WEAK-TURBULENCE SPECTRUM IN A PLASMA WITHOUT A MAGNETIC FIELD

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A kinetic equation describing four-plasmon processes in a plasma without a magnetic field is derived and it is shown that it has an exact solution. The solution may be interpreted as the turbulence spectrum in the universal equilibrium region. It is demonstrated that the turbulence is of a local-isotropic nature.

1. INTRODUCTION

As is well known, two types of weak plasma turbulence are possible.[1,2] The first is due to the scattering of waves by the plasma particles; it was investigated in a number of papers.[1,2] The second is due to processes of decay, coalescence, and neutral scattering of waves without energy exchange between the particles and the waves. We shall show that weak turbulence of this type has properties analogous in many respects to those of ordinary hydrodynamic turbulence. Namely, regions of wave numbers can be separated in k-space, such that the turbulence has a universal power-law spectrum determined only by the magnitude of the energy flux in the region of large k.

In hydrodynamic turbulence, the proof of this fact is based on considerations of dimensionality and on the hypothesis of local turbulence, that is, on the assumption that only spatial scales of one order of magnitude interact with one another.[3,4] In spite of the numerous attempts to prove this statement, it has remained heuristic so far. The difficulty lies in the fact that it is impossible to construct a closed system of equations with which to describe hydrodynamic turbulence. Much more progress can be made in the theory of weak turbulence because weak turbulence is described by the kinetic equation for waves.

We shall consider one of the simplest cases of weak turbulence—a system of interacting Langmuir plasmons in an isothermal plasma without a magnetic field, when the main plasmon interaction is their scattering by one another. Since this process does not depend on the detailed structure of the electron-velocity distribution function, we can use the hydrodynamic equations to describe the plasma. Starting from these equations, we obtain a kinetic equation for the plasmons. An analysis of the kinetic equation shows that it has an exact solution, $n_k = \text{const} \cdot k^{-13/3}$, which can be interpreted as the spectrum in the region of universal equilibtium. The same solution is obtained also from dimensionality considerations, with const $\sim p^{1/3}$, where p is the energy flux if k is large. It is simultaneously possible to prove the local isotropy of the turbulence.

Analogous results were obtained earlier for one model problem[5] and for weak turbulence of waves on the surface of a liquid.[6]

2. KINETIC EQUATION FOR PLASMONS

We start with the system of hydrodynamic equations for an electronic liquid in the presence of a positively-charged background

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -3\omega_p^2 r_D^2 \nabla n - \frac{e}{m}\nabla\varphi,$$
$$\partial n / \partial t + \operatorname{div}(n\mathbf{v}) = 0,$$
$$\Delta\varphi = 4\pi e(n - n_0). \tag{1}$$

Here n and $\mathbf{v}$ are respectively the density and velocity of the electrons, n_0 the unperturbed density, φ the electrostatic potential, and ω_p and r_D the Langmuir frequency and the Debye radius of the plasma.

Equations (1) are suitable for the description of motions with small gradients ($\nabla n/n \ll 1/r_D$). Under these assumptions the electrostatic pressure is much larger than the gas-kinetic pressure. Therefore the term with the gas-kinetic pressure is linearized in terms of n.

We take the Fourier transform with respect to the coordinates in the normalization volume V and go over to new variables a_k by means of the formulas

$$\mathbf{v} = \frac{1}{V^{1/2}} \sum_k \left(\frac{\omega_k}{2mn_0}\right)^{1/2} \frac{\mathbf{k}}{k}(a_k - a^*_{-k}) e^{i(\mathbf{kr})},$$

$$\delta n = n - n_0 = \frac{1}{V^{1/2}} \sum_k \left(\frac{n_0}{2m\omega_k}\right)^{1/2} k(a_k + a^*_{-k}) e^{i(\mathbf{kr})}. \tag{2}$$

Here $\omega_k = \omega_p + {}^3/_2\,\omega_p(kr_D)^2$ is the law of dispersion of the Langmuir waves. We shall assume throughout that $kr_D \ll 1$.

