POTENTIAL SENSITIVITY OF RADIANT POWER DETECTORS

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The causes of errors in radiant power measurements are analyzed for both natural and artificial sources. The potential accuracy of isotropic-radiation temperature measurements with nonideal detectors is ascertained. The maximum accuracy of radiant power measurements is found for low-divergence beams, and its dependence on the statistical properties of some artificial source types, on background temperature, and on the temperature and absorptive properties of detectors is learned. The radiation field is divided formally into classical and quantum parts. The feasibility of this decomposition is proved in the Appendix for statistical states described by Glauber's P representation of the density operator.

1. INTRODUCTION

FIELD fluctuations are the principal cause of accuracy limitations on measurements of quantities pertaining to electromagnetic fields. In modern terminology these fluctuations are divided into quantum and wave fluctuations. The former result from noncommutation of field operators. The associated limitation on measurement accuracy is usually interpreted in classical language as the result of additive noise that is always present in a radiation field. However, this is not an exhaustive interpretation for the general case, and it can be applied to specific problems only on the basis of additional assumptions.

The wave fluctuations, on the other hand, are associated with randomness of the field-generating sources and should be absent from the radiation emitted by determinate sources. Radiation from such sources is customarily designated as coherent, in contrast with the sources of incoherent (such as thermal) fields that are generated by the random motion of microscopic charges. Following Glauber,[1-3] the term "coherence" is here used in an unusually narrow sense that embraces the behavior of both second and higher moments of fields.

Radiation emitted by artificial sources occupies an intermediate position between completely coherent and completely incoherent radiation, and can approximate either type, depending on the design and operation of the emitters. For example, a stable gas laser operating in the fundamental resonator mode emits a field that appears to be very close to the coherent type, whereas multimode emission can in many instances belong to the random type with respect to its statistical properties.

Until recently the division into wave and quantum fluctuations was largely heuristic.[4] An important advance in the understanding of radiation fluctuations was the method, proposed by Glauber[2] and by Sudarshan,[5] of presenting the field density operator ρ as the basis of the so-called coherent states (eigenstates of the fraction of field operators representing photon annihilation). The main result of this concept was the important conclusion that mean values of normal products of the field operators correspond to the moments of some classical field; these moments are calculated using the "distribution function" given by the field density operator ρ. (Normal products of operators are those in which all photon annihilation operators stand to the right of the creation operators.) Since the normal products do not depend on the order of their factors, their association with classical analogs is unique, like the determination of a "distribution function" from the field density operator. We note, however, that this "distribution function" can lack some classical probability properties. For example, it can be negative at some points, and its use to calculate normal moments is to some extent a formal procedure.

From the foregoing point of view it is reasonable to assume that the wave fluctuations of a field are described only by physical quantities that are the mean values of normal products (such as the mean rate of energy absorption). Moments not expressed in terms of normal mean values, can be found if they are first converted to the normal form by means of the field commutation relations. Any given moment thus becomes the sum of normal moments of the initial order plus a linear combination of lower-order normal moments. It is reasonable to designate this linear combination as the contribution of quantum fluctuations to the mean value in which we are interested here. Thus quantum fluctuations depend, in the general case, on the order of the operators in the moment of present interest; therefore no description as additive classical noise is adequate.[1)]

In the present work the approach to mean-value calculations of field operators differs from the aforediscussed procedure. It has been shown[6] that the classical equations of electrodynamics can be used directly to calculate moments of any order consisting of thermal field operators. Both the quantum and wave fluctuations then result from beats between different frequencies in the classical field. A radiation signal often possesses statistics differing greatly from blackbody radiation statistics. A similar procedure can also be applied in such cases if for field moments in mixed states a symbolic formula is used, for which our derivation is given in the Appendix.

In studying the problem of the accuracy of signal measurements we shall employ the simplest criterion of detection, based on an evaluation of the signal-to-noise ratio. A signal increment will be considered detectable if its ratio to the mean square fluctuation exceeds unity. This is a quite reliable criterion, although it yields only qualitative results.

The first part of the present paper discusses accuracy in the temperature measurement of external equilibrium radiation by means of a nonideal thermal detector having an arbitrary temperature and possessing an arbitrary dielectric constant. The result obtained here is compared with that given in [7], where the analogous problem was investigated for an ideal (cooled and perfectly black) detector. The second part is devoted to the measurement of radiant power from a small source against an equilibrium radiation background of temperature T. The detector is again taken to be nonideal with a temperature T_1. In studying radiation from nonthermal sources we confine ourselves to certain limiting cases for which the determination of detailed statistical properties is not necessary.

