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LINE SHAPE OF RADIO FREQUENCY SIZE EFFECT IN METALS

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A theory is developed for the line shape of the radio-frequency size effect due to cutoff of the extremal electron trajectories in a metal plate. It is shown that the line shape depends directly on the nature of attenuation of the electromagnetic waves in the skin layer. The inverse problem of determining the field in a metal from the experimental data is solved. The line shape is calculated for exponential radio-wave attenuation.

1. INTRODUCTION

SEVERAL recent experimental papers[1-7] are devoted to the observation and investigation of radio-frequency size effects in metals. These effects consist in the fact that the surface impedance Z of a thin plate is a nonmonotonic function of the external magnetic field **H**, namely, singularities of various types appear on the plot of Z against H. The size effects are observed in pure single crystals of metals at low temperatures under the conditions of the anomalous skin effect.

The singularities of the impedance of the plate may be connected with two different phenomena. One of them is the anomalous penetration of the electromagnetic field deep into the sample, to a depth much larger than the depth δ of the skin layer. In many cases[4,8-10] the electromagnetic field in a semi-infinite metal is an aggregate of narrow slowly-attenuating peaks located at distances nD_0 from the surface ($n = 1, 2, 3, \ldots$ is an integer and D_0 is the characteristic dimension of the electron orbit in the magnetic field, $D_0 \gg \delta$). It is clear that in a plane-parallel plate of finite thickness d the emergence (or vanishing) of the next succeeding peak on the opposite face of the sample ($d = nD_0$) leads to a corresponding singularity in the surface impedance. Another purely geometrical effect is also possible. In a magnetic field parallel to the surface of the metal, the motion of the electrons in a plane perpendicular to H is finite. In a sufficiently strong magnetic field the trajectories of the electrons lie wholly in the sample. With decreasing magnetic field, the diameter of the electron orbit D increases and at a certain value of H, equal to H_1, it coincides with the thickness of the plate d. Owing to the diffuse character of the reflection of the electrons from the boundaries of the sample, a "cutoff" takes place, namely, the contribution of such electrons to the current density turns out to be insignificant. With this, a singularity appears on the plot of the impedance of the plate against the field;[11] the character of this singularity depends on the form of the extremal sections of the Fermi surface. A similar size effect was observed at radio frequencies by Gantmakher in single crystals of tin[2,3] and was subsequently used to investigate the Fermi surface of a number of metals[5-7]. At the present time much experimental material is available on size effects. There are still no theoretical calculations of the line shape.

The line shape can be affected by various factors, namely, the inhomogeneity in the distribution of the alternating field in the skin layer, random deviations of the thickness of the plate from the average value, and the inhomogeneity of the constant magnetic field. The line broadening of the size effect due to the latter two factors usually plays no essential role; for example, etching of the samples with acid does not lead to noticeable changes in the line shape[3,6,7]. It can therefore be assumed that the main cause of the line broadening is the inhomogeneity of the electromagnetic wave in the skin layer.

Inasmuch as the shape of the size-effect lines is determined by the character of the damping of the electromagnetic wave in the metal, we can attempt to reconstruct, by using the line shape, the distribution of the field in the skin layer. It must be emphasized that an investigation of the distribution of the high-frequency field in the skin layer is an important problem in metal physics. Prior to discovery of radio-frequency size effects there was no direct experimental method for investigating the structure of the skin layer. A study of the

Doppler broadening of the cyclotron-resonance lines in an inclined magnetic field[12] yields only the mean distances over which the amplitude and the phase of the wave change in the metal. The line shape of the size effect reflects directly the law governing the field distribution in the skin layer.

We report in this paper a theoretical investigation of the line shape of the geometrical size effect at radio frequencies, and show the feasibility in principle of constructing the field distribution in the skin layer from the experimental data.

2. SYSTEM OF EQUATIONS

To construct the theory of the line shape it is necessary to solve Maxwell's equations and the kinetic equation for the electron distribution function in the plate. These equations can be written in the form

$$\frac{\partial^2 E_\alpha(z)}{\partial z^2} = -i\frac{4\pi\omega}{c^2} j_\alpha(z) \quad (\alpha = x, y), \tag{1}$$

$$-i\omega f + v_z\frac{\partial f}{\partial z} + \Omega\frac{\partial f}{\partial \tau} + \nu f = eE_\alpha v_\alpha \frac{\partial f_0}{\partial \varepsilon}, \tag{2}$$

$$j_\alpha = -\frac{2e}{h^3}\int v_\alpha f d^3p. \tag{3}$$

The coordinate system is chosen as follows: the x axis coincides with the direction of a constant magnetic field **H** (the vector **H** is parallel to the surface of the plate), and the z axis is parallel to the inward normal to the surface of the plate z = 0. E_α and j_α are the tangential components of the electric field and current in the metal and ω is the frequency of the external field; the time dependence of all the quantities is in the form $\exp(-i\omega t)$; f is the equilibrium addition to the Fermi distribution function $f_0(\epsilon)$; **v**, **p**, and ϵ are the electron velocity, momentum, and energy, respectively; $\Omega = eH/mc$ is the cyclotron frequency, and m is the effective mass, which depends on ϵ and on p_x; $2\pi m(\epsilon, p_x) = \partial S(\epsilon, p_x)/\partial\epsilon$, where $S(\epsilon, p_x)$ is the area of the intersection of the equal-energy surface with the plane p_x = const; τ is the dimensionless time of motion of the electron along the orbit in p-space; ν is the frequency of collisions between the electrons and the scatterer (the reciprocal relaxation time); e is the absolute value of the electron charge, c the velocity of light, and h Plancks constant.