In terms of the variables a_k, Eqs. (1) take the form

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i\sum_{k',k''}\Gamma_{kk'k''}(a_{k'}a_{k''}\delta(\mathbf{k},\mathbf{k}'+\mathbf{k}'')$$
$$+ 2a_{k'}{}^* a_{k''}\delta(\mathbf{k},\mathbf{k}''-\mathbf{k}') + a_{k'}{}^* a_{k''}^*\delta(\mathbf{k},-\mathbf{k}'-\mathbf{k}''). \quad (3)$$

When $kr_D \ll 1$, the matrix element Γ takes the form

$$\Gamma_{kk_1k_2} = (\omega_p / 8Vn_0m)^{1/2}Q_{kk_1k_2},$$
$$Q_{kk_1k_2} = \frac{(\mathbf{k}_1\mathbf{k}_2)k}{k_1k_2} + \frac{(\mathbf{k}\mathbf{k}_1)k_2}{kk_1} + \frac{(\mathbf{k}\mathbf{k}_2)k_1}{kk_2}. \quad (4)$$

Equation (3) can be obtained in variational form

$$\partial a_k / \partial t = -i\partial H / \partial a_k^*, \quad (5)$$

where the Hamiltonian H is

$$H = \sum_k \omega_k a_k a_k^* + \frac{1}{3}\sum_{k_1k_2}\Gamma_{kk_1k_2}(a_k a_{k_1} a_{k_2}$$
$$+ a_k^* a_{k_1}^* a_{k_2}^*)\delta(\mathbf{k},-\mathbf{k}_1-\mathbf{k}_2) + \sum_{k_1,k_2}\Gamma_{kk_1k_2}(a_k^* a_{k_1} a_{k_2}$$
$$+ a_k a_{k_1}^* a_{k_2}^*)\delta(\mathbf{k},\mathbf{k}_1+\mathbf{k}_2).$$

In the normalization chosen by us, H coincides with the total energy of the plasma, given by the formula

$$H = \int d\mathbf{r}\left[\frac{m}{2}n(\mathbf{v})^2 + \frac{1}{8\pi}(\nabla\varphi)^2\right].$$

Let us simplify (3). To this end we represent a_k in the form

$$a_k = [A_k(t) + f(k,t)]e^{-i\omega_p t}. \quad (6)$$

Here A_k is a slowly varying function with a characteristic variation time much larger than $1/\omega_p$; the value of f changes appreciably within a time on the order of $1/\omega_p$. We substitute (6) in (3) and eliminate f. To this end we retain in the right side only the terms that are quadratic in A, and then, using the slowness of the variation of A, we integrate the obtained equation with respect to the time. In addition, we neglect in the integration the thermal corrections to the dispersion of ω_k. We obtain

$$f = \frac{1}{\omega_p}\sum_{k_1k_2}\Gamma_{kk_1k_2}(A_{k_1}A_{k_2}e^{-i\omega_p t}\delta(\mathbf{k},\mathbf{k}_1+\mathbf{k}_2)$$
$$- 2A_{k_1}{}^* A_{k_2}\, e^{i\omega_p t}\delta(\mathbf{k},\mathbf{k}_2-\mathbf{k}_1)$$
$$- {}^1/_3 A_{k_1}{}^* A_{k_2}{}^* e^{3i\omega_p t}\delta(\mathbf{k},-\mathbf{k}_1-\mathbf{k}_2).$$

Substituting f in the right side of (3) and considering only the terms that do not contain rapid exponentials, we obtain for A_k

$$\frac{\partial A_k}{\partial t} + i\Omega_k A_k = -\frac{i}{8Vmn_0}\sum_{k_1,k_2,k_3} S_{kk_1k_2k_3}A_{k_1}{}^* A_{k_2}A_{k_3}\cdot$$
$$\cdot\,\delta(\mathbf{k}+\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_3). \quad (7)$$