2. ACCURACY OF THE TEMPERATURE MEASUREMENTS OF EXTERNAL EQUILIBRIUM RADIATION

The cause considered here for errors in temperature measurements consists in fluctuations of the radiation exchanged between the detector and a cavity. Let a change ΔT of the cavity temperature T correspond to a change in the mean energy flux $\langle I \rangle$ that is represented by

$$\Delta I = \Delta T \partial \langle I \rangle / \partial T, \tag{1}$$

where

$$\langle I \rangle = \frac{\hbar\omega^3\sigma}{4\pi^2c^2}\varepsilon\Delta\omega(n-n_1),$$

$$n = n(T) = \frac{1}{2}\left(\operatorname{cth}\frac{\hbar\omega}{2kT} - 1\right), \tag{2*}$$

$n_1 = n(T_1)$, σ is the detector aperture, $\Delta\omega$ is the high-frequency absorption band, ϵ is the efficiency expressed in terms of the absorption coefficients $A_i(\theta)$, which depend on the angle of incidence θ and on the polarization (i = 1, 2):

$$\varepsilon = \int (A_1 + A_2)\cos\theta\, do/2\pi. \tag{3}$$

Substituting $x = \hbar\omega/2kT$, we obtain from (1) and (2):

$$\Delta I = \frac{\hbar\omega^3\sigma\varepsilon\Delta\omega}{8\pi^2c^2T}\frac{x}{\operatorname{sh}^2 x}\Delta T. \tag{4†}$$

According to the aforementioned criterion, the increment of the background temperature is detectable subject to fulfillment of the inequality

$$\Delta I \geqslant \langle \Delta I^2 \rangle^{1/2}. \tag{5}$$

An expression for $\langle \Delta I^2 \rangle$ was derived in [6]:

$$\langle \Delta I^2 \rangle = \frac{\hbar^2\sigma\omega^4\varepsilon\Delta\omega\Delta\Omega}{4\pi^3c^2}[(n+n_1+2nn_1)+\alpha(n-n_1)^2], \tag{6}$$

where

$$\alpha = \int (A_1^2 + A_2^2)\cos\theta\, do \Big/ \int (A_1 + A_2)\cos\theta\, do,$$

$$\Delta\Omega = \int_0^\infty |F(\omega)|^2 d\omega, \tag{7}$$

$F(\omega)$ is the low-frequency characteristic of the detector. Substituting (6) into (5), we obtain the maximum detectable temperature increment:

$$\Delta T = \frac{4\sqrt{\pi}\,cT}{\omega\sqrt{\sigma\varepsilon}}\left(\frac{\Delta\Omega}{\Delta\omega}\right)^{1/2}\frac{\operatorname{sh}^2 x}{x} \times [\alpha(n-n_1)^2 + n + n_1 + 2nn_1]^{1/2}. \tag{8}$$

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1)The Appendix contains some additional aspects of this question.

*cth ≡ coth.

†sh ≡ sinh.

We note that when $A_1 = A_2$ and the surface obeys Lambert's law we have $A_1 = \epsilon = \alpha$.

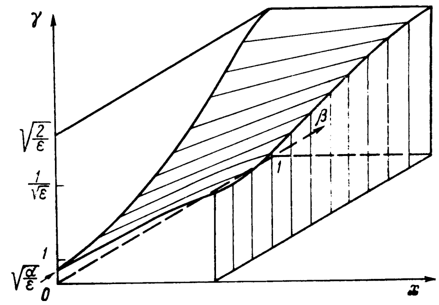
As already mentioned, an expression for ΔT derived in [7] has the form

$$\Delta T = \frac{2\sqrt{\pi}\, cT}{\omega\sqrt{\sigma}} \left(\frac{\Delta\Omega}{\Delta\omega}\right)^{1/2} \frac{\operatorname{sh} x}{x} \qquad (9)$$

in our notation and differs from (8) by a factor depending on T, T_1, α, and ϵ, which we denote by γ.[2)] Introducing $\beta = n_1/n$, we obtain from (8) and (9)

$$\gamma = \varepsilon^{-1/2}\{(1+\beta) - (1-\beta)[1-(1-\beta)\alpha]e^{-2x}\}^{1/2}. \qquad (10)$$

The topography of this function of x and β is represented schematically in the accompanying figure.



It is easily seen that the sensitivity threshold calculated in [7] is the minimum possible threshold only for detectors obeying Lambert's law. In this case $\alpha = \epsilon$, and we obtain $\gamma = 1$ only if $x = 0$ and $\beta = 0$. For real detectors the ratio α/ϵ can be smaller than unity, but the accompanying gain of the signal-to-noise ratio is realizable only at sufficiently high radiation temperatures ($x \ll 1$). In all other cases a real detector has a lower than ideal sensitivity. For example, at low temperatures ($x \gtrsim 1$) for equal sensitivities it is required that the aperture of a real detector be larger than that of an ideal detector by at least the factor $1/\epsilon$.

3. SPATIALLY COHERENT SIGNALS

Since the field that will be investigated is not assumed to be thermal we must first find a suitable method of calculating moments for a field state representing the superposition of a thermal field and a signal field. Furthermore, we must select statistics that will describe satisfactorily the properties of the real signal type that concerns us here.

