown in [2] that the solution for dust matter will have a singularity on the caustic in such cases when it is possible to choose a "synchronous-comoving" reference system (i.e., when the matter does not rotate).

Let us thus write the expansion of the metric tensor in the neighborhood of the caustic hypersurface in the form proposed in [3]:

$$\begin{aligned} -g_{ab} &= a_{ab} + b_{ab}\tau + c_{ab}\tau^2 + d_{ab}\tau^3 + \ldots, \\ -g_{a3} &= a_{a3}\tau^2 + b_{a3}\tau^3 + \ldots, \\ -g_{33} &= a_{33}\tau^2 + b_{33}\tau^3 + \ldots, \end{aligned} \quad (2.1)$$

where $\tau = x^0 - x^3$, and all coefficients are some functions of the three spatial coordinates x^1, x^2, x^3. The metric determinant g vanishes like τ^2 for $\tau \to 0$.

Let us try to "inscribe" in the metric (2.1) a medium with an arbitrary equation of state such that a physical singularity occurs for $\tau = 0$. The consideration of this problem automatically leads to the result that this is possible only if $p = 0$. As was to be expected, the corresponding solution contains eight physically arbitrary functions of three variables.

Thus let us write the equations of the gravitational field in the form

$$G^{\mu\nu} = -T^{\mu\nu}, \quad (2.2)$$

where $G^{\mu\nu}$ is the Einstein tensor, and $T^{\mu\nu}$ is the energy-momentum tensor of the medium [$T^{\mu\nu}$

$= (\epsilon + p)u^{\mu}u^{\nu} - pg^{\mu\nu}]$. The quantities p and ϵ are connected by the equation of state $p = f(\epsilon)$.

The calculation of the left-hand side of (2.2) shows that

$$G^{00} \sim \tau^{-3}, \quad G^{0a} \sim \tau^{-1}, \quad G^{03} \sim \tau^{-3},$$
$$G^{ab} \sim \tau^{-3}, \quad G^{a3} \sim \tau^{-3}, \quad G^{33} \sim \tau^{-4},$$

Let us determine the order of the quantity p. To this end we transform (2.2) with the indices $\mu = a$, $\nu = b$ and $\mu = \nu = 3$ to the form

$$G^{ab} = -\frac{G^{0a}G^{0b}}{G^{00} - p} + pg^{ab}, \tag{2.3}$$

$$G^{33} = -\frac{G^{03}G^{03}}{G^{00} - p} + pg^{33}. \tag{2.4}$$

It is seen from (2.4) that p is not higher than $(1/\tau)^2$ in the order of $1/\tau$. In this case it follows from (2.3) that at least the main term in the expansion of G^{ab} must be equal to zero. Since

$$G^{ab} = -{}^1/_2 a^{33}(a^{ab}a^{cd} - a^{ac}a^{bd})\,(a'_{cd} - b_{cd})\tau^{-3} + \ldots$$

(the prime indicates differentiation with respect to x^3) this leads to the relation

$$b_{ab} = a'_{ab}. \tag{2.5}$$

The calculation of $G^{\mu\nu}$ with account of (2.5) shows that except for G^{33}, all $G^{\mu\nu} \sim \tau^{-1}$, while $G^{33} \sim \tau^{-2}$. It then follows from (2.4) that p is of no lower than zeroth order in τ. With any physically reasonable equation of state for non-dustlike matter ϵ and p should have the same order in τ. Hence $\epsilon \sim \tau^0$ in this case (although the gravitational equations also admit $\epsilon \sim \tau^{-1}$ for $p \sim \tau^0$). An exception to this is dust-like matter since in this case p is always equal to zero, and there is no contradiction with the equation of state when ϵ goes to infinity for $\tau \to 0$.

Let us therefore consider a space filled with dust-like matter. In this case (2.3) leads immediately to (2.5). Let us assume further that $\epsilon \sim \tau^{-1}$ and all $u^i \sim \tau^0$. Equation (2.2) with the indices $\mu = \nu = 0$ is conveniently written in the form

$$R^{00} = {}^1/_2(\varepsilon - p) - (\varepsilon + p)\,(u^0)^2. \tag{2.6}$$

Then we obtain

$$R^{00(-1)} \equiv -\,b_3^3 = -\,\varepsilon^{(-1)}(u^{0(0)2} - {}^1/_2), \tag{2.7}$$

$$G^{0a(-1)} \equiv {}^1/_2(3b^{a3} - a^{a3}\,b_3^3 - b^{ab}\,\overset{3}{a_b} - 2a^{33}\,a^{ab}\,a'_{b3} + a^{ai}\,k_i)$$
$$= -\varepsilon^{(-1)}u^{0(0)}u^{a(0)}, \tag{2.8}$$