In (1)—(3) we have neglected the field component E_z. This component should be obtained from the condition of electric quasineutrality of the metal, $\rho' = 0$, where ρ' is the uncompensated charge density. From the continuity equation it follows that the condition $\rho' = 0$ is identical with the equation $j_z(z) = 0$. In several papers (for example,[13]) it is shown that in the anomalous skin effect the field component E_z obtained from the quasineutrality condition leads only to inessential small corrections in the system of equations (1)—(3). This is due to the fact that the main contribution of the current density is made by electrons moving almost parallel to the surface of the metal. We can therefore assume that $E_z = 0$, and disregard the equation $j_z(z) = 0$.

The solution of the kinetic equation (2) for the case when the magnetic field is parallel to the surface of the plate is known to be (see[14])

$$f = \frac{e}{\Omega}\frac{\partial f_0}{\partial \varepsilon}\int_{\lambda(z,\tau)}^{\tau} d\tau' v_\beta(\tau') E_\beta\left[z + \frac{1}{\Omega}\int_{\tau}^{\tau'} v_z(\tau'')d\tau''\right] \times \exp\left[\frac{\nu - i\omega}{\Omega}(\tau' - \tau)\right]. \tag{4}$$

Here $\lambda(z, \tau)$ denotes the instant of the last collision of the electron with one of the boundaries of the plate. For an electron which does not collide with the surface of the sample $\lambda(z, \tau) = -\infty$. The quantity $\lambda(z, \tau)$ is defined as the root closest to τ of one of the following equations:

$$z + \frac{1}{\Omega}\int_{\tau}^{\lambda} v_z d\tau'' = 0, \quad z + \frac{1}{\Omega}\int_{\tau}^{\lambda} v_z d\tau'' = d. \tag{5}$$

Expression (4) corresponds to the condition of diffuse scattering of electrons from the surface of the metal. It is easy to show [14] that such a definition of the function $\lambda(z, \tau)$ ensures the vanishing of the nonequilibrium part of the distribution function f for electrons scattered by the boundaries of the plate. In other words,

$$\lambda(0, \tau) = \tau \quad (v_z(\tau) > 0), \quad \lambda(d, \tau) = \tau \quad (v_z(\tau) < 0). \tag{6}$$

The solution (4) has a simple physical meaning. Electrons that do not collide with the boundaries are described by the distribution function characteristic of the unbounded metal. The presence of a boundary affects the distribution of those electrons whose trajectories cross at least one of the surfaces of the plate during each revolution. Owing to the diffuse character of the reflection, the electron is "knocked out of the game" as a result of the collision. Therefore in the case of a strong magnetic field, when the inequality

$$\omega, \nu \ll \Omega \tag{7}$$

is satisfied, the contribution to the current density from such electrons is negligibly small. The electrons that do not collide with the boundary can return many times to the skin layer (Ω/ν times), and interact effectively with the high-frequency field. It is precisely these electrons which determine the current density in the metal.

The region of the states of these electrons in phase space can be readily obtained in the following manner. We write out the unperturbed equations of motion of the electron:

$$\dot{\mathbf{p}} = -\frac{e}{c}[\mathbf{vH}], \tag{8}*$$

$$\dot{\mathbf{r}} = \mathbf{v} \tag{9}$$

(the dot denotes the derivative with respect to time). The motion along the normal to the surface of the plate is described by the equation

$$\dot{z} = -\frac{c}{eH}\dot{p}_y, \tag{10}$$

the solution of which is

$$z(\tau) = -\frac{c}{eH}p_y(\tau) + z_0. \tag{11}$$

The integration constant depends only on p_x and ϵ. The condition under which the trajectory of a given electron is fully contained in the plate is

$$z_{max} - z_{min} < d, \tag{12}$$

where

$$z_{max} - z_{min} = \frac{c}{eH}(p_{ymax} - p_{ymin}) \equiv D \tag{13}$$

(minimum and maximum with respect to τ).

Inasmuch as the motion of the electrons in the yz plane is finite, the quantity $D(\epsilon, p_x)$ represents the maximum dimension of the electron trajectory (with given ϵ and p_x) along the z axis. The electrons colliding with the boundaries of the plate are those for which

$$z_{min} > 0, \quad z_{max} < d, \tag{14}$$

that is,

$$\frac{c}{eH}p_{ymax} < z_0 < d + \frac{c}{eH}p_{ymin}. \tag{15}$$

Substituting here expression z_0 from formula (11), and we obtain

$$z_1(\tau, p_x) \equiv \frac{c}{eH}(p_{ymax} - p_y) < z < d - \frac{c}{eH}(p_y - p_{ymin}) \equiv z_2(\tau, p_x). \tag{16}$$

3. SURFACE IMPEDANCE OF THE PLATE

Let us consider the case of unilateral excitation of the plate by a high-frequency field, when the external wave is incident on the surface z = 0. We are interested in the properties of the reflected wave. Its characteristics are described by the surface-impedance tensor $Z_{\alpha\beta}$, which is defined by

$$Z_{\alpha\beta} = \frac{4\pi i\omega}{c^2}\frac{\partial E_\alpha(0)}{\partial E_\beta'(0)}. \tag{17}$$

Here $E_\alpha(0)$ and $E'_\beta(0)$ are the tangential components of the electric field and their normal derivatives on the surface z = 0.

To find the tensor $Z_{\alpha\beta}$ it is necessary to solve Maxwell's equations (1). It is convenient to solve them in the Fourier representation. To this end we continue the function $E_\alpha(z)$ formally to the region outside the plate in the following manner:

$$E_\alpha(z) = 0 \quad (z > d), \quad E_\alpha(-z) = E_\alpha(z). \tag{18}$$

From Maxwell's equations (1) it is obvious that the continuation of the current density $j(z)$ is similar.

We change over in (1) to Fourier components:

$$k^2\mathscr{E}_\alpha(k) + 2E_\alpha'(0) - 2E_\alpha'(d)\cos kd - 2kE_\alpha(d)\sin kd = 4\pi i\omega c^{-2}j_\alpha(k), \tag{19}$$

$$\mathscr{E}_\alpha(k) = 2\int_0^d dz\, E_\alpha(z)\cos kz,$$

$$E_\alpha(z) = \pi^{-1}\int_0^\infty dk\,\mathscr{E}_\alpha(k)\cos kz. \tag{20}$$

From the conditions for the continuity of the tangential components of the electric and magnetic fields at z = d it follows that $E'_\alpha(d) = i\omega c^{-1}E_\alpha(d)$, and therefore the third term in the left side of (19) is $kc/\omega \sim c/\omega\delta$ times smaller than the last term. The term $2kE_\alpha(d)\sin kd$ describes the change in the electromagnetic field on the surface z = 0 due to the reflection of the wave from the opposite face of the sample. This term must be taken into account in those cases when anomalous penetration of the alternating field in the metal takes place.

If the thickness d of the plate is considerably larger than the depth of penetration δ, that is,

$$\delta \ll d, \tag{21}$$

then the term with $E_\alpha(d)$ in (19) can be neglected in first approximation. In this case the influence of the shape and finite thickness of the sample is manifest only to the extent to which the Fourier component of the current density $j_\alpha(k)$ is altered. The change in $j_\alpha(k)$ is due to the aforementioned "cutoff" phenomenon. In a bulky sample, whose thickness is much larger than the characteristic dimensions D_0 of the electron orbits, there is no "cutoff." In this case the field distribution and the surface impedance of the plate have the same form

*[vH] ≡ v × H.

in the first approximation in δ/d as in a semi-infinite metal.

As shown in[11], the change in the surface impedance of the plate $Z(d)$ at $d < D_0$, due to the "cutoff" of the electrons near the extremal section, is determined by the formula

$$\frac{Z(d)-Z(\infty)}{Z(\infty)}=a\left(1-\frac{d^2}{D_0^2}\right)^{1/2}, \qquad (22)$$

where a is a constant of the order of unity ($a = 4/3\pi$ for a spherical Fermi surface). The derivative of the impedance with respect to the magnetic field has a singularity of the type $(1 - d^2/D_0^2)^{-1/2}$. $\partial Z/\partial H$ becomes infinite at $d = D_0$ because no account is taken in (22) of the smearing of the singularity due to the inhomogeneity of the field in the skin layer. It is quite obvious (as confirmed by a subsequent exact calculation, see (58) and (59)), that the damping of the wave in the skin layer leads to the estimate

$$\left|\frac{\partial \ln Z}{\partial \ln H}\right|_{d=D_0}\sim\left(\frac{d}{\delta}\right)^{1/2}. \qquad (23)$$

Let us examine the influence of the anomalous penetration of the field on the impedance of the plate. It is shown in[9] that the peaks of alternating field in a metal attenuate rapidly when the vector H is parallel to the surface of the sample. At a depth $z = nD_0$ the amplitude of the peak is of the order of $(\delta/D_0)^{n/2}$. The amplitude of the peak decreases rapidly with increase in depth because the electrons producing the bursts constitute a relatively small fraction (of the order of $(\delta/D_0)^{1/2}$) of the total number of the electrons. It is obvious that when $d \approx D_0$ the relative change in the field on the surface $z = 0$ due to the existence of a peak near the second boundary is of the order of δ/D_0. Inasmuch as the spatial width of the singularities of the field is of the order of δ, the relative change in the derivative of the impedance (17) is

$$|\partial \ln Z/\partial \ln H| \sim 1. \qquad (24)$$

Thus, under unilateral excitation of the plate, the character of the singularity and the line shape of the size effect are determined by the electron "cutoff." Therefore, to separate the geometrical size effect from the phenomena due to anomalous field penetration it is necessary to irradiate the plate and measure its impedance from the same side. Such experiments were carried out at microwave frequencies to study the "cutoff" of cyclotron resonances[1]. Under these conditions, however, it is difficult to study the line shape of the size effect because of the presence of cyclotron resonances. No experiments with unilateral excitation were made at radio frequencies (in the range of several MHz). In this range, a different procedure is usually employed: the plate is placed inside an alternating-current coil and is excited from both sides; the measured characteristic of the skin effect is the real or imaginary part of the depth of penetration of the alternating magnetic field into the sample

$$\delta_\alpha=\frac{1}{H_\beta(0)}\int_0^d dz H_\beta(z)=-\frac{E_\alpha(0)-E_\alpha(d)}{E_\alpha'(0)},\quad \beta\neq\alpha. \qquad (25)$$

The observed anomalies of the depth of penetration are due both to "cutoff" and to peaks of field in the metal.

The term $-E_\alpha(0)/E'_\alpha(0)$ describes a size effect of the geometric type. The term $E_\alpha(d)/E'_\alpha(0)$ is connected with the anomalous penetration. Unlike the case of unilateral excitation, the field peaks give a change of impedance of the same order of magnitude as electron "cutoff," since the amplitude of the transmitted wave is measured on the opposite side of the plate.

In this paper we consider size effects under unilateral excitation of the plate and confine ourselves to the case of low frequencies

$$\omega \ll \nu. \qquad (26)$$

4. CURRENT DENSITY

Let us calculate the Fourier component of the current density $j_y(k)$. The size effect takes place in the case of "cutoff" of electrons with extremal diameter D. In the simplest case of a singly-connected and convex Fermi surface, the extremum of the function $D(p_x) = D_0$ is attained on the central section $p_x = 0$. The high-frequency current produced by the electrons with $D(p_x) \approx D_0$ is directed along the y axis.

Substituting in (3) the expression for the distribution function (4) and going over to the Fourier representation, we obtain

$$j_y(k)=\frac{2e^2}{\pi h^3}\int\frac{m}{\Omega}dp_x\theta(d-D)\int_0^{2\pi}d\tau v_y(\tau,p_x)$$
$$\times\int_{z_1(\tau,p_x)}^{z_2(\tau,p_x)}dz\cos kz\int_{-\infty}^{\tau}d\tau'\,v_y(\tau',p_x)\exp[\gamma(\tau'-\tau)]$$
$$\times\int_{-\infty}^{\infty}dk'\mathscr{E}_y(k')\cos k'\left[z+\frac{c}{eH}(p_y(\tau,p_x)-p_y(\tau',p_x))\right]. \qquad (27)$$

Here $\gamma = \nu/\Omega \ll 1$. The unit step function $\theta(x)$ is defined in the usual manner: $\theta(x) = 1$ $(x > 0)$, $\theta(x) = 0$ $(x < 0)$. By using the function $\theta(d - D)$ and by varying the limits of integration in the integral with respect to z in accordance with formula (15), we take into account only those electrons which do not collide with the boundaries of the plate.

The integral with respect to z in (27) can be calculated in elementary fashion. In order to simplify the calculation of the integrals with respect to τ and τ', we shall assume that the projection of the electron trajectories on the xz plane are circles. Then

$$v_y(\tau) = v_\perp(p_x)\sin\tau, \qquad p_y(\tau) = m(p_x)v_y(\tau, p_x). \quad (28)$$

In the limiting case of small γ we obtain the following formula for the Fourier component of the current density:

$$j_y(k) = \frac{4e^2}{h^3\nu}\int_{-\infty}^{\infty} dk' \frac{\mathscr{E}_y(k')}{k-k'} \int_{p_{x\,min}}^{p_{x\,max}} dp_x m v_\perp^2 \theta(d-D)$$
$$\times J_1(kR)J_1(k'R)\cdot[\sin(k-k')(d-R) - \sin(k-k')R], \quad (29)$$

where $J_1(z)$ is a Bessel function and $2R = D(p_x)$.

5. FOURIER COMPONENT OF ELECTRIC FIELD AND IMPEDANCE

The field distribution $\mathscr{E}_y(k)$ in the plate is described by the solution of the integral equation (1) in which the current $j_y(k)$ is determined by (29). The kernel of this equation can be simplified by using the condition under which the skin effect is anomalous, $\delta \ll V_0$. The wave number k is of the order of δ^{-1}, and consequently

$$kD_0 \gg 1. \quad (30)$$

Using the known asymptotic expression for Bessel functions, we write $j_y(k)$ in the form

$$j_y(k) = \frac{2e^2}{\pi h^3\nu}\int_{p_{x\,min}}^{p_{x\,max}} dp_x \frac{mv_\perp^2}{R}\theta(d-D)\int_0^\infty dk'\mathscr{E}_y(k')$$
$$\times\left\{\frac{\sin(k-k')(d-D)}{k-k'} + \frac{\sin(k-k')d}{k-k'} - \frac{\sin(k-k')D}{k-k'} + \frac{\sin^2(k+k')R}{k+k'}\right\}. \quad (31)$$

In formula (31) we retained only the main terms of the resultant products of the rapidly oscillating functions. It can be readily verified[13] that in the limit of large k and k', the terms omitted produce insignificant errors. If we are interested, on the other hand, in the line shape of the size effect and assume that $d - D \gg \delta$, then all functions of the type $(k - k')^{-1}\sin(k - k')x$ can be replaced by $\pi\delta(k - k')$, and $\sin^2(k + k')R$ can be replaced by its mean value 1/2. Azbel' and one of the authors[13,8] have shown that the last term in (31) can be disregarded when calculating the field distribution and the impedance.

Thus, the equation for the Fourier component of the field $\mathscr{E}_y(k)$ becomes algebraic, and its solution is

$$\mathscr{E}_y(k) = \frac{-2E_y'(0)}{k^2 - i4\pi\omega c^{-2}\sigma(k)}, \quad (32)$$

where

$$\sigma(k) = \frac{A}{|k|}; \quad A = \frac{2e^2}{h^3\nu}\int_{p_{x\,min}}^{p_{x\,max}} dp_x \frac{mv_\perp^2}{R}\theta(d - D(p_x)). \quad (33)$$

For a spherical Fermi surface, the value of A can be written in the form

$$A = \frac{3Ne^2}{4mvD_0}\begin{cases} 1 & (d > D_0) \\ \frac{1}{\pi}(2\theta_0 - \sin 2\theta_0) & (d < D_0) \end{cases} \quad (34)$$

where $N = 8\pi p^3/3h^3$ is the electron density, and $\sin\theta_0 = d/D_0$. For a Fermi surface with arbitrary shape

$$A = \frac{e^2}{h^3\nu}\int v_y^2\Omega\delta(\varepsilon - \varepsilon_F)\delta(v_z)\theta(d-D)d^3p. \quad (35)$$

It is easy to see that A as a function of the magnetic field has kinks at those values of H, where

$$d = D_0, \quad (36)$$

D_0 is the value at which $D(p_x)$ is extremal with respect to p_x. The surface impedance for a given polarization

$$Z \equiv R - iX = \frac{4i\omega}{c^2E_y'(0)}\int_0^\infty dk\,\mathscr{E}_y(k) \quad (37)$$

is inversely proportional to the cube root of A. Consequently, the derivative of the impedance with respect to H has a singularity of the type $(1 - d^2/D_0^2)^{-1/2}$ at the points (36) (see also (24)).

Near the singularity, when $|d - D_0| \sim \delta$, the function $(k - k')^{-1}\sin(k - k')(d - D)$ cannot be replaced by a δ function. Usually in the study of the size effect one measures not the impedance itself, but its derivative with respect to the magnetic field. Since the relative width of the line is small ($\Delta H/H \sim \delta/d$), the singularities of the derivative of the impedance with respect to H (accurate to a constant factor) are well described by the derivative with respect to the thickness of the plate. Differentiating (37) with respect to d, we obtain

$$Z' \equiv \frac{\partial Z}{\partial d} = \frac{4i\omega}{c^2E_y'(0)}\int_0^\infty dk\frac{\partial\mathscr{E}_y(k)}{\partial d}. \quad (38)$$

The equation for the function $\partial\mathscr{E}_y(k)/\partial d$ can be obtained by differentiating with respect to d equation (19), in which the current density $j_y(k)$ is given by formula (29). As a result we get

$$\left(k^2 - i\frac{4\pi\omega}{c^2}\sigma(k)\right)\frac{\partial\mathscr{E}_y}{\partial d} = \frac{4\pi i\omega}{c^2}\frac{4e^2}{h^3\nu}\int dp_x mv_\perp^2\theta(d-D)$$

$$\times\int_{-\infty}^{\infty} dk'\mathscr{E}_y(k')\cos(k-k')(d-R)J_1(kR)J_1(k'R). \quad (39)$$

In the region of strong fields ($d > D_0$) the Fourier component of the field in the plate coincides with $\mathscr{E}_y(k)$ for a half-space and is determined by formula (32). Therefore relation (39) makes it possible to express $\partial\mathscr{E}_y/\partial d$ (and consequently Z′) in terms of $\mathscr{E}_y(k)$:

$$Z' = \left(\frac{4\omega}{c^2}\right)^2\frac{4\pi e^2}{h^3\nu[E_y'(0)]^2}\int_0^{p_{x\,max}} dp_x mv_\perp^2\theta(d-D)B^2, \quad (40)$$

where

$$B = 2\int_0^{\infty} dk\,\mathscr{E}_y(k)\sin k(d-R)J_1(kR). \quad (41)$$

Replacing the Bessel function $J_1(kR)$ by its asymptotic expression for large values of the argument, we obtain

$$B = -\left(\frac{2}{\pi R}\right)^{1/2}\int_0^{\infty}\frac{dk}{k^{1/2}}\mathscr{E}_y(k)\left\{\sin\left[k(d-D)-\frac{\pi}{4}\right]\right.$$

$$\left.+\sin\left(kd+\frac{\pi}{4}\right)\right\}. \quad (42)$$

The second term in the curly brackets contains a rapidly oscillating function, whereas the first term changes much more slowly within the limits of the line ($|d-D| \sim 1/k$). Therefore the second term in (42) can be neglected. As a result (40) reduces to

$$Z'(d-D_0) = \left(\frac{4\omega}{c^2}\right)^2\frac{8\pi e^2}{h^3\nu[E_y'(0)]^2}\int_0^{p_{x\,max}} dp_x\frac{mv_\perp^2}{R}$$

$$\times\theta(d-D)\psi^2(d-D), \quad (43)$$

where

$$\psi(x) = -\frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{dk}{k^{1/2}}\mathscr{E}_y(k)\sin\left[kx-\frac{\pi}{4}\right]. \quad (44)$$

6. SOLUTION OF THE INVERSE PROBLEM

In this section we show how to reconstruct the field distribution in the layer from the experimental data. It is easy to see that if we know the function $\psi(x)$, then we can find the field $E_y(z)$. To this end we go over in (44) to the coordinate representation:

$$\psi(x) = -\frac{2}{\sqrt{\pi}}\int_0^d dz\,E_y(z)\int_0^{\infty}\frac{dk}{k^{1/2}}\cos kz\sin\left(kx-\frac{\pi}{4}\right). \quad (45)$$

The integral with respect to k in (45) can be calculated in elementary fashion, and as a result we obtain

$$\psi(x) = \int_x^d dz\frac{E_y(z)}{(z-x)^{1/2}}. \quad (46)$$

Equation (46) is Azbel's equation, the solution of which is known, namely

$$E_y(z) = -\frac{d}{dz}\frac{1}{\pi}\int_z^d dx\frac{\psi(x)}{(x-z)^{1/2}}. \quad (47)$$

Thus, the field distribution is determined by the function $\psi(x)$ for positive values of the argument. We shall use Eq. (43) to find this function. As indicated above, the relative width of the singularities of the impedance is small, and the line shape is determined by the electrons near the extremal sections. Let us consider first the case when the diameter $D(p_x)$ reaches a relative maximum on the central section $p_x = 0$. Expanding $D(p_x)$ in a series about $p_x = 0$, we obtain the following equation for $\psi^2(x)$:

$$Z'(d-D_0) = \left(\frac{4\omega}{c^2}\right)^2\frac{8\pi e^2}{h^3\nu[E_y'(0)]^2}\left(\frac{mv_\perp^2}{R}\right)_0$$

$$\times\int_0^{\infty} dp_x\psi^2\left(d-D_0+\frac{D_0''}{2}p_x^2\right)$$

$$= M\int_{d-D_0}^{\infty} dx\frac{\psi^2(x)}{[x-(d-D_0)]^{1/2}}. \quad (48)$$

where

$$M = \left(\frac{4\omega}{c^2}\right)^2\frac{8\pi e^2}{h^3\nu[E_y'(0)]^2}\left[\frac{mv_\perp^2}{R|2D''|^{1/2}}\right]_{p_x=0}. \quad (49)$$

Hence

$$\psi^2(x) = \frac{1}{M\pi}\left(-\frac{d}{dx}\right)\int_x^{\infty}\frac{Z'(u)}{\sqrt{u-x}}du. \quad (50)$$

For the minimal diameter we get

$$\psi^2(x) = \frac{1}{M\pi}\frac{d}{dx}\int_0^x\frac{Z'(u)}{\sqrt{x-u}}du. \quad (51)$$

Formulas (47), (50), and (51) give the solution of the inverse problem—reconstruction of the field in the skin layer from the experimental data on the line shape of the size effect. This solution, obviously, is stable against small variations of Z′, inasmuch as the kernels of the integral equations (46) and (50) are singular.

In connection with the obtained solution of the inverse problem, we must make one more remark. Formula (43) contains the function $\psi(x)$ with positive values of the argument. Consequently, if we regard (43) as an integral equation for the function $\psi(x)$, then we can determine $\psi(x)$ from it only for $x > 0$. Therefore in the actual application of formulas (50) and (51) it is necessary to use the wing of the size-effect line on the side of the strong magnetic field. This circumstance is connected with the obvious fact that in the presence of "cutoff" the contribution to the current density is made by electrons whose trajectories lie wholly inside the plate. We emphasize once more that the proposed method of reconstructing the field in the skin layer is suitable only for unilateral excitation of the plate.

The obtained distribution of the field in the skin layer can be compared with the function E(z) for a semi-infinite space:

$$-\frac{\pi E(z)}{E'(0)\delta} = 2\int_0^\infty \frac{x\cos(xz/\delta)}{x^3 - i}\,dx, \qquad (52)$$

$$\delta = (c^2/4\pi\omega A)^{1/3}. \qquad (53)$$

Figure 1 shows the calculated curves for the real and imaginary parts of the function (52). We see from the plots that a field of noticeable magnitude exists at distances of the order of $(3-4)\delta$ from the surface of the metal.

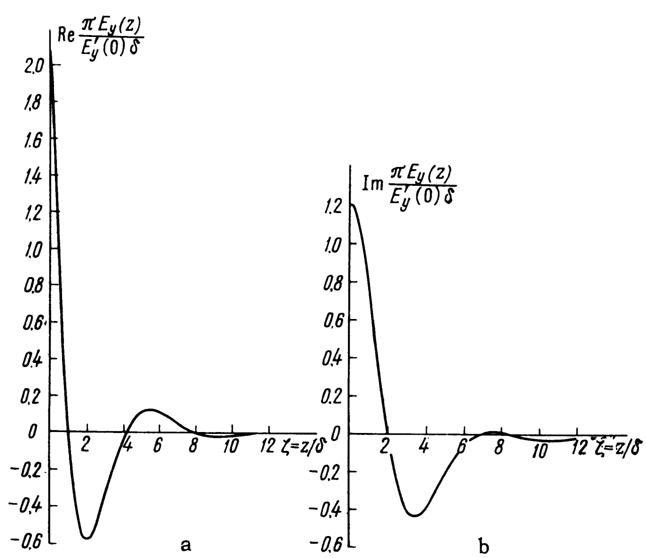

FIG. 1. Spatial distribution of electromagnetic field in metal: a: $\mathrm{Re}(-\pi E(z)/E'(0)\delta)$, b: $\mathrm{Im}(-\pi E(z)/E'(0)\delta)$.

7. APPROXIMATE CALCULATION OF THE LINE SHAPE

Formulas (43), (44), and (32) yield the solution of the problem of the line shape of the size effect. Unfortunately, it is impossible to carry through all the integrations to conclusion in general form. By way of an example we present the calculation of the line shape for a simplified distribution of the field in the metal (compared with the true distribution). Namely, we replace the exact expression (52) for E(z) by one exponential or by a sum of two exponentials:

$$1)\ E_y(z) = E_y'(0)\,k_{eff}^{-1}\exp(-k_{eff}z); \qquad (54)$$

$$2)\ E_y(z) = E_y'(0)\sum_{\alpha=1}^{2} k_\alpha^{-1}\exp\left(-k_\alpha z - \frac{\pi i}{2}\right). \qquad (55)$$

An exponential approximation of the field of the type (54) was used, for example, by Mina and Khaĭkin[12] in the interpretation of the experimental data on the Doppler splitting of the cyclotron-resonance lines in an inclined magnetic field.

In case (1) the effective damping decrement of the wave is determined by the relations

$$-i\frac{4\pi\omega}{c^2}\sigma(k_{eff}) = k_{eff}^2 \quad \text{or} \quad k_{eff} = \delta^{-1}\exp\left(-\frac{\pi i}{6}\right). \qquad (56)$$

Here

$$\psi^2(d-D) = \pi[E'(0)]^2 k_{eff}^{-3}\exp[-2k_{eff}(d-D)]. \qquad (57)$$

Size Effect at Maximum Diameter. We substitute (57) in (43) and expand all the functions in powers of p_x about $p_x = 0$, where $D(p_x)$ has a maximum. Then we obtain on the strong-field size ($D_{max} < d$)

$$Z'(d - D_{max}) = C\exp[-2k_{eff}(d - D_{max})], \qquad (58)$$

where

$$C = \left(\frac{8\pi\omega}{c^2}\right)^2 \frac{\sqrt{\pi}\,e^2}{h^3 v k_{eff}^{7/2}}\left(\frac{mv_\perp^2}{R|D''|^{1/2}}\right)\Bigg|_{p_x=0}$$

$$|C| \sim Z\frac{|k_{eff}d|^{1/2}}{d}, \qquad \arg C = \frac{7\pi}{12}. \qquad (59)$$

At the point of the singularity the derivative $|\partial \ln Z/\partial \ln d|$ is of the order of $(d/\delta)^{1/2}$ and is much larger than unity.

On the weak-field side ($d < D_{max}$) we have

$$Z'(d - D_{max}) = C\exp[-2k_{eff}(d - D_{max})] \times\{1 + \Phi[i\sqrt{2k_{eff}(D_{max} - d)}]\}, \qquad (60)$$

where

$$\Phi(z) = i2\pi^{-1/2}\int_0^z dx\exp(x^2)$$

is the known probability function.

Figure 2a shows plots of the real and imaginary parts of the derivative of the impedance with respect to thickness (or with respect to field), for the case of the maximal diameter and for the field approximated by a single exponential.

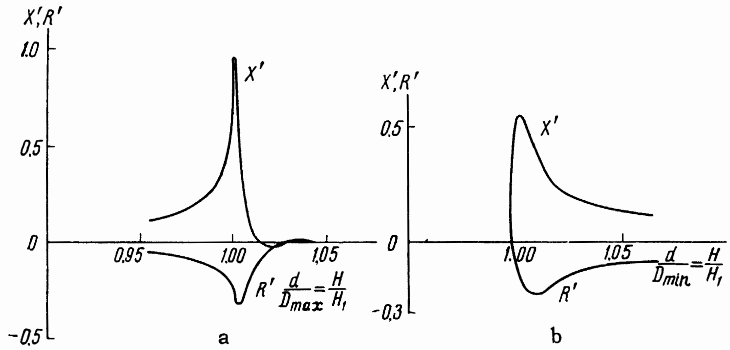

FIG. 2. Line shape of size effect for real and imaginary parts of the impedance when the field is approximated by a single exponential (case 1)): a—maximum section, b—minimum section. The ordinates R' and X' are measured in units of $|C|$. The calculation is made for $d/\delta = 100$.

Size Effect at Minimal Diameter. On the strong-field side ($d > D_{min}$) we have

$$Z'(d - D_{min}) = Cw[2k_{eff}(d - D_{min})]^{1/2}, \qquad (61)$$

where

$$w(z) = 2\pi^{-1/2}\int_0^z \exp(x^2 - z^2)\,dx.$$

In weak fields, when $d < D_{min}$, we have in this approximation $Z' = 0$, since all the electrons collide during each revolution with the boundaries of the plate.

Figure 2b shows plots of R' and X' vs. the magnetic field for the minimal diameter.

We have also calculated the line shape for the more complicated approximation of the field in the metal by a sum of two exponentials (case 2). We have made here the following substitution:

$$[k^2 - i4\pi\omega c^{-2}\sigma(k)]^{-1} = {}^1/_2\{(k^2 + k_1^2)^{-1} + (k^2 + k_2^2)^{-1}\}. \qquad (62)$$

Here

$$|k_1| = |k_2| = \delta^{-1}; \quad \arg k_1 = \pi/3; \quad \arg k_2 = 0. \qquad (63)$$

The choice of the arguments k_1 and k_2 is dictated by the fact that the phase of the impedance must be $-\pi/3$. Calculation of the line shape in this case is perfectly analogous to that for case 1). The results are given in Fig. 3.

It follows from Figs. 2 and 3 that the size-effect lines are asymmetrical. For the maximum diameter the lines fall off more steeply in the direction of the stronger fields ($H > H_1$), whereas for the minimal diameter there is a sharper change on the weak-field side. The different character of the asymmetry of the lines makes it possible to distinguish the minimal and maximal cross sections of the Fermi surface directly by means of the shape of the experimental curves. From a comparison of Figures 2a, 3a, and 1, and from (58), it follows that the size-effect line shape for $H > H_1$ duplicates directly the distribution of the field in the skin layer.

Attention is called to the fact that in case 2) the line has a more complicated structure than in the case 1). It is characteristic that the regions of sharp variation of the real (or imaginary) part of Z' lie near the extrema of the imaginary (or real) part. This fact was recently noted by Krylov[6]. In his paper, devoted to a study of the line shape of the size effect in indium under bilateral excitation of the plate, it is shown that the position of the maximum of the function $\partial R/\partial H$ coincides with the region of the sharp variation of $\partial X/\partial H$, and vice versa.

In this connection, he advances the hypothesis that some integral connection exists in the size effect between R' and X', similar to the Kramers-

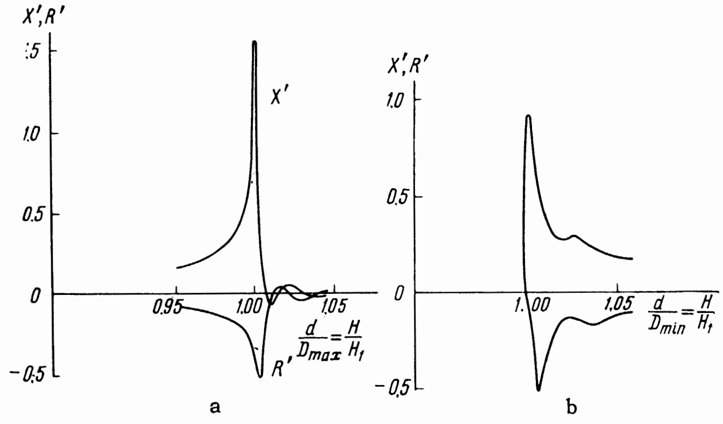

FIG. 3. Line shape of size effect for real and imaginary parts of the impedance when the field is approximated by a sum of two exponentials (case 2)): a—maximum section, b—minimum section; $d/\delta = 100$, R' and X' are in units of $|C|$.

Kronig dispersion relations. Actually, the Kramers-Kronig relations, which describe the analytic properties of the dielectric constant and of the surface impedance as functions of the frequency, yield no information whatever on the line shape of the size effect. The latter is determined by the shape of the extremal cross sections of the Fermi surface and by the character of the damping of the electromagnetic wave in the metal. From the formulas obtained by us it follows that there is apparently no such universal connection in the general case.

However, such a relation is satisfied approximately. For example, if we separate the real and imaginary parts in (58) we obtain

$$R' = |C| \exp\left(-2\frac{d-D_{max}}{\delta_r}\right) \cos\left[\frac{2(d-D_{max})}{\delta_i} - \frac{7\pi}{12}\right],$$
$$X' = |C| \exp\left(-2\frac{d-D_{max}}{\delta_r}\right) \sin\left[\frac{2(d-D_{max})}{\delta_i} - \frac{7\pi}{12}\right]. \tag{64}$$

Because the oscillating factors in R′ and X′ are shifted in phase by $\pi/2$, the regions of rapid variation of one of the functions coincide approximately with the extrema of the other. It can be expected that a similar connection will take place also for the impedance of the plate under bilateral excitation.

In conclusion, we consider it our pleasant duty to thank M. Ya. Azbel' for valuable remarks.

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