Let the field be generated by two statistically independent sources such that when the first source is switched off the field state is described by the density operator ρ_2, and by the operator ρ_1 when the second source is switched off.[3)] Then in a state corresponding to the simultaneous operation of both sources we have the following symbolic formula, which is derived in the Appendix, for the moments of the field operators:

$$\langle A_i \ldots A_n\rangle = \langle : \langle (A_i^1 + A_i^2) \ldots (A_n^1 + A_n^2)\rangle_1 : \rangle_2. \qquad (11)$$

Here the symbol $\langle\ \rangle_1$ denotes averaging over the statistical operator ρ_1 and applies only to the operators A_l^1; the symbol $\langle : : \rangle_2$ applies only to A_l^2 and denotes averaging of normal products over ρ_2.

The spectrum of power flux to the detector is expressed through fourth moments of the electric and magnetic fields, or, as was shown in [6], through fourth-order central moments of the space- and frequency-dependent Fourier components of the electric field E. These moments are denoted by the symbol $\langle\ \rangle_0$ and are defined as follows:

$$\langle EE'E_1E_1'\rangle_0 \equiv {}^1/_4[\langle (EE' + E'E)(E_1E_1' + E_1'E_1)\rangle - \langle EE' + E'E\rangle\langle E_1E_1' + E_1'E_1\rangle].$$

Applying Eq. (11), we obtain in a mixed state

$$\langle\{\}\rangle_0 = \langle\{\}\rangle_{01} + \langle:\{\}:\rangle_{02} + \langle EE_1\rangle_{+1}\langle:E'E_1':\rangle_2 + \langle EE_1'\rangle_{+1}\langle:E'E_1:\rangle_2 + \langle E'E_1\rangle_{+1}\langle:EE_1':\rangle_2 + \langle E'E_1'\rangle_{+1}\langle:EE_1:\rangle_2, \qquad (12)$$

where, for example $\langle EE_1'\rangle_{+1} = {}^1/_2\langle EE_1' + E_1'E\rangle$, and $\{\} = EE'E_1E_1'$.

We shall understand ρ_1 to be the density operator of the thermal field. For a state defined by this operator the second and fourth moments appearing in (12) can be calculated by means of the classical field equations, and are given in [6]. The field of the detected signal will be described by ρ_2.

In accordance with (12), the spectrum $S(\omega)$ of power fluctuations in the detector output is the sum

$$S(\omega) = S_0(\omega) + S_1(\omega) + S_2(\omega), \qquad (13)$$

where S_0 is the fluctuation spectrum in the absence of a signal and corresponds to the first term on the right-hand side of (12). S_1 results from the second term, and S_2, which results from beats between the signal and thermal fields, corresponds to the last four terms in (12). We note that S_2 does not vanish at zero temperature of the thermal sources. It then defines the so-called shot (or quantum) noise of the radiation and corresponds to

[2)]In the cited reference the coefficient 2 of Eq. (9) was omitted.

[3)]It will not be important henceforth to know exactly how the sources are switched off, since this could affect only the zero-point field a few wavelengths from a source.

beats between the signal and the zero-point field of thermal emission.

Each of the quantities S_0, S_1, and S_2 depends essentially on the relations between the low-frequency passband $\Delta\Omega$, the detector absorption band $\Delta\omega$, and the signal band $\delta\omega$. In studying real detectors operating in the very short wavelength region, we shall henceforth assume arbitrarily that

$$\Delta\Omega \ll \Delta\omega \ll \omega, \qquad \delta\omega \ll \Delta\omega. \tag{14}$$

Under the first of these conditions we have, in accordance with [6],

$$S_0(\omega) = F(\omega)\,\frac{\hbar^2\omega^4\sigma\varepsilon\Delta\omega}{8\pi^3c^2} \times [(\alpha(1-\beta)^2+2\beta)n^2+(1+\beta)n]. \tag{15}$$

If for the purpose of calculating S_0 and S_2 only the thermal and signal power spectra are required, then S_1, corresponding to the second term in (12), depends essentially on the signal statistics, which we shall subsequently give using the aforementioned classical analogs of signals (now to be explained).

The classical analog of a signal is a statistical ensemble of classical signals in which the mean values of the orders that concern us agree sufficiently accurately with the mean values of normal products for the quantum field.[4)]

In the present work we shall not be concerned with field moments higher than the fourth order. It will be seen that first and third moments drop out. Second moments make up the coherence matrix and can be obtained, at least in principle, by measuring band intensities with a Young interferometer. The determination of higher moments generally requires an analysis of the source emission mechanism or must be based on suitable experimental results. We are thus far acquainted only with the Brown and Twiss type of experiments, which have yielded some information about the statistical properties of the signal envelope.

Under certain conditions a detailed analysis is not required. For example, a large class of quantum fields can be described by Gaussian classical analogs wherein all higher moments are expressed in terms of second moments. Furthermore, some types of artificial sources emit radiation having a very stable amplitude, as is known from the Brown and Twiss type of experiments. The only classical analog of this field will be an oscillation with the time-dependent factor

$$(1+r(t))\cos(\omega t+\varphi(t)), \quad r \ll 1.$$

If we confine ourselves to terms that are quadratic in r we shall subsequently find that the moments of energy terms do not depend on the distribution function of the corresponding signal. A further natural extension is the superposition of such oscillations to describe the operation of a multimode laser, for example. When we assume independence of the modes we thus, of course, neglect their interactions, which are associated with nonlinear effects in matter, and also their possible coupling through the pumping mechanism.