$$G^{03(-1)} \equiv -{}^1/_2\,a^{33}(3d_a^a + {}^1/_2 b^{ab}\,\pi_{ab} + {}^1/_2 b_3^3\,\pi_a^a - a^{ab}\,c'_{ab} + a_3^a k_a)$$
$$= -\varepsilon^{(-1)}u^{0(0)}u^{3(0)}, \tag{2.9}$$

$$G^{ab(-1)} \equiv -{}^1/_2\,a^{33}(3d^{ab} - 3c^{ab\prime} - 3c_c^b b^{ac} - 3c_c^a b^{bc} + {}^1/_2 b_3^3 \pi^{ab}$$
$$- {}^1/_2 k_3\pi^{ab} + a_{3;}^{a\;\;b} + a_{3;}^{b\;\;a} + b^{ab\prime} + b_c^a b^{bc} + b_c^b b^{ac} - a_{33}b^{ab})$$
$$= -\varepsilon^{(-1)}\,u^{a(0)}\,u^{b(0)}, \tag{2.10}$$

$$G^{a3(-1)} \equiv {}^1/_2\,a^{33}(3b_3^a - b_{33}a^{a3} + \pi_{c;}^{a\;\;c} - \pi_{c;}^{c\;\;a} + {}^1/_2\,k^a\pi_c^c$$
$$- {}^1/_2\,k^c\pi_c^{a\prime} + a_3^a b_c^c - a_3^c b_c^a) = -\varepsilon^{(-1)}\,u^{a(0)}\,u^{3(0)}, \tag{2.11}$$

$$G^{33(-2)} \equiv {}^1/_2(K + {}^1/_2\,b_3^3 b_a^a - 2c_a^a - {}^1/_4\,a^{33}\,\pi^{ab}\pi_{ab}$$
$$+ {}^1/_4\,a^{33}\pi_a^a\,\pi_b^b - {}^1/_4\,b_a^a\,b_b^b + {}^3/_4\,b_{ab}b^{ab} + 2a^{a3}a_{a3}) = 0. \tag{2.12}$$

Here the raising and lowering of indices[2)] and the covariant differentiation are carried out according to the rules formulated in [3]. Furthermore, $k_i = \partial \ln a_{33}/\partial x^i$, K is the two-dimensional curvature, and $\pi_{ab} = a''_{ab} - 2c_{ab}$.

There are 20 functions, a_{ik}, b_{i3}, c_{ab}, d_{ab}, $\epsilon^{(0)}$ $u^{0(0)}$, $u^{a(0)}$, $u^{3(0)}$ and some derivatives (recall that $b_{ab} = a'_{ab}$) which enter in (2.7) to (2.12) and the identity $u^{\mu}u_{\mu} = 1$. Hence 11 functions can be expressed in terms of the remaining 9, which thus remain arbitrary.[3)] As the latter we can choose 9 of the 10 functions $\epsilon^{(0)}$, $u^{a(0)}$, $u^{3(0)}$, a_{ik}, connected by one relation. Indeed, $u^{0(0)}$ can be eliminated with the help of the identity. The quantities b_{a3} can be found from (2.8), and d_{ab} from (2.10). Then the c_{ab} are expressed in terms of arbitrary functions with the help of (2.9) and (2.11). Equation (2.12) establishes a connection between the 10 functions mentioned above. Taking into account that the solution contains one nonphysical arbitrary function, we find that the number of physically arbitrary functions is equal to eight.

We note that the dust-like matter can also be "inscribed" in the metric (2.1) in a different way such that the energy density ϵ remains finite for $\tau = 0$ (here $\epsilon \sim \tau^0$, $u^0 \sim \tau^0$, $u^a \sim \tau^0$, $u^3 \sim \tau^{-1}$). The corresponding solution will also contain eight physically arbitrary functions. The natural interpretation of this result seems to be the following. Both solutions are actually the same solution expanded in the neighborhoods of two different hypersurfaces, where the (true) singularity occurs only on one of the hypersurfaces.

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[1] A. Z. Petrov, Prostranstva Eĭnshteĭna (Einstein Spaces) Fizmatgiz (1961).

[2)] The indices a, b, c, d run through the values 1, 2.

[3)] The metric (2.1) contains one nonphysical arbitrary function connected with the possibility of choosing a hypersurface from which the time coordinate is to be reckoned.

[2] E. M. Lifshitz and I. M. Khalatnikov, UFN **80**, 391 (1963), Soviet Phys. Uspekhi **6**, 495 (1964).
[3] E. M. Lifshitz, V. V. Sudakov, and I. M. Khalatnikov, JETP **40**, 1847 (1961), Soviet Phys. JETP **13**, 1298 (1961).
[4] L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), 4th ed., Fizmatgiz (1962), p. 331.

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