COMMUTATION RELATIONS FOR THE COMPONENTS OF THE CURRENT DENSITY

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The nucleon current density in a symmetric pseudoscalar meson theory with pseudovector coupling is defined such that it does not commute with the canonical momentum of the meson field. Some properties of the theory are discussed.

THE interesting results in elementary particle physics which have recently been obtained with the help of the current algebra approach have revived the interest in the study of equal-time commutators. In this connection the problem of the consistency of the canonical commutation relations became important. The investigation of this point in different types of field theories may be useful.

Some time ago, Schwinger showed that the vacuum expectation value of the commutator of the density of a conserved current with the charge density cannot be equal to zero.[1] He also obtained a definition of the current such that this result is consistent with the equal-time commutation relations (cf. also [2]). Very recently, Okubo showed that Schwinger's result is a consequence only of covariance and the spectral property and is also valid for nonconserved currents.[3] Considering a pseudoscalar meson theory with pseudovector coupling, Okubo found that the above-mentioned requirement is inconsistent with the canonical commutation relations for this theory. He concluded from this that the theory is internally inconsistent. However, this assertion of Okubo's is incorrect. The apparent inconsistency noted in [3] is avoided by a more accurate definition of the current density, similar to what is done in quantum electrodynamics.

A system of spinor and pseudoscalar fields with pseudovector coupling is described by the isospin-invariant Lagrangian

$$L = \bar{\psi}(i\hat{\partial} - m_0)\psi + {}^1/_2\partial_\mu\varphi^\alpha\partial_\mu\varphi^\alpha - {}^1/_2\mu_0^2\varphi^\alpha\varphi^\alpha + j_\mu{}^\alpha\partial_\mu\varphi^\alpha. \quad (1)$$

(In our notation $\hat{\partial} = \gamma_\mu\partial_\mu = \gamma_0\partial_0 - \gamma_m\partial_m$; $\gamma_0^+ = \gamma_0$; $\gamma_m^+ = -\gamma_m$; $\gamma_5^+ = \gamma_5$; $\Box = -\partial_\mu\partial_\mu$. The upper indices refer to the isotopic degrees of freedom.) The question of the correct definition of the axial spinor current $j_\mu^\alpha(x)$ will be discussed in detail below.

For the following we need the canonical momentum of the scalar field

$$\pi^\alpha(x) = \varphi^\alpha(x) + j_0{}^\alpha(x), \quad (2)$$

the equations of motion

$$(\Box - \mu_0^2)\varphi^\alpha(x) = \partial_\mu j_\mu{}^\alpha(x) \quad (3)$$

and the equal-time commutation relations

$$[\varphi^\alpha(\mathbf{x}), \pi^\beta(0)] = i\delta^{\alpha\beta}\delta(\mathbf{x}), \quad \{\psi(\mathbf{x}), \bar{\psi}(0)\} = \gamma_0\delta(\mathbf{x}),$$
$$[\varphi^\alpha(\mathbf{x}), \varphi^\beta(0)] = \{\psi(\mathbf{x}), \psi(0)\}$$
$$= [\psi(\mathbf{x}), \varphi^\alpha(0)] = [\psi(\mathbf{x}), \pi^\alpha(0)] = 0. \quad (4)$$

If, following Schwinger,[1, 3] we define the spinor current by the limit

$$j_\mu{}^\alpha(x) = \frac{g_0}{2}\lim_{\varepsilon\to 0}\left[\bar{\psi}\left(x - \frac{\varepsilon}{2}\right), \gamma_5\gamma_\mu\tau^\alpha\psi\left(x + \frac{\varepsilon}{2}\right)\right], \quad \varepsilon^2 < 0, \quad (5)$$

then we find in accordance with what has been said before

$$\langle[j_m{}^\alpha(\mathbf{x}), j_0{}^\beta(0)]\rangle = \frac{g_0^2}{2}\delta^{\alpha\beta}\partial_n\delta(\mathbf{x})\lim_{\varepsilon\to 0}\varepsilon_n$$
$$\times\left\langle\left[\bar{\psi}\left(\mathbf{x} - \frac{\boldsymbol{\varepsilon}}{2}\right), \gamma_m\psi\left(\mathbf{x} + \frac{\boldsymbol{\varepsilon}}{2}\right)\right]\right\rangle \neq 0. \quad (6)$$

Nevertheless, the definition (5) is not quite satisfactory. Indeed, the general requirements of covariance and the spectral property lead, together with the canonical commutation relations, to the representation

$$\langle[j_\mu{}^\alpha(x), \varphi^\beta(0)]\rangle = -i\delta^{\alpha\beta}\int d\varkappa^2\tilde{\rho}(\varkappa^2)\partial_\mu\Delta(x, \varkappa^2), \quad (7)$$

where

$$\int d\varkappa^2\tilde{\rho}(\varkappa^2) = 0. \quad (8)$$

It follows from this that $\langle[j_\mu^\alpha(\mathbf{x}), \partial_0\varphi^\beta(0)]\rangle = 0$, and hence

$$\langle[j_m{}^\alpha(\mathbf{x}), \pi^\beta(0)]\rangle = \langle[j_m{}^\alpha(\mathbf{x}), j_0{}^\beta(0)]\rangle \neq 0. \quad (9)$$

On the other hand, the operators $j_m^{\alpha}(\mathbf{x})$ and $\pi^{\beta}(0)$ should commute owing to (4) and (5). Noting this contradiction, Okubo concluded that the theory under consideration is inconsistent.