We shall confine ourselves to the case of linearly polarized radiation impinging normally on the detector and shall assume spatial coherence of the signal field throughout the aperture, i.e., the relation $\delta_0 \ll \lambda^2/\sigma$ is fulfilled for the solid angle of the source.

We shall first consider the last term in (13), corresponding to beats between the thermal field and the signal; this term does not depend on the signal statistics, and is calculated by means of Eq. (35) of [6]:

$$S_2(\omega) = \frac{1}{2\pi}\int \exp(-i\omega\tau)\,\Psi_2(\tau)\,d\tau,$$

$$\Psi_2(\tau) = \left(\frac{c}{4\pi}\right)^2 \int [A^2\langle E_I E_{I1}\rangle_{+1}\langle E_I' E_{I1}'\rangle_2 + 4A\langle E_T' E_{T1}'\rangle_{+1}\langle E_I E_{I1}\rangle_2] \times \exp i[(\omega+\omega')t+(\omega_1+\omega_1')t' + (\varkappa+\varkappa')\mathbf{r}+(\varkappa_1+\varkappa_1')\mathbf{r}'] \cdot \times F(\omega+\omega')F(\omega_1+\omega_1')\,d\omega\ldots d\omega_1'\,d\varkappa\ldots d\varkappa_1'\,d\mathbf{r}\,d\mathbf{r}'. \tag{16}$$

Here $E_I = E_I(\kappa, \omega)$ and $E_T = E_T(\kappa, \omega)$ are the space- and frequency-dependent Fourier amplitudes of the incident field and the detectors' thermal field, respectively. Since the aperture is much smaller than the transverse correlation radius of the field and the process is stationary in time, the signal field is

$$\langle E_I E_{I1}\rangle_2 = C(\omega)\,\delta(\varkappa)\,\delta(\varkappa_1)\,\delta(\omega+\omega_1), \tag{17}$$

and the correlation functions for the background and thermal field of the detector are[6]

$$\langle E_I E_{I1}\rangle_{+1} = \frac{\hbar\omega}{8\pi^2 c\cos\theta}\,\mathrm{cth}\,\frac{\hbar\omega}{2kT}\,\delta(\varkappa+\varkappa_1)\,\delta(\omega+\omega_1),$$

$$\langle E_T E_{T1}\rangle_{+1} = \frac{A(\theta)\hbar\omega}{8\pi^2 c\cos\theta}\,\mathrm{cth}\,\frac{\hbar\omega}{2kT_1}\,\delta(\varkappa+\varkappa_1)\,\delta(\omega+\omega_1) \tag{18}$$

Substituting (17) and (18) into (16), after a simple integration we obtain

4)This analogy is not unique in general. Examples of quantum states without classical analogs can also be cited, although these are apparently not realized.

$$S_2(\omega)=\frac{A\hbar\omega}{2\pi}I_0|F(\omega)|^2\left[2n\left(1+\frac{1-A}{A}\beta\right)+\frac{1}{A}\right], \quad (19)$$

where I_0 is the average power absorbed by the detector.

The magnitude of S_1 is determined by the statistics of the signal. Since the signal field is independent of the coordinates within the aperture we have from [6]

$$S_1(\omega)=\frac{1}{2\pi}\int\exp(-i\omega\tau)\Psi_1(\tau)\,d\tau,$$

$$\Psi_1(\tau)=\left(\frac{c}{4\pi}\right)^2\sigma^2A^2\int\langle E_I E_I' E_{I1} E_{I1}'\rangle_{02}$$
$$\times\exp\{i[(\omega+\omega')t+(\omega_1+\omega_1')t']\}$$
$$\times F(\omega+\omega')F(\omega_1+\omega_1')\,d\omega\ldots d\omega_1', \quad (20)$$

where E_{inc} is the time-dependent Fourier amplitude of the signal field. In the calculation of (20) we shall consider the following simplest models of signals.

4. NARROW-BAND (COMPARED WITH DETECTOR ABSORPTION BAND) RANDOM SIGNAL OF CONSTANT AMPLITUDE

For this classical analog the power absorbed by a detector is constant in time and is proportional to

$$\int E_I E_I'\exp\{i(\omega+\omega')t\}F(\omega+\omega')\,d\omega\,d\omega',$$

where $F(\omega+\omega')$ vanishes when ω and ω' have like signs. In addition, S_1, which describes the fluctuations of this power, also vanishes. This result applies specifically to a sinusoidal signal, which can be assumed in the case of an ideally stabilized laser.