Here

$$\Omega_k = {}^3/_2\omega_p(kr_D)^2, \quad (8)$$

$$S_{k,k_1,k_2,k_3} = 2Q_{k+k_1,k,k_1}Q_{k_2+k_3,k_2,k_3}$$
$$- {}^1/_3 Q_{-k-k_1,k,k_1}Q_{-k_2-k_3,k_2,k_3}$$
$$- 2Q_{k,k_2,k-k_2}Q_{k_1,k_3,k_3-k_1} - 2Q_{k,k_3,k-k_3}Q_{k_1,k_2,k_2-k_1}$$
$$-2Q_{k,k_2,k_2-k}Q_{k_1,k_3,k_1-k_3} - 2Q_{k,k_3,k_3-k}Q_{k_1,k_2,k_1-k_2}. \quad (9)$$

The function S_{k,k_1,k_2,k_3} satisfies the symmetry conditions

$$S_{k,k_1,k_2,k_3} = S_{k_1,k,k_2,k_3} = S_{k,k_1,k_3,k_2} = S_{k_2,k_3,k,k_1}. \quad (10)$$

For (8) to be valid it is necessary to satisfy the condition $f \ll A$, which leads to the requirement that the nonlinearity be small, $\delta n/n \ll 1$. Equation (8) is perfectly analogous to the Heisenberg equations of motion for a Bose gas, with A_k and A_k^* the classical analogs of the quantum annihilation and creation operators. Equation (8) conserves the energy, the momentum, and the total number of plasmons

$$N = \sum_k A_k A_k^*.$$

Corresponding to (7) is the Hamiltonian

$$\tilde{H} = \sum_k \Omega_k A_k A_k^* + \frac{1}{16mVn_0}\sum_{k,k_1,k_2,k_3} S_{kk_1k_2k_3}A_k^* A_{k_1}{}^* A_{k_2}A_{k_3}$$
$$\times\,\delta(\mathbf{k}+\mathbf{k}^1\;\mathbf{k}_2+\mathbf{k}_3).$$

We now proceed to a statistical description of the system. We assume that the oscillation phases corresponding to different k are random, and change to a new variable—the particle number density

$$n_k = \langle A_k A_k^*\rangle.$$

The angle brackets denote averaging. For n_k we obtain the kinetic equation

$$\frac{\partial n_k}{\partial t} = \frac{\pi}{32V^2n_0{}^2m^2}\sum_{k_1,k_2,k_3}|S_{kk_1k_2k_3}|^2\delta(\mathbf{k}+\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_3)$$
$$\times(n_{k_1}n_{k_2}n_{k_3} + n_k n_{k_2}n_{k_3} - n_k n_{k_1}n_{k_3} - n_k n_{k_1}n_{k_2})$$
$$\times\,\delta(\Omega_k + \Omega_{k_1} - \Omega_{k_2} - \Omega_{k_3}) + \gamma_k n_k. \quad (11)$$

The kinetic equation describes plasmon collisions. In addition, there exists also a collective plasmon interaction, which leads to a shift in the plasmon frequency. The frequency shift is given by

$$\Delta\Omega_k = \frac{1}{16mVn_0}\sum_{k_1} S_{k,k_1,k,k_1}n_{k_1}. \quad (12)$$

Equation (11) is valid if $\Delta\Omega_k \ll \Omega_k$ and contains an additional term $\gamma_k n_k$, which describes the interaction of the plasmons with the plasma electrons. This term, which is calculated in [1], conserves the number of plasmons, but leads to a loss of energy by the plasmons.