However, a completely analogous situation prevails in quantum electrodynamics, where owing to the equation div $\mathbf{E} = j_0$ the current density also should not commute with the electric field intensity. In order to guarantee that they do not commute, it suffices to define the current in an explicitly gauge invariant way:[1, 2]

$$j_\mu(x) = \frac{e_0}{2}\lim_{\varepsilon\to 0}\Big[\bar{\psi}\Big(x-\frac{\varepsilon}{2}\Big),$$

$$\gamma_\mu \exp\Big(ie_0\int\limits_{x-\varepsilon/2}^{x+\varepsilon/2} d\xi_\mu A_\mu(\xi)\Big)\psi\Big(x+\frac{\varepsilon}{2}\Big)\Big]. \tag{10}$$

An analogous result can also be found in our case. The gauge invariance of the operator (10) can be interpreted as an invariance under rotations about a fixed axis in charge space. In a charge-symmetric theory it is natural to define each component of the current such that it is invariant under rotations about the corresponding axis in isotopic spin space:

$$\psi(x)\to\exp\{ig_0\gamma_5\tau^\alpha\lambda(x)\}\psi(x),$$

accompanied by a transformation of the corresponding component of the meson field

$$\varphi^\alpha(x)\to\varphi^\alpha(x)-\lambda(x).$$

Therefore one must adopt the definition

$$j_\mu{}^\alpha(x) = \frac{g_0}{2}\lim_{\varepsilon\to 0}\Big\{\bar{\psi}\Big(x-\frac{\varepsilon}{2}\Big),\ \gamma_5\gamma_\mu\tau^\alpha \exp + \frac{\varepsilon}{2}\Big)$$

$$\times\Big[ig_0\gamma_5\tau^\alpha\Big[\varphi^\alpha\Big(x-\varphi^\alpha\Big(x-\frac{\varepsilon}{2}\Big)\Big]\Big]\psi\Big(x+\frac{\varepsilon}{2}\Big)\Big\} \tag{11}$$

(there is no summation over α in the exponent). It is easy to see that this definition of the current avoids the contradiction observed in [3], since now

$$\langle[j_m{}^\alpha(\mathbf{x}),\ \pi^\beta(0)]\rangle$$

$$=\frac{g_0{}^2}{2}\delta^{\alpha\beta}\partial_n\delta(\mathbf{x})\lim_{\varepsilon\to 0}\varepsilon_n\Big\langle\Big[\bar{\psi}\Big(\mathbf{x}-\frac{\boldsymbol{\varepsilon}}{2}\Big),\ \gamma_m\psi\Big(\mathbf{x}+\frac{\boldsymbol{\varepsilon}}{2}\Big)\Big]\Big\rangle \tag{12}$$

in correspondence with (6) and (9).

Let us now turn to formula (8). Using the spectral representation

$$\langle[\varphi^\alpha(x),\varphi^\beta(0)]\rangle = i\delta^{\alpha\beta}\int d\varkappa^2\rho(\varkappa^2)\Delta(x,\varkappa^2),$$

$$\int d\varkappa^2\rho(\varkappa^2) = 1 \tag{13}$$

and Eq. (3), we easily find that

$$\varkappa^2\tilde{\rho}(\varkappa^2) = (\varkappa^2-\mu_0{}^2)\rho(\varkappa^2). \tag{14}$$

If there is no state with vanishing mass and the quantum numbers of the meson, then substitution of (14) in (8) gives

$$\frac{1}{\mu_0{}^2} = \int d\varkappa^2\frac{\rho(\varkappa^2)}{\varkappa^2}. \tag{15}$$

For the neutral version of the theory this result was obtained by Hellman.[4] This relation is generally characteristic of a theory with vector-type coupling.[2, 5]

Let us now write the propagator for the meson field in the form

$$D(k) = \int d\varkappa^2\frac{\rho(\varkappa^2)}{\varkappa^2-k^2} = \frac{1}{\mu_0{}^2-k^2+\Pi(k^2)}, \tag{16}$$

where $\Pi(k^2)$ is the polarization operator. Assuming that the integration over κ^2 does not begin at zero and setting $k^2 = 0$, we obtain with the help of (15)

$$\Pi(0) = 0. \tag{17}$$

The case $\mu_0 = 0$ requires special consideration. From (14) and the sum rules (13) and (8) it merely follows that

$$\rho(\varkappa^2) = \delta(\varkappa^2)+\tilde{\rho}(\varkappa^2). \tag{18}$$

No further conclusions can be drawn from this. In particular, the renormalization of the mass of the meson field can be different from zero. For this it suffices that $\tilde{\rho}(\kappa^2)$ have the form

$$\tilde{\rho}(\varkappa^2) = -\delta(\varkappa^2)+Z\delta(\varkappa^2-\mu^2)+\sigma(\varkappa^2). \tag{19}$$

The opposite assertion, made by Hellman and Roman,[6] is unfounded. These authors did not take account of the fact that the equality $\kappa^2\bar{\rho}(\kappa^2) = \mu_0^2\rho(\kappa^2)$ [cf. relation (16) in [4]; it is equivalent to our formula (14)] with $\mu_0^2 = 0$ only implies that $\bar{\rho}(\kappa^2) = b\delta(\kappa^2)$ but not (15). On the other hand, it is quite probable that (17) holds true also for $