5. NORMAL SIGNAL

Applying to this case the known expression for the fourth moment of normally distributed classical quantities, we obtain for the analogs

$$\langle E_I E_I' E_{I1} E_{I1}'\rangle_{02}=\langle E_I E_{I1}\rangle_2\langle E_I' E_{I1}'\rangle_2$$
$$+\langle E_I E_{I1}'\rangle_2\langle E_I' E_{I1}\rangle_2. \quad (21)$$

Since the process is stationary in time, we have

$$\langle E_I E_I'\rangle_2=C(\omega)\delta(\omega+\omega'),\quad C(\omega)\geqslant 0. \quad (22)$$

Substituting this expression into (21) and also in (20), after an elementary calculation we obtain

$$S_1(\omega)=2\left(\frac{\sigma Ac}{4\pi}\right)^2|F(\omega)|^2\int_{-\infty}^{\infty}C(\omega')C(\omega'-\omega)\,d\omega'. \quad (23)$$

For a low-frequency band,

$$\Delta\Omega\equiv\int_0^\infty|F(\omega)|^2\,d\omega\ll\delta\omega,$$

we can assume $\omega = 0$ in the integrand of (23), thus obtaining

$$S_1(\omega)=2\left(\frac{\sigma Ac}{4\pi}\right)^2|F(\omega)|^2\int_{-\infty}^{\infty}C^2(\omega)\,d\omega. \quad (24)$$

We have not yet defined the signal band $\delta\omega$; we now write

$$\delta\omega=\left[\int_{-\infty}^{\infty}C(\omega)\,d\omega\right]^2\Big/\int_{-\infty}^{\infty}C^2(\omega)\,d\omega, \quad (25)$$

which enables us to put (24) into the form

$$S_1(\omega)=2|F(\omega)|^2I_0^2/\delta\omega. \quad (26)$$

For the mean square fluctuation we obtain from (26):

$$\langle\Delta I_1^2\rangle=4I_0^2\Delta\Omega/\delta\omega. \quad (27)$$

In the opposite case of a low-frequency broad band ($\Delta\Omega\gg\delta\omega$) we obtain

$$\langle\Delta I_1^2\rangle=I_0^2. \quad (28)$$

6. RANDOM NARROW-BAND PROCESS WITH SMALL FLUCTUATIONS OF THE ENVELOPE

The radiation of a single-mode laser is described more accurately by this classical analog than by a signal of constant amplitude. The signal in the present case is

$$E_I=E(1+r), \quad (29)$$

where E is a frequency-modulated signal and $r\ll 1$. Neglecting the fourth power of r, we easily obtain from (20):

$$S_1(\omega)=4I_0^2R(\omega)|F(\omega)|^2, \quad (30)$$

where $R(\omega)$ is the power spectrum of the fluctuations of $r(t)$ ($\langle r(t)\rangle = 0$).

For the mean square fluctuation in a band $\Delta\Omega$ that is much narrower than the $R(\omega)$ band we obtain[5)]

$$\langle\Delta I_1^2\rangle=4\nu I_0^2\Delta\Omega/\delta'\omega, \quad (31)$$

where the band width $\delta'\omega$ for this case is defined as

$$\delta'\omega=\int_0^\infty R(\omega)\,d\omega/R(0), \quad (32)$$

and ν is the mean square degree of modulation:

$$\nu=\langle r^2(t)\rangle=\int_{-\infty}^{\infty}R(\omega)\,d\omega. \quad (33)$$

5)$R(\omega)$ is assumed to be a continuous spectrum.

In the opposite case of a quite broad band $\Delta\Omega \gg \delta'\omega$, the mean square fluctuation is given by

$$\langle \Delta I_1^2 \rangle = 4\nu I_0^2. \quad (34)$$

We note that $R(\omega)$ in (30) is not expressed in terms of the signal spectrum for the general case. However, if amplitude modulation results from additive noise with the power spectrum $D(\omega)$, the fluctuation spectrum will be

$$S_1(\omega) = 4\left(\frac{\sigma c A}{4\pi}\right)^2 |F(\omega)|^2 \int_{-\infty}^{\infty} C(\omega' - \omega) D(\omega') d\omega', \quad (35)$$

where $C(\omega)$ is the signal spectrum. For a band $\Delta\Omega$ that is much broader than $C(\omega)$ or $D(\omega)$ we obtain

$$\langle \Delta I_1^2 \rangle = 2\nu' I_0^2, \quad (36)$$

and for a band $\Delta\Omega$ that is very narrow in the same sense we have

$$S_1(\omega) = 4\nu' |F(\omega)|^2 I_0^2 / \delta''\omega,$$

$$\langle \Delta I_1^2 \rangle = 8\nu' I_0 \Delta\Omega / \delta''\omega. \quad (37)$$

The band width $\delta''\omega$ and the degree of modulation ν' are here defined by

$$\delta''\omega = \int_{-\infty}^{\infty} C(\omega) d\omega \int_{-\infty}^{\infty} D(\omega) d\omega \Big/ \int_{-\infty}^{\infty} C(\omega) D(\omega) d\omega,$$

$$\nu' = \int_{-\infty}^{\infty} D(\omega) d\omega \Big/ \int_{-\infty}^{\infty} C(\omega) d\omega. \quad (38)$$