S_{k,k_1,k_2,k_3} is a homogeneous function of second degree. Let us investigate its behavior when one of the arguments tends to zero. Let, this argument, for example, be k. Calculations show that although some of the terms of (8) remain finite as $k \to 0$, the principal terms cancel out, and the function S_{k,k_1,k_2,k_3} has an asymptotic behavior

$$S_{k,k_1,k_2,k_3} \approx kW(k_1, k_2, k_3) \quad \text{as} \quad k \to 0. \quad (13)$$

$W(k_1, k_2, k_3)$ is a homogeneous function of first degree.

A similar property is possessed by the function S_{k,k_1,k,k_1}. If one of its arguments is much larger than the other, its asymptotic value is

$$S_{k,k_1,k,k_1} \sim 16[(\mathbf{k}\mathbf{k}_1) - kk_1].$$

We note also that in the one-dimensional case, when all the vectors $\mathbf{k}$, $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ are parallel and have the same direction, the function S_{k,k_1,k_2,k_3} vanishes identically.

3. DIMENSIONAL ANALYSIS

Let the normalization volume V tend to infinity, and let us multiply the variable n_k by $(2\pi)^3/V$; the new variable, which now has the meaning of the particle-number density in six-dimensional phase space $(\mathbf{k}, \mathbf{r})$, will be denoted by the same letter. We replace summation by integration. The equations take the form

$$\frac{\partial n_k}{\partial t} = \frac{1}{(2\pi)^6}\frac{\pi}{32}\frac{1}{m^2n_0^2}\int |S_{k,k_1,k_2,k_3}|^2\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$$
$$\times\delta(\Omega_k+\Omega_{k_1}-\Omega_{k_2}-\Omega_{k_3})\Big|(n_{k_1}n_{k_2}n_{k_3}+n_kn_{k_2}n_{k_3}$$
$$-n_kn_{k_1}n_{k_2}-n_kn_{k_1}n_{k_3})\,d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3+\gamma_kn_k, \quad (14)$$

and the condition for its applicability is

$$\frac{1}{4(2\pi)^3mn_0}\int S_{k,k_1,k,k_1}n_{k_1}\,d\mathbf{k}_1 \ll \Omega_k. \quad (15)$$

The principal term of the kinetic equation is of the order of

$$(kr_D)^2(\mathscr{E}/nT)^2n_k, \text{ where } \mathscr{E} \sim \omega_0\int n\,dk$$

is the oscillation energy density, and the order of $\gamma_k n_k$ is[1]

$$\gamma_kn_k \sim (kr_D)^3(\mathscr{E}/nT)n_k.$$

It follows therefore that scattering of plasmons by electrons can be neglected when $k \ll k_s$, where $k_s = \mathscr{E}/r_DnT$.

In this region, equation (14) has three integrals of motion, viz., momentum $P = \int \mathbf{k}n_k d\mathbf{k}$, particle number $N = \int n_k d\mathbf{k}$, and the integral of motion $T = \int \Omega_k n_k d\mathbf{k}$, which by analogy with a Bose gas will be called the plasmon kinetic energy.

We now consider the evolution of a wave packet with average wave number k. The packet is assumed to be essentially multidimensional, but not necessarily isotropic. Let the average wave number of the packet be $k_0 \ll k_S$. Then the entire phase space breaks up into three regions—that containing the energy ($k \sim k_0$), the scattering region ($k \sim k_S$), and the intermediate region ($k_0 < k < k_S$). Assume that at the initial instant of time the wave packet has filled the energy-containing region. Under the influence of the collisions, some of the plasmons go to the intermediate region. Owing to the particle-number and kinetic-energy conservation laws, the rms wave number remains unchanged, so that the entire packet as a whole drifts to the region of small k. The probability that the plasmon will fall into the scattering region is low, and therefore the particle-number spectrum falls off quite steeply in the region of large k. Plasmons falling in the scattering region lose their kinetic energy and return to the region of small k, but their contribution to the total particle-number balance is insignificant.

Thus, a flux of plasmon kinetic energy is established in the region of large k, along with a systematic drift of the wave packet as a whole towards smaller k.

