INCOHERENT ELECTROMAGNETIC WAVE SCATTERING IN A COULOMB FIELD

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Scattering of circularly polarized waves in a Coulomb field, involving the coalescence of two wave quanta to form one quantum of double frequency, is considered by employing the exact Green's functions for an electron in the field of a plane electromagnetic wave. The scattering matrix element is investigated for a low intensity wave (perturbation theory) and for the case of low frequencies and comparatively arbitrary intensities.

1. IN connection with the development of laser beams containing a large number of photons that are in the same state, it is of interest to consider the scattering of such a beam in a constant external field. We are interested in a process wherein two (or some other even number) quanta of the incident wave merge into one. Then the scattered quantum will have a frequency equal to double (quadruple, etc.) the frequency of the beam quanta.

The incident wave, regarded as a classical electromagnetic field, is taken into account exactly in the sense that exact Green's functions of the electron in the field of a plane electromagnetic wave are used. The external field and the radiated quantum are taken into account in the lowest order of perturbation theory. We confine ourselves to examination of scattering in a Coulomb field. Moreover, we assume that the wave is circularly polarized. The latter circumstance makes possible the merging of only two quanta of the wave into one.

2. The Green's function of the electron in the field of a plane electromagnetic wave with vector potential $A_\mu(nx)$, where $n\omega_0 = k_0$ is the wave vector of the quantum, can be obtained in the usual manner[1] as the difference between the chronological and normal products of the operators Ψ of the electron-positron field, which are expanded in a complete orthonormal system of functions $\{\varphi^{(-)}, \varphi^{(+)}\}$. The functions $\varphi^{(-)}$ and $\varphi^{(+)}$ are exact solutions of the Dirac equations in the field of the plane electromagnetic wave,[2] pertaining respectively to the electron and positron states:

$$\varphi^{(-)} = \left[1 + \frac{e}{2fn}\hat{n}\hat{A}(nx)\right]u^{(r)}(\mathfrak{f})\,e^{iS^{(-)}},$$

$$\varphi^{(+)} = \left[1 - \frac{e}{2fn}\hat{n}\hat{A}(nx)\right]v^{(r)}(-\mathfrak{f})\,e^{-iS^{(+)}},$$

$$S^{(-)}(e) = S^{(+)}(-e) = fx + \frac{e}{fn}\int^{nx} fA\,dy - \frac{e^2}{2fn}\int^{nx} A^2 dy. \quad (1)$$

Then

$$G(x, x' \mid eA) \equiv T[\Psi(x)\Psi(x')] - N[\Psi(x)\Psi(x')]$$
$$= -\int \frac{d^4 f}{(2\pi)^4 i}\left[1 + \frac{e}{2fn}\hat{n}\hat{A}(nx)\right]\frac{i\hat{f} - m}{f^2 + m^2}$$
$$\times\left[1 - \frac{e}{2fn}\hat{n}\hat{A}(nx')\right]$$
$$\times\exp\left\{if(x - x') + i\frac{e}{fn}\int_{nx'}^{nx} fA\,dy - i\frac{e^2}{2fn}\int_{nx'}^{nx} A^2 dy\right\}. \quad (2)$$

Expression (2) for the Green's function was obtained by Lovell et al.[3] as a solution of the equation

$$\{i[\hat{p} - e\hat{A}(nx)] + m\}G(x, x') = -i\delta(x - x').$$

The Green's function (2), for the case of a monochromatic circularly-polarized wave

$$A_x = A_0\cos k_0 x, \quad A_y = A_0 \sin k_0 x, \quad A_z = A_4 = 0 \quad (3)$$

can be represented in the momentum representation in the form

$$G(pp' \mid eA_0) = \sum_{s=-\infty}^{\infty} (2\pi)^4 \delta(p - p' - sk_0)\, G_s(p);$$

$$G_s(p) = ie^{is\varphi}\sum_{l=-\infty}^{\infty}\left\{J_l(z) + \frac{eA_0}{4pk_0}\hat{k}_0\right.$$
$$\times\left[J_{l-1}(z)e^{-i\varphi}\hat{a} + J_{l+1}(z)e^{i\varphi}\hat{a}^*\right]\Big\}$$
$$\times\frac{i\left(\hat{p} - l\hat{k}_0 + \dfrac{e^2A_0^2}{2pk_0}\hat{k}_0\right) - m}{(p - lk_0)^2 + m^{*2}}$$
$$+\left\{J_{l-s}(z) - \frac{eA_0}{4pk_0}\hat{k}_0\left[J_{l-s-1}(z)e^{i\varphi}\hat{a}^* + J_{l-s+1}(z)e^{-i\varphi}\hat{a}\right]\right\};$$

$$ze^{i\varphi} = eA_0\frac{pa}{pk_0}, \quad ze^{-i\varphi} = eA_0\frac{pa^*}{pk_0}. \quad (4)$$

The vectors a and a* describe the states of polarization of the wave (for example in our case

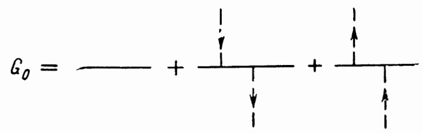

FIG. 1.