7. SUM OF INDEPENDENT PROCESSES WITH STABLE AMPLITUDES AND DIFFERENT CENTRAL FREQUENCIES

To some approximation this signal describes the situation in superheterodyne reception or multimode laser operation. We shall first consider two independent processes. It is obvious that when a low-frequency band $\Delta\Omega$ exceeds the difference $|\omega_1 - \omega_2|$ between the central frequencies of two oscillations, then S_1 is determined mainly by beats with the difference frequency $|\omega_1 - \omega_2|$. If the spectra of the two oscillations are here described by the same function $C(\omega)$, we then easily obtain

$$S_1(\omega) = 4\left(\frac{\sigma A c}{4\pi}\right)^2 |F(\omega_1 - \omega_2)|^2 \times \int_0^{\infty} C(\omega') C(\omega' - |\omega_1 - \omega_2| - |\omega|) d\omega'. \quad (39)$$

The mean square fluctuation, which is the integral of (39) over all frequencies, is then

$$\langle \Delta I_1^2 \rangle = |F(\omega_1 - \omega_2)|^2 I_0^2 / 2, \quad (40)$$

which equals approximately half of the corresponding quantity for a normal signal [Eq. (28)].

When the central frequency of low-frequency beats lies outside the band $\Delta\Omega$, then S_1 is determined only by beats between neighboring frequencies of each line. In this case [Eq. (30)]

$$S_1(\omega) = 2I_0^2 R(\omega) |F(\omega)|^2. \quad (41)$$

The mean square fluctuation $\langle \Delta I_1^2 \rangle$ is defined as the ratio between $\Delta\Omega$ and $\delta'\omega$ and is given in the limiting cases by

$$\langle \Delta I_1^2 \rangle = 2\nu I_0^2 \Delta\Omega / \delta'\omega, \quad \Delta\Omega \ll \delta'\omega,$$

$$\langle \Delta I_1^2 \rangle = 2\nu I_0^2, \quad \Delta\Omega \gg \delta'\omega. \quad (42)$$

The relative fluctuation is then one-half of that for a single narrow-band signal.

A basis exists for assuming that fluctuations in separate laser modes can be coupled. A simultaneous investigation of two coupled modes must involve the four-dimensional probability density distribution, whose form is unknown. However, the situation is simplified when $\Delta\Omega \ll |\omega_1 - \omega_2|$ or $\Delta\Omega \gg |\omega_1 - \omega_2|$. In the first of these two cases the beat frequency is not passed through a low-frequency filter, so that fluctuations of the instantaneous frequency are not reflected in the signal output. It is easily shown that then

$$S_1(\omega) = 2I_0^2 (R(\omega) + \tilde{R}(\omega)) |F(\omega)|^2, \quad (43)$$

where $\tilde{R}(\omega)$ is the cross-term power spectrum of r_1 and r_2. Specifically, for completely correlated amplitude fluctuations $S_1(\omega) = 4 I_0^2 R(\omega) |F(\omega)|^2$, which coincides with (30).

When $\Delta\Omega \gg |\omega_1 - \omega_2|$ the magnitude of S_1 is determined by the fluctuations of the instantaneous frequencies in both oscillations. For completely correlated fluctuations of the instantaneous frequencies we obtain

$$S_1(\omega) = I_0^2 |F(\omega_1 - \omega_2)|^2 \delta(|\omega| - |\omega_1 - \omega_2|), \quad (44)$$

in contrast with (39) for independent fluctuations. The investigation of intermediate cases involves the two-dimensional distribution function of the instantaneous frequencies. For $\Delta\Omega \gg |\omega_1 - \omega_2|$ the mean square value $\langle \Delta I_1^2 \rangle$ is independent of mode coupling and is given by (40).

The extension to a larger number of independent oscillations leads to the quite obvious conclusion that when a few lines of $C(\omega)$ are located inside the band $\Delta\Omega$, the signal distribution can be regarded as normal, permitting use of the formulas in Sec. 5. If only one line of $C(\omega)$ is located inside this band, or only a part of a line, we obtain for the mean square fluctuation:

$$\langle \Delta I_1^2 \rangle = 4I_0^2 \Delta\Omega / m\delta'\omega, \quad \Delta\Omega \ll \delta'\omega,$$
$$\langle \Delta I_1^2 \rangle = 4I_0^2 / m, \quad \Delta\Omega \gg \delta'\omega, \qquad (45)$$

where m is the number of lines.

8. ACCURACY OF POWER MEASUREMENTS

Before writing a final expression for the error of signal power measurements, we introduce dimensionless variables that characterize the power of the signal and background. The average number of photons traversing the aperture in the "build-up time" $\tau = \pi/\Delta\Omega$ for the band $\Delta\Omega$ of a low-frequency filter, will be denoted by M for the signal:

$$M = \pi I_0 / A\hbar\omega\Delta\Omega,$$

and by N for the background:

$$N = n\sigma\omega^2\Delta\omega / 4\pi c^2 \Delta\Omega.$$

Denoting by δM the minimum detectable change in the photon flux of the signal, we assume $\delta M = \langle \Delta M^2 \rangle^{1/2}$, as in the first part of the present article; here

$$\langle \Delta M^2 \rangle = \pi^2 \int_{-\infty}^{\infty} S(\omega)\, d\omega / (\hbar\omega)^2 (\Delta\Omega)^2. \qquad (46)$$