Further, in analogy with hydrodynamic turbulence,[3,4] we advance the hypothesis that the turbulence spectrum in the region of the intermediate wave numbers is determined by a single quantity p—the flux of plasmon kinetic energy in the region of large k, so that the intermediate region is a region of universal equilibrium.

The flux can be readily expressed in terms of the characteristics of the packet in the energy-containing region. Let n_{k_0} be the characteristic density of the particles in the energy-containing region. Then from (14) we have

$$p = \int \Omega_k\frac{\partial n_k}{\partial t}\,d\mathbf{k} \sim \frac{1}{m^2n_0^2}k_0^4(n_{k_0}k_0^3)^3. \quad (16)$$

On the other hand, in the intermediate region the flux should be expressed in terms of n_k and k, from which we get

$$p \sim k^4(n_kk^3)^3/m^2n_0^2.$$

Hence

$$n_k \sim (pn_0^2m^2)^{1/3}k^{-13/3} \sim n_{k_0}(k_0/k)^{13/3}. \qquad (17)$$

Expression (16) is perfectly analogous to the formula for the Kolmogorov spectrum in ordinary turbulence, $\mathscr{E}_k \sim \epsilon^{2/3}k^{-11/3}$, where ϵ is the energy flux in the region of large k.

From (15) we can estimate the rate of motion of wave packet in the region of small k. Bearing in mind that the total number of particles is conserved, we can obtain

$$\frac{d}{dt}(r_Dk_0)^2 \approx \frac{p}{N} \approx \omega_p\left(\frac{\mathscr{E}}{nT}\right)^2 (r_Dk_0)^4.$$

Hence $(r_Dk_0)^2 \sim \tau/t$, where $\tau^{-1} \sim \omega_p(\mathscr{E}/nT)^2$. The rms wave number can be treated as the plasmon-gas temperature. We see from the foregoing that the collisions cause cooling of the plasmon gas, the temperature decreasing in proportion to $1/t$.

The number of particles is conserved in this case, so that no appreciable dissipation of the Langmuir-oscillation energy takes place. This dissipation can occur only as a result of Coulomb collisions.

4. EXACT SOLUTION OF THE KINETIC EQUATION

We now show that the obtained spectral density $n_k = \text{const}\cdot k^{-13/3}$ causes the collision term of the kinetic equation to vanish:

$$\int |S_{kk_1k_2k_3}|^2\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,\delta(\Omega_k+\Omega_{k_1}-\Omega_{k_2}-\Omega_{k_3}) \times (n_{k_1}n_{k_2}n_{k_3}+n_kn_{k_2}n_{k_3}-n_kn_{k_1}n_{k_2}-n_kn_{k_1}n_{k_3}) \times d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3 = 0. \qquad (18)$$

We assume that n_k depends only on the modulus of k and use the relation

$$\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) = \frac{1}{(2\pi)^3}\int \exp\{i(\lambda, \mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\}\,d\lambda.$$

We multiply (18) by k^2 and integrate over the angles in k-space. We obtain

$$\int V_{k,k_1,k_2,k_3}\delta(k^2+k_1^2-k_2^2-k_3^2)(n_{k_1}n_{k_2}n_{k_3} + n_kn_{k_2}n_{k_3}-n_kn_{k_1}n_{k_2}-n_kn_{k_1}n_{k_3})\,dk_1\,dk_2\,dk_3 = 0.$$

Here

$$V_{k,k_1,k_2,k_3} = k^2k_1^2k_2^2k_3^2\int \exp\{i(\lambda, \mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\} \times |S_{k,k_1,k_2,k_3}|^2 d\Omega d\Omega_1 d\Omega_2 d\Omega_3, \qquad (19)$$

Ω_i is the solid-angle element in the k_i space.

It is obvious that V_{k,k_1,k_2,k_3} is a homogeneous function of ninth degree. It has the same symmetry properties as the function S_{k,k_1,k_2,k_3}.

We now change over to the variable $\omega = k^2$ and multiply (18) by $dk/d\omega = 1/2\sqrt{\omega}$ to conserve the symmetry of the kernel. We obtain

$$\int T(\omega, \omega'+\omega''-\omega, \omega', \omega'')(n_{\omega'}n_{\omega''}n_{\omega'+\omega''-\omega} + n_\omega n_{\omega'}n_{\omega''} - n_\omega n_{\omega'}n_{\omega'+\omega''-\omega} - n_\omega n_{\omega''}n_{\omega'+\omega''-\omega})\,d\omega'd\omega'' = 0. \qquad (20)$$

Here

$$T(\omega, \omega_1, \omega_2, \omega_3) = (\omega\omega_1\omega_2\omega_3)^{1/2}V(\sqrt{\omega}, \sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}).$$

T satisfies the symmetry relations (9) and is a homogeneous function of the order $5/2$. Integration is over the shaded region in the ω', ω'' plane (see the figure). We seek the solution in the form $n_\omega = c\omega^s$. We break up the region of integration into four regions (I, II, III, IV) and by change of variable we map each of the regions (II, III, IV) on region I.

The formulas for the change of variables are as follows: for region II

$$\omega' \to \frac{\omega''\omega}{\omega'+\omega''-\omega}, \quad \omega'' \to \frac{\omega'\omega}{\omega'+\omega''-\omega}; \qquad (21)$$

for region III

$$\omega' \to \frac{(\omega'+\omega''-\omega)\,\omega}{\omega''}, \quad \omega'' \to \frac{\omega^2}{\omega''};$$

for region IV

$$\omega' \to \frac{\omega^2}{\omega'}, \quad \omega'' \to \frac{(\omega'+\omega''-\omega)\,\omega}{\omega'}.$$

Let us consider the transformation of the integrand, using region II as an example. We note that under transformation (21), $\omega' + \omega'' - \omega$ goes over into $\omega^2/(\omega' + \omega'' - \omega)$, so that

$$T_{\omega,\omega'+\omega''-\omega,\omega',\omega''} \to T_{\omega,\alpha\omega,\alpha\omega'',\alpha\omega'} = \left(\frac{\omega}{\omega'+\omega''-\omega}\right)^{5/2} T_{\omega'+\omega''-\omega,\omega,\omega'',\omega'} = \left(\frac{\omega}{\omega'+\omega''-\omega}\right)^{5/2} T_{\omega,\omega'+\omega''-\omega,\omega',\omega''},$$

where $\alpha \equiv \omega/(\omega' + \omega'' - \omega)$. We have used here the homogeneity and symmetry properties of the function T. We transform analogously the expression in the parentheses containing the products n_ω. The Jacobian of the transformation is $\omega^3/(\omega' + \omega'' - \omega)^3$. Carrying out the transformation in all regions and gathering all terms, we obtain

$$\int\limits_{(I)} \frac{T_{\omega,\omega'+\omega''-\omega,\omega',\omega''}}{\omega'^s\omega''^s(\omega'+\omega''-\omega)^s}\left[1+\left(\frac{\omega}{\omega'+\omega''-\omega}\right)^s - \left(\frac{\omega}{\omega'}\right)^s - \left(\frac{\omega}{\omega''}\right)^s\right]\left[1+\left(\frac{\omega}{\omega'+\omega''-\omega}\right)^{11/2+3s} - \left(\frac{\omega}{\omega'}\right)^{11/2+3s} - \left(\frac{\omega}{\omega''}\right)^{11/2+3s}\right]d\omega'd\omega''. \qquad (22)$$

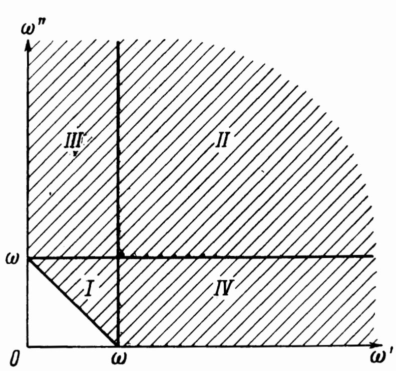
The integration is carried out over region (I). The function T is positive. The brackets under the integral sign in (22) vanish when s = −1 and $^{11}/_2 + 3s = -1$, yielding two solutions:

$$n_\omega = \text{const}\cdot\omega^{-1} \text{ and } n_\omega = \text{const}\cdot\omega^{-13/6}.$$

Going over to the variables k, we get the solutions $n_k = \text{const}\cdot k^{-2}$ and $n_k = \text{const}\cdot k^{-13/3}$. The first solution is the Rayleigh-distribution. It obviously is not applicable in our case. The second solution is the spectrum obtained by us earlier from dimensionality considerations. It is easy to verify that (17) has no other power-law solutions.

It is also necessary to prove the convergence of the integrals in (17) when the obtained solution is substituted in it. We consider first the region of small k.

The dangerous regions are those in which each of the arguments k_1, k_2, k_3 vanishes separately, and also the regions where two arguments, k_1, k_2 or k_1, k_3, vanish simultaneously. Let us consider the region where k_1 vanishes ($k_2k_3 \neq 0$). In this region, according to (9), the kernel of the integral equation takes the form $k_1^2[W(k_2, k_3)]^2$, so that the convergence of the integral in this region is assured. Similarly, the integral converges in regions where the arguments k_2 and k_3 vanish.



We now consider the vicinity of the straight line $k_1 = 0$, $k_2 = 0$. Near this line, the expression that diverges most strongly is

$$n_{k_1}n_{k_2}(n_{k_3} - n_k) \approx n_{k_1}n_{k_2}(k_1 - k_2)\partial n_k / \partial k.$$

Taking into account the asymptotic behavior of the kernel of the integrand, we can conclude by counting powers, that the integral converges in this region, too. We consider similarly the region in which k_1 and k_3 vanish. Let us examine the convergence at large k. Owing to the conservation of the kinetic energy, two arguments tend to infinity simultaneously, for example, k_1 and k_2. For large k, the term of the kinetic equation that decreases most slowly is proportional to

$$n_k n_{k_3}(n_{k_1} - n_{k_1+k-k_3}) = (k - k_3)n_k n_{k_3}\partial n_{k_1} / \partial k_1.$$

Recognizing that for large k_1 the kernel is proportional to k_1^2, we have for the principal term $k_1^2\partial n_{k_1}/\partial k_1 \sim \partial k_1^{-3-1/3}$, so that convergence is assured.

Actually, the solution in the energy-containing region differs from $\text{const}\cdot k^{-13/3}$. However, the convergence of the integral at small values of k causes the contribution from the energy-containing region to the region of intermediate wave numbers to be proportional to $(k_0/k)^{1/3}k^{-5}$, whereas the contribution of the region of wave numbers of order k is proportional to k^{-5}. Thus, the contribution from the energy-containing region should be neglected. Analogously, the contribution from the damping region to the region of intermediate wave numbers is of the order of $(k/k_s)^{1/3}k^{-5}$, and this contribution to the intermediate region can also be neglected. These considerations prove the local isotropy of the turbulence.

Thus, the solution is of the order of n_{k_0} when $k \sim k_0$ and $n_{k_0}(k_0/k)^{13/3}$ when $k \gg k_0$. Substituting this solution into (15), we note that the main contribution is made by integration over the energy containing region. We obtain the inequality

$$n_{k_0} \ll \frac{1}{k_0^3}\frac{nT}{\omega_0},$$

which limits the region in which the kinetic equation can be used for the description of the plasma turbulence.

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