$a = \{1, -i, 0, 0\}$, $a^* = \{1, i, 0, 0\}$), $m^* = (m^2 + e^2A_0^2)^{1/2}$ is the effective mass, p and p′ are the quasimomenta of the electron, and J_s is a Bessel function.

The "partial" Green's function $G_s(p)$ describes a virtual particle, which has initially a quasimomentum $-sk_0$ and finally a quasimomentum p, where s is the difference between the number of absorbed and emitted quanta of the wave k_0. Expanding in powers of the field (in the case of a weak field), we can readily verify, for example, that $G_0(p)$ corresponds to the aggregate of diagrams shown in Fig. 1.

Under a charge-conjugation transformation[1] with the aid of the operator $C = i\gamma_4\gamma_2$, the Green's function (2) and (4) satisfy the relations

$$\begin{aligned} C^{-1}G_s(p)C &= (-1)^s\tilde{G}_s(-p+sk_0), \\ C^{-1}G(x, x'|eA_0)C &= \tilde{G}(x', x|-eA_0), \end{aligned} \tag{5}$$

where the tilde denotes the transposed matrix. This property of the Green's function ensures satisfaction of the Furry theorem[1] for diagrams containing closed electron loops.

3. The scattering matrix element of interest to us corresponds to the diagram shown in Fig. 2, where the continuous lines in the loop are the Green's functions (2), and takes in the coordinate representation the form

$$M = -\frac{e^2}{\sqrt{2\omega}}e_\mu^{(\lambda)}\int d^4x_1d^4x_2e^{-ikx_1}$$

$$\times \mathrm{Sp}\,\{\gamma_\mu G(x_1, x_2)\hat{A}^{(e)}(x_2)\,G(x_2, x_1)\}. \tag{6}$$

Here $k = \{\mathbf{k}, \omega\}$ is the momentum of the emitted quantum with polarization λ, and $A^{(e)}$ is the potential of the external field.

We shall henceforth assume that the wave is circularly polarized (3), and its amplitude A_0 is so normalized that $A_0^2 = \rho/\omega_0$, where ρ is the density of the quanta, and we choose as the external field a Coulomb field whose Fourier component of the potential is

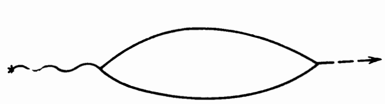

FIG. 2.

$$A_4^{(e)}(q) = -i\frac{Ze}{\mathbf{q}^2}2\pi\delta(q_0). \tag{7}$$

Substituting in (6) the expressions for the Green's function (2) and integrating with respect to x_1 and x_2, we obtain

$$M = \sum_{n=1}^{\infty} e^2\frac{e_\mu^{(\lambda)}}{\sqrt{2\omega}}\frac{A_4^{(e)}(k-2nk_0)}{(2\pi)^4}i\pi^2\left(\frac{eA_0}{2}\right)^2 I_{\mu4}. \tag{8}$$

The scattering tensor $I_{\mu 4}$[1] is of the form

$$I_{\mu4} = \frac{1}{i\pi^2}\left(\frac{2}{eA_0}\right)^2\sum_{s=-\infty}^{\infty}(-1)^s\int d^4pe^{i2n\varphi}\,\mathrm{Sp}$$

$$\times\left\{\left[\left(\gamma_\mu - \frac{e^2A_0^2}{2pk_0\cdot pk_0}k_{0\mu}\hat{k}_0\right)J_s(z)\right.\right.$$

$$+\frac{eA_0}{4}e^{i\varphi}J_{s+1}(z)\left(\gamma_\mu\hat{k}_0\frac{\hat{a}^*}{pk_0}+\frac{\hat{a}^*}{p_1k_0}\hat{k}_0\gamma_\mu\right)$$

$$\left.+\frac{eA_0}{4}e^{-i\varphi}J_{s-1}(z)\times\left(\gamma_\mu\hat{k}_0\frac{\hat{a}}{pk_0}+\frac{\hat{a}}{p_1k_0}\hat{k}_0\gamma_\mu\right)\right]$$

$$\times\frac{i(\hat{p}+e^2A_0^2\hat{k}_0/2pk_0)-m}{p^2+m^{*2}}$$

$$\times\left[\left(\gamma_4 - \frac{e^2A_0^2}{2pk_0\cdot p_1k_0}k_{04}\hat{k}_0\right)J_{2n-s}(z) - \frac{eA_0}{4}e^{i\varphi}J_{2n-s+1}\right.$$

$$\times\left(\gamma_4\hat{k}_0\frac{\hat{a}^*}{p_1k_0}+\frac{\hat{a}^*}{pk_0}\hat{k}_0\gamma_4\right)-\frac{eA_0}{4}e^{-i\varphi}J_{2n-s-1}(z)$$

$$\left.\times\left(\gamma_4\hat{k}_0\frac{\hat{a}}{p_1k_0}+\frac{\hat{a}}{pk_0}\hat{k}_0\gamma_4\right)\right]$$

$$\times\frac{i(\hat{p}_1+e^2A_0^2\hat{k}_0/2p_1k_0)-m}{p_1^2+m^{*2}}, \tag{9}$$

where $J_s(z)$—Bessel function, a and a^*—wave polarization vectors, $p_1 = p - k + sk_0$, and finally

$$ze^{i\varphi} = eA_0\left(\frac{pa}{pk_0} - \frac{p_1a}{p_1k_0}\right),$$

$$ze^{-i\varphi} = eA_0\left(\frac{pa^*}{pk_0} - \frac{p_1a^*}{p_1k_0}\right) \tag{10}$$

As seen from expressions (8) and (7), an even number of quanta k_0 is absorbed from the wave, this being a consequence of the Furry theorem.

Further calculation of the scattering tensor is very cumbersome, and therefore we present only a few explanations. After calculating the trace, the terms of the sum with different s are regrouped, using the properties of the Bessel functions. This increases the number of factors of the type $p^2 + m^{*2}$ in the denominator, a result important for the calculation of the integral with respect to d^4p.

This is followed by a Feynman parametrization.[1] The argument of the Bessel functions z, and also $\exp(\pm i\varphi)$ from (10), depend only on the scalar products pk_0, pa, and pa^*. As a result the calculation reduces to taking integrals of the following types:

$$P_1 = \int \frac{d^4p}{(p^2+\Lambda-i\delta)^r}\,\Phi(pk_0, pa, pa^*), \qquad r = 2, 3, 4,$$

$$P_2 = \int \frac{d^4p}{(p^2+\Lambda-i\delta)^r}\,p_\mu\Phi(pk_0, pa, pa^*), \qquad r = 3, 4,$$

$$P_3 = \int \frac{d^4p}{(p^2+\Lambda-i\delta)^r}\,p_\mu p_\nu\Phi(pk_0, pa, pa^*), \qquad r = 4 \tag{11}$$

and Λ does not depend on p.

Let, for example, $\mathbf{k}_0$ be directed along the z axis, when three out of the integrations are possible (with respect to p_0, p_z, and the azimuthal angle φ; details are given in the appendix). It then turns out that the only powers of pa remaining in the integrals for the harmonics of the matrix element (8) with $n \geq 2$ are those causing the integrals to vanish. Therefore only the term with $n=1$ remains.

The impossibility of merging of four (six, etc.) circularly-polarized quanta into one can be explained in the following manner. We have to form a scalar containing four fields $F^0_{\mu\nu}$ corresponding to four absorbed quanta of the wave, the field of the emitted quantum $F_{\mu\nu}$, and the external field $F^{(e)}_{\mu\nu}$. The tensors $F^0_{\mu\nu}$ cannot be contracted with one another even with respect to one index, since $k_0^2 = 0$, $k_0a = 0$, and $a^2 = 0$, where a is the polarization vector. Therefore the four fields pertaining to the wave make up a tensor of eighth rank. In order to form a scalar, it is necessary to introduce four more derivatives, for example,

$$F^0_{\alpha\beta}F^0_{\gamma\delta}F^0_{\lambda\sigma}\partial_\alpha\partial_\gamma F_{\lambda\sigma}F^0_{\mu\nu}\partial_\beta\partial_\delta F^{(e)}_{\mu\nu}$$
$$= F^0_{\alpha\beta}F^0_{\gamma\delta}F^0_{\lambda\sigma}k_\alpha k_\gamma F_{\lambda\sigma}F^0_{\mu\nu}k_\beta k_\delta F^{(e)}_{\mu\nu} = 0,$$

owing to the antisymmetry of the tensor $F_{\mu\nu}$.

Then the scattering tensor (9) can be representsented in the form

$$I_{\mu 4} = \left[\frac{k_{0\mu}k_{04}(ka)^2}{(kk_0)^2} - \frac{a_\mu k_{04}(ka)}{(kk_0)}\right]Q,$$

$$Q = -8\sum_{s=-\infty}^{\infty}\int_0^1 \frac{dy}{\sqrt{1-y}}\int_0^\infty x\,dx\,J_s^2\left(\frac{eA_0}{m^*}\frac{2}{\alpha y}x\right)$$
$$\times\left[\frac{b}{\alpha^2y^2}\ln\left(1-\frac{\alpha^2y^2}{b^2}\right)+\frac{1}{b}\right],$$

$$b = 1+\alpha(s+1)y+x^2-i\delta, \quad \alpha = -\frac{kk_0}{2m^{*2}},$$

$$m^{*2} = m^2 + e^2A_0^2. \tag{12}$$

It must be noted that the conclusion that only two circularly-polarized quanta can merge into one is valid only in first order of perturbation theory with respect to the external field. In higher orders, merging of a larger number of quanta of the wave into one become possible.

4. We proceed to consider the case of a weak field. Letting formally $A_0 \to 0$, we retain in the infinite sum of (12) only the term with $s = 0$, and replace J_0 with unity. Calculating the integrals, we obtain the following expression for the scattering tensor:

$$I_{\mu 4} = \frac{ka}{m^4}\left[k_{0\mu}k_{04}(ka) - a_\mu k_{04}(kk_0)\right]R(\alpha);$$

$$R(\alpha) = \frac{1}{\alpha^2} - \frac{1}{\alpha^4}\sqrt{2\alpha(1+2\alpha)}\ln\left(\sqrt{2\alpha}+\sqrt{1+2\alpha}\right)$$
$$+\frac{2}{\alpha^4}\sqrt{\alpha(1+\alpha)}\ln\left(\sqrt{\alpha}+\sqrt{1+\alpha}\right)$$
$$+\frac{1+4\alpha}{4\alpha^4}\{F[2\sqrt{2\alpha}(\sqrt{2\alpha}+\sqrt{1+2\alpha})]$$
$$+F[2\sqrt{2\alpha}(\sqrt{2\alpha}-\sqrt{1+2\alpha})]\} -$$
$$-\frac{1+4\alpha}{2\alpha^4}\{F[2\sqrt{\alpha}(\sqrt{\alpha}+\sqrt{1+\alpha})]$$
$$+F[2\sqrt{\alpha}(\sqrt{\alpha}-\sqrt{1+\alpha})]\}, \tag{13}$$

where

$$\alpha = -\frac{kk_0}{2m^2}, \qquad F(x) = \int_0^x \frac{\ln(1+t)}{t}\,dt.$$

Then the scattering matrix element (8), where $I_{\mu 4}$ is of the form (13), corresponds exactly to the diagrams of Fig. 3, as can be verified by a direct check.

The corresponding differential scattering cross section with allowance for (7), (8) and (13), summed over the polarizations of the outgoing quantum, is

$$d\sigma = \frac{1}{8}r_0^2\xi_0^2\left(\frac{Ze^2}{4\pi}\right)^2\left(\frac{\omega_0}{m}\right)^4\sin^2\theta R^2(\alpha)\frac{do}{4\pi}, \tag{14}$$

where $r_0 = e^2/4\pi m$ is the classical radius of the electron, $\xi_0^2 = e^2A_0^2/4\pi m^2$, θ is the angle of emission of the quantum of frequency $\omega = 2\omega_0$, and $e^2/4\pi = 1/137$.

At low frequencies $\omega_0 \ll m$, corresponding to $\alpha \ll 1$, the function $R(\alpha)$ turns into a constant

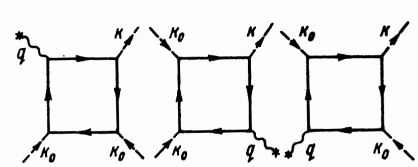

FIG. 3.

$$R(\alpha) = {}^4/_{15}, \quad (15)$$

and the differential cross section is equal to

$$d\sigma = \frac{2}{15^2} r_0^2 \xi_0^2 \left(\frac{Ze^2}{4\pi}\right)^2 \left(\frac{\omega_0}{m}\right)^4 \sin^2\theta \frac{do}{4\pi}. \quad (16)$$

In this case the total cross section

$$\sigma = \frac{4}{3\times 15^2} r_0^2 \xi_0^2 \left(\frac{Ze^2}{4\pi}\right)^2 \left(\frac{\omega_0}{m}\right)^4, \quad \omega_0 \ll m, \quad (17)$$

is proportional to the fourth power of the frequency.

Let us integrate (14) over the angles of emission of the quantum. Then

$$\sigma = \frac{1}{8} r_0^2 \xi_0^2 \left(\frac{Ze^2}{4\pi}\right)^2 \int_0^{2\omega_0^2/m^2} x\left(1 - \frac{m^2}{2\omega_0^2} x\right) R^2(x)\, dx. \quad (18)$$

If the frequency is high, $\omega_0 \gg m$, then the upper limit of integration can be replaced by infinity, since R(x) decreases sufficiently rapidly as $x \to \infty$, $R(x) \approx 1/x^2$, and the integrals converge well. In this limiting case

$$\sigma = c r_0^2 \xi_0^2 \left(\frac{Ze^2}{4\pi}\right)^2, \ \omega_0 \gg m; \ c = \frac{1}{8}\int_0^\infty x R^2(x)\, dx \approx 0{,}03. \quad (19)$$

The frequencies of modern lasers are low ($\omega_0/m \sim 10^{-6}$), and therefore the scattering cross section (17) is also small. The position becomes aggravated by the fact that the wavelength is in this case much larger than the dimensions of the atom, and the Coulomb potential of the nucleus is completely screened. Allowance for screening leads to the appearance in formula (17) of an additional factor $(a_0/\lambda)^4$, where a_0 is the radius of the atom and λ the wavelength of the quantum. If ultrarelativistic protons (with energy $\gtrsim 10^9$ MeV) are irradiated with a laser beam, then the frequency in the coordinate system connected to the protons is $\omega_0 \gtrsim m$, and the cross section will be maximal (19).

5. Let us return to the exact expression (12) for the scattering tensor, which we now investigate for the case of low frequencies $\omega_0 \ll m$ (as in the real situation). We consider the contribution to Q [Eq. (12)] from the terms of the sum with $|s| \sim 1$. The logarithm can be expanded in a series. Then these terms take the form

$$Q_s = 4\alpha^2 \int_0^1 \frac{y^2 dy}{\sqrt{1-y}} \int_0^\infty dx \frac{x}{[1 + \alpha(s+1)y + x^2]^3} \times J_s^2\left(\frac{eA_0}{m^*}\frac{2}{\alpha y} x\right)$$

$$= 4\alpha^2 \int_0^1 \frac{y^2 dy}{\sqrt{1-y}} \left(\frac{eA_0}{m^*}\frac{2}{\alpha y}\right)^4 \frac{1}{2}\frac{\partial^2}{\partial z^2} I_s(\sqrt{z}) K_s(\sqrt{z}),$$

where I_s and K_s are Bessel functions of imaginary arguments,[4] and

$$z = \left(\frac{eA_0}{m^*}\frac{2}{\alpha y}\right)^2 (1 + \alpha y + \alpha s y).$$

In addition,

$$\gamma \equiv \frac{eA_0}{m^* \alpha} \sim \frac{m^2}{\omega_0^2} \xi \sqrt{1+\xi^2} \gg 1,$$

if $\xi \equiv eA_0/m \gg \omega_0^2/m^2$, which is satisfied even for contemporary beams. Then, using the asymptotic values of the Bessel functions I_s and K_s,[4] we get $Q_s \sim \alpha^2/\gamma$, i.e., the contribution of the terms with $|s| \sim 1$ is small, and the principal role is played by the terms with $|s| \gg 1$.

Let us change the integration variable

$$x \to \frac{|s|}{\gamma}\frac{y}{2} x.$$

We go over to the asymptotic expression for the Bessel function in terms of the Airy function:[4]

$$J_n(nx) \approx \frac{1}{\sqrt{\pi}}\left(\frac{2}{n}\right)^{1/3} \frac{1}{x^{1/3}} \Phi\left[\left(\frac{n}{2}\right)^{2/3} \frac{2(1-x)}{x^{1/3}}\right], \quad n \gg 1 \quad (21)$$

and break up Q [Eq. (12)] into three parts:

$$Q \equiv Q^{(1)} + Q^{(2)} + Q^{(3)} \equiv \sum_{s=-\infty}^{\infty} \int_1^\infty dx + \sum_{s=-\infty}^{-s_0-1} \int_0^1 dx + \sum_{s=-s_0}^{\infty} \int_0^1 dx. \quad (22)$$

Here s_0 is such that $1 \le \alpha s_0 \le 1 + \alpha$, i.e, $s_0 \approx 1/\alpha \gg 1$ and

$$b = 1 + \alpha(s+1)y + \frac{1}{4}(\alpha s y)^2 \left(\frac{m^*}{eA_0}\right)^2 x^2 - i\delta.$$

Re b has a minimum equal to $1 - x^{-2}(eA_0/m^*)^2$, which is reached when s is negative. If $x \ge 1$, this minimum is positive and

$$\alpha / (\mathrm{Re}\, b)_{min} \lesssim \omega_0^2 / m^2 \ll 1,$$

i.e., the logarithm entering in Q can be expanded in a series. Re B can vanish for negative s such that $|s| \ge s_0 + 1$.

Let us now consider $Q^{(1)}$ [Eq. (22)]. Expanding the logarithm and replacing the summation over s with integration, we represent $Q^{(1)}$ after simple transformations in the form

$$Q^{(1)} = 4\alpha^2 \frac{2}{\pi} \int_0^1 \frac{y^2 dy}{\sqrt{1-y}} \left(\frac{\gamma}{y}\right)^{1/3} \int_0^1 dx \int_0^\infty ds$$

$$\times s^{4/3} \Phi^2 \left[-2x\left(\frac{\gamma s}{y}\right)^{2/3}\right] \left\{ \left[1 + 2\frac{eA_0}{m^*}(1-x)s + s^2\right]^{-3} + \left[1 - 2\frac{eA_0}{m^*}(1-x)s + s^2\right]^{-3} \right\}, \quad (23)$$

where Φ is the Airy function, and $\gamma \equiv eA_0/m^*\alpha$.

The essential region in the integral is $s \sim 1$. In the second denominator, when $\xi \gg 1$, the values $x \sim \xi^{-2}$ are significant. Then

$$\xi^{-2}\gamma^{2/3} \sim (\alpha\xi^3)^{-2/3} \gg 1, \quad (24)$$

if $\xi \ll m^2/\omega_0^2$, which we shall assume satisfied. We can then use the asymptotic value of Φ for large negative values of the argument[4]

$$\Phi(-z) \approx \frac{1}{z^{1/4}} \sin\left(\frac{2}{3} z^{3/2} + \frac{\pi}{4}\right).$$

Replacing $\sin^2 z$ by $1/2$ in the integration, we obtain

$$Q^{(1)} = 4\alpha^2 \varphi_1(eA_0/m^*), \quad (25)$$

The quantity $Q^{(2)}$ has a real and an imaginary

$$\varphi_1\left(\frac{eA_0}{m^*}\right) = \frac{2\sqrt{2}}{15\pi} \int_0^1 \frac{dx}{\sqrt{x}} \left[\frac{2+a^2}{(1-a^2)^2} + \frac{3a}{(1-a^2)^{5/2}} \arcsin a\right],$$

$$a = \frac{eA_0}{m^*}(1-x). \quad (25')$$

part. In analogy with the procedure used above, we obtain for the sum $Q^{(3)} + \mathrm{Re}\, Q^{(2)}$ the expression

$$Q^{(3)} + \mathrm{Re}\, Q^{(2)} = 4\alpha^2 \frac{2}{\pi} \int_0^1 \frac{y^2 dy}{\sqrt{1-y}} \left(\frac{\gamma}{y}\right)^{1/3} \int_0^{m^*/eA_0 - 1} dx \int_0^\infty ds$$

$$\times s^{4/3} \Phi^2 \left[2x\left(\frac{\gamma s}{y}\right)^{2/3}\right] \left\{ \left[1 + 2\frac{eA_0}{m^*}(1+x)s + s^2\right]^{-3} + \left[1 - 2\frac{eA_0}{m^*}(1+x)s + s^2\right]^{-3} \right\}. \quad (26)$$

An important role in the integral is played by $s \sim 1$ and $x \sim \gamma^{-2/3}$ since Φ decreases exponentially at large positive values of the argument. Therefore, owing to the condition (24), we can neglect the dependence on x in the denominators, and replace the upper limit of integration with respect to x by $+\infty$. Then

$$Q^{(3)} + \mathrm{Re}\, Q^{(2)} = 4\alpha^2 \gamma^{-1/3} \varphi_2\left(\frac{eA_0}{m^*}\right); \quad (27)$$

$$\varphi_2 = \frac{7\cdot 16\cdot 3^{1/3}}{5\cdot 11\cdot 17\sqrt{\pi}} \int_0^\infty ds \cdot s^{2/3} \left[\left(1 + 2\frac{eA_0}{m^*}s + s^2\right)^{-3} + \left(1 - 2\frac{eA_0}{m^*} s + s^2\right)^{-3}\right]. \quad (27')$$

Thus, the real part of the scattering tensor $I_{\mu 4}$ [Eq. (12)] is

$$\mathrm{Re}\, I_{\mu 4} = \frac{ka}{m^4}[k_{0\mu}k_{04}(ka) - a_\mu k_{04}(kk_0)] \times \left(\frac{m}{m^*}\right)^4 \left[\varphi_1\left(\frac{eA_0}{m^*}\right) + \left(\frac{eA_0}{m^*\alpha}\right)^{-1/3} \varphi_2\left(\frac{eA_0}{m^*}\right)\right], \quad (28)$$

where φ_1 and φ_2 are determined by formulas (25′) and (27′).

Expression (28) is valid under the following conditions:

$$\omega_0 \ll m, \quad \omega_0^2/m^2 \ll \xi \ll m^2/\omega_0^2. \quad (29)$$

The functions φ_1 and φ_2 can be calculated in final form, but the expressions obtained are unwieldy, so that we shall consider only limiting cases.

For a weak field, when $\xi \ll 1$, the functions φ_1 and φ_2 turn into constants, and the scattering tensor (28) is

$$\mathrm{Re}\, I_{\mu 4} = \frac{ka}{m^4}[k_{0\mu}k_{04}(ka) - a_\mu k_{04}(kk_0)] \left[c_1 + \left(\frac{\alpha_0}{\xi}\right)^{1/3} c_2\right], \quad (30)$$

where

$$c_1 \cong \frac{8\sqrt{2}}{15\pi}, \quad c_2 \cong \frac{4\cdot 7^2\cdot 3^{1/3}\sqrt{\pi}}{5\cdot 9\cdot 11\cdot 17}, \quad \alpha_0 = -\frac{1}{2}\frac{kk_0}{m^2}.$$

In the case of a strong field with $\xi \gg 1$

$$\varphi_1 \sim \xi^4, \quad \varphi_2 \sim \xi^5$$

and

$$\mathrm{Re}\, I_{\mu 4} = \frac{ka}{m^4}[k_{0\mu}k_{04}(ka) - a_\mu k_{04}(kk_0)][c_3 + (\alpha_0\xi)^{1/3} c_4], \quad (31)$$

where

$$c_3 \cong \frac{4}{15}, \quad c_4 \cong \frac{42\cdot 3^{1/3}\cdot\sqrt{\pi}}{5\cdot 11\cdot 17}.$$

Finally, $\mathrm{Im}\, I_{\mu 4}$ is determined by the imaginary part of $Q^{(2)}$ [Eq. (22)]:

$$\mathrm{Im}\, Q^{(2)} = 2\pi \sum_{s=s_0+1}^\infty \int_0^1 \frac{y^2 dy}{\sqrt{1-y}} \alpha^2 s^2 \left(\frac{m^*}{eA_0}\right)^2 \int_0^1 x\, dx J_s^2(sx) \times \left\{\int_0 dt\cdot t[\delta(b + \alpha yt) + \delta(b - \alpha yt)] - \delta(b)\right\},$$

$$b = 1 + \alpha(1-s)y + 1/4(\alpha sy)^2 (m^*/eA_0)^2 x^2. \quad (32)$$

In the case of a strong field $1 \ll \xi \ll m^2/\omega_0^2$:

$$\operatorname{Im} Q^{(2)} \cong \frac{3}{2}\sqrt{\frac{\pi}{2}}\frac{\alpha_0^2}{\sqrt{\alpha_0\xi}}\exp\left\{-\frac{4}{3}\frac{1}{\alpha_0\xi}\right\}. \qquad (33)$$

The occurrence of an imaginary part in the scattering tensor $I_{\mu 4}$ is connected with the fact that several quanta of the wave k_0 can create a real electron-positron pair, which then annihilates with formation of the quantum k and several quanta k_0.

Thus, as can be seen from (30), (31), and (33), the scattering cross section in the case of low frequencies will be determined by formulas (16) and (17), in which $\xi = \sqrt{4\pi}\,\xi_0$ can change down to values $\xi \ll m^2/\omega_0^2$.

This is connected with the fact that although in strong fields it is necessary to take into account the diagrams in which a large number of quanta of the wave participate, their contribution to the matrix element almost completely cancel each other.

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APPENDIX

Let us consider the well known relation[1]

$$\int_{+0}^{\infty} e^{-i(A+xy)t-\delta t}dt = \frac{-i}{A+xy-i\delta}, \qquad \delta \to +0.$$

Let us differentiate it once with respect to A, and then integrate with respect to y from $-\infty$ to $+\infty$. We then obtain

$$P_0 \equiv \int_{-\infty}^{\infty}\frac{dy}{(A+xy-i\delta)^2} = -\int_{+0}^{\infty} te^{-iAt-\delta t}\,dt\int_{-\infty}^{\infty} e^{-ixyt}\,dy$$

$$= -2\pi\int_{+0}^{\infty} te^{-iAt-\delta t}\delta(xt)\,dt.$$

but since the integral with respect to t is taken from +0, we get

$$\delta(xt) = t^{-1}\delta(x),$$

and

$$P_0 = -2\pi\delta(x)\int_{+0}^{\infty} e^{-iAt-\delta t} = \frac{2\pi i}{A-i\delta}\delta(x).$$

Thus, we have the following relation:

$$P_0 \equiv \int_{-\infty}^{\infty}\frac{dy}{(A+xy-i\delta)^2} = \frac{2\pi i}{A-i\delta}\delta(x). \qquad (A.1)$$

We now proceed to calculate the integrals (11). We consider the integral P_1 for $r = 2$:

$$P_1 = \int\frac{d^4p}{(p^2+\Lambda-i\delta)^2}\Phi(pk_0, pa, pa^*).$$

We assume that $\mathbf{k}_0$ is directed along the z axis, therefore $pk_0 = \omega_0(p_z - p_0)$, since $k_0^2 = 0$; then $pa = p_x - ip_y$, and $pa^* = p_x + ip_y$. The square of the momentum is written as follows: $p^2 = (pa)(pa^*) + (p_z - p_0)(p_0 + p_0)$. We proceed to new integration variables $x = p_z - p_0$ and $y = p_z + p_y$, and then

$$P_1 = \int dp_x\,dp_y\frac{1}{2}\int_{-\infty}^{\infty} dx\Phi(\omega_0 x, pa, pa^*)$$

$$\times\int_{-\infty}^{\infty}\frac{dy}{[(pa)(pa^*)+\Lambda+xy-i\delta]^2}$$

Using (A.1) we obtain ultimately

$$P_1 = \pi i\int dp_x\,dp_y\frac{\Phi(0, pa, pa^*)}{(pa)(pa^*)+\Lambda-i\delta}. \qquad (A.2)$$

It is necessary to retain the negative imaginary addition in the denominator, since Λ can in general be negative. Integrals of the type P_1 with $r = 3, 4$, etc. are obtained in (A.2) by differentiating with respect to Λ.

We now consider the integral P_2. The vector p_μ can be represented in the form

$$p_\mu = \frac{1}{2}(pa^*)a_\mu + \frac{1}{2}(pa)a_{\mu^*} + \frac{pk_0}{\omega_0}\delta_{\mu 3}$$

$$-\frac{1}{2}\frac{pk_0}{\omega_0^2}k_{0\mu} + \frac{1}{2}\frac{p_z+p_0}{\omega_0}k_{0\mu}.$$

The first four terms in p_μ lead to integrals of the type P_1, so that it remains to consider the integral of the fifth term

$$\tilde{P}_2 \equiv \int\frac{d^4p}{(p^2+\Lambda-io)^3}\frac{1}{2}\frac{p_z+p_0}{\omega_0}\Phi(pk_0, pa, pa^*)$$

$$= \int dp_x\,dp_y\frac{1}{4\omega_0}\int_{-\infty}^{\infty} dx\Phi(\omega_0 x, pa, pa^*)$$

$$\times\int_{-\infty}^{\infty}\frac{y\,dy}{[(pa)(pa^*)+\Lambda+xy-i\delta]^3}$$

The resultant integral with respect to y can be obtained from (A.1) by differentiation with respect to x, and this leads to the appearance of a derivative of a function. Then

$$\tilde{P}_2 = \frac{\pi i}{4}\int\frac{dp_x\,dp_y}{(pa)(pa^*)+\Lambda-i\delta}\left.\frac{\partial\Phi(pk_0, pa, pa^*)}{\partial(pk_0)}\right|_{pk_0=0}. \qquad (A.3)$$

The integral P_3 is calculated in similar fashion.

We note in conclusion that we have assumed in the calculation that all the integrals converge and have no singularities at the point $pk_0 = 0$, which certainly holds true in our case. When diverging integrals are calculated in this manner, the obtained main diverging term turns out to be correct, but the finite addition to it is in general incorrect because of the non-invariance of the integration region.

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