Substituting now the previously obtained expressions for S_0, S_1, and S_2, we obtain

$$(\delta M)^2 = N \frac{\varepsilon}{A^2} [n(\alpha(1-\beta)^2 + 2\beta) + (1+\beta)]$$
$$+ 2Mn\left(1 - \frac{1-A}{A}\beta\right) + (\delta M_0)^2, \qquad (47)$$

where δM_0 is the smallest detectable increment of the photon flux in the absence of a background when the detector temperature is zero ($n = 0$, $\beta = 0$):

$$\delta M_0 = M(1/A + qM). \qquad (48)$$

The factor q here depends only on the statistical properties of the signal. As we have seen previously, for a signal of constant amplitude we have $q = 0$, for a signal with small amplitude fluctuations $q = 4\nu\Delta\Omega/\delta'\omega$ when $\Delta\Omega \ll \delta'\omega$, and $q = 4\nu$ when $\Delta\Omega \gg \delta'\omega$. With decreasing band width of a normal signal, q approaches unity, while with broadening of this band q approaches the value $4\Delta\Omega/\delta\omega$.

Equation (47) shows that the dependence of the minimum detectable flux on the signal statistics becomes significant only for measurements of quite strong narrow-band signals by means of detectors having sufficiently high sensitivity ($MA \gg 1$). It can be assumed in the optical region that the type of statistics is important only for laser power measurements.

The type of signal statistics is often not known with sufficient accuracy. In such instances the effects of the background and detector emission can be taken into account through Eq. (47) if δM_0 is known from the proper experiments. We shall confine ourselves here to an examination of the cases in which these effects are clearly small. For the sake of simplicity we shall consider only a detector obeying Lambert's law. Then $\epsilon = \alpha = A$ and (47) becomes

$$(\delta M)^2 = N\left[n\left((1-\beta)^2 + \frac{2\beta}{A}\right) + \frac{1+\beta}{A}\right]$$
$$+ 2Mn\left(1 + \frac{1-A}{A}\beta\right) + (\delta M_0)^2. \qquad (49)$$

Let us first consider a detector having zero temperature ($\beta = 0$). Then for a very weak signal ($MA \ll 1$) with

$$N \lesssim M\left(\frac{1-2nA}{1+nA}\right) \qquad (50)$$

the background can be neglected. On the other hand, for a sufficiently strong signal ($MA \gg 1$), instead of (50) we have

$$N \lesssim MA \frac{qM - 2n}{An + 1}. \qquad (51)$$

Let us assume, furthermore, that there is no thermal background but that the detector has a finite temperature T_1 ($n \to 0$, $\beta \to \infty$, $n_1 = \frac{1}{2}[\coth(\hbar\omega/2kT_1) - 1]$). It then follows from (49) that the detector temperature effect can be neglected when for a weak signal we have fulfillment of the condition

$$N_1 \lesssim M\left[1 - \frac{n_1(2-A)}{An_1 + 1}\right], \qquad (52)$$

and for a strong signal

$$N_1 \lesssim M \frac{qMA + 2n_1(1-A)}{An_1 + 1}. \qquad (53)$$

In (52) and (53), N_1 is the value of N taken at the detector temperature T_1.

Finally, when the detector is in equilibrium with the background, we substitute $\beta = 1$ in (49); then instead of (50)–(53) we have for weak and strong signals, respectively,

$$N \lesssim M(1 - 2N) / 2(n+1),$$
$$N \lesssim M(qAM - 2) / 2(n+1). \qquad (54)$$

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APPENDIX

CALCULATION OF FIELD MOMENTS FOR A MIXTURE OF INDEPENDENT STATES

To prove the symbolic relation (11) we shall use some results given in [2]. As already mentioned, it was there shown that the mean values of the normal products of field operators can be calculated from classical formulas if we use the "distribution function" represented by the state density operator. We are interested in calculating mean values of the form $\langle A_i \ldots A_n \rangle$, where the field operators are not normally ordered. With application to the present work in mind, we assume the A_l to be Hermitian operators. In a microscopic analysis we encounter mean values of the field amplitudes for both positive and negative frequencies. However, no use is made of Hermiticity in the derivation that follows; therefore (11) will apply to moments of the given type.

It is always possible by means of commutation relations to reduce any given moment to a sum of normal moments down to the first order:

$$\langle A_i \ldots A_n \rangle = \sum_{jk} c_{jk} \langle : A_j \ldots A_k : \rangle, \tag{55}$$

where c_{jk} are determinate numbers whose values are unimportant for the sequel. Normal moments of different orders in the right-hand side of (55) will be considered as moments of the corresponding classical quantities in the mixed state of present interest. As has been shown in [2], the corresponding "distribution function" is given by a convolution of the distributions corresponding to the different states of the mixture. In the classical statistics this convolution describes the distribution of the sum of two independent random quantities. Therefore, considering each of the A_l as a number, we can divide it into the sum of two independent terms:

$$A_l = A_l^1 + A_l^2.$$

In quantum-mechanical language this division corresponds to the symbolic notation

$$\langle A_i \ldots A_n \rangle = \left\langle \left\langle : \sum_{j,k} c_{jk} (A_j \ldots A_k) : \right\rangle_1 \right\rangle_2 \tag{56}$$

where $A_l = A_l^1 + A_l^2$, the symbol $\langle\,\rangle_1$ pertains only to the operators A_l^1 and denotes averaging over ρ_1, while the symbol $\langle\,\rangle_2$ pertains to the operators A_l^2 and denotes averaging over ρ_2. We note that to calculate the right-hand side of (56) we do not require the commutation relations between the newly introduced operators A_l^1 and A_l^2, because the normal products do not depend on the order of the factors.

The next step must be the inverse transformation of the expression

$$: \sum_{j,k} c_{jk} (A_j \ldots A_k) :$$

in (56) into a conventional operator product. The result of this transformation depends only on the commutation relations for the "sum" field A_l. If these are the same as the previous relations, then the commutation relations between A_l^1 and A_l^2 do not affect the transformation. Making use of this fact, we find it useful to write

$$[A_i^1, A_k^2] = 0, \quad [A_i^1, A_k^2] = [A_i, A_k],$$
$$[A_i^2, A_k^2] = 0, \tag{57}$$

without violating the commutation relations for the "sum" field A_l. By performing this transformation in (51) we arrive at (11).

We note an interesting deduction from this formula. Let us assume zero temperature of the thermal field corresponding to the statistical operator ρ_1. It then follows from (11) that the radiation field can be considered formally as the sum of a classical part A_l^2 and of a quantum part A_l^1 representing zero-point oscillations. In the general case a calculation of the moments of this quantum part cannot be interpreted classically.

The situation becomes simpler, however, when we are interested in the mean values of the actually measurable physical quantities. We then need to know only the real parts of these mean values; when these parts are calculated with any given order of the operators a classical interpretation is admissible. Let us consider the real part of any even-order moment of the zero-point field (the odd moments vanish). Since this field (like the thermal field) formally obeys the normal distribution law, we have[9]

$$\operatorname{Re} \langle A_i^1 \ldots A_n^1 \rangle = \operatorname{Re} \sum \langle A_i^1 A_j^1 \rangle \ldots \langle A_k^1 A_m^1 \rangle$$
$$= \operatorname{Re} \sum (1 + iG_{ij}) \langle A_i A_j \rangle_+ \ldots (1 + iG_{km}) \langle A_k A_m \rangle_+. \tag{58}$$

Here $\langle A_i A_j \rangle_+ \equiv \frac{1}{2} \langle A_i^1 A_j^1 + A_j^1 A_i^1 \rangle$, and G_{ij} is the Hilbert transformation operator with respect to the variable $\tau = t_i - t_j$:

$$G_{ij} \langle A_i A_j \rangle_+ = \frac{1}{2i} \langle A_i^1 A_j^1 - A_j^1 A_i^1 \rangle \equiv \frac{1}{i} \langle A_i A_j \rangle_-,$$

and the sum is taken over all permutations of the indices among second moments; in each of these the order of the operators should remain the same as for the original moment.

Thus in accordance with (58) the calculation of zero-point field moments is reduced to deriving the symmetrized moments $\langle A_i A_j \rangle_+$. The latter are derived from classical electrodynamical equations if the given correlation functions of random non-electromagnetic sources are consistent with the fluctuation-dissipation theorem at zero field temperature.[8] Therefore in formulating the problem a quantum-mechanical analysis is required only when operators are placed in correspondence with the observable quantities in which we are interested.

The zero-point field moments $\langle A_i A_j \rangle_+$ can best be investigated for free space; their space- and frequency-dependent Fourier components are used in the present work [see Eq. (18)]. Omitting consideration of a quasistationary field localized near absorptive surfaces and apparently playing no large role in optical and infrared regions, we can assume that these Fourier components are unchanged in form for a field contained within a space of dimensions much larger than λ. For systems having dimensions that are comparable with the wavelength, to derive these moments we can use the already existing solutions of a large number of thermal emission problems by simply assuming zero temperature of all bodies.

We now finally discuss the moments (mentioned at the beginning of this Appendix) of positive-frequency (A_j^+) and negative-frequency (A_j^-) field amplitudes. The expansion (58) also holds for these moments. Furthermore, in the zero-point state we have

$$\langle A_i^- A_k^- \rangle = \langle A_i^+ A_k^+ \rangle = \langle A_i^+ A_k^- \rangle = 0,$$

we so that we are only concerned with calculating the moment $\langle A_i^- A_k^+ \rangle$. In $\langle A_i A_k \rangle$ and $\langle A_i A_k \rangle_-$ we separate the operators into positive- and negative-frequency parts:

$$\langle A_i A_k \rangle = {}^1/_2 \langle A_i^- A_k^+ + A_k^- A_i^+ \rangle,$$
$$\langle A_i A_k \rangle = {}^1/_2 \langle A_i^- A_k^+ - A_k^- A_i^+ \rangle,$$

whence it follows that

$$\langle A_i^- A_k^+ \rangle = \langle A_i A_k \rangle_+ + \langle A_i A_k \rangle_- \qquad (59)$$

which solves our problem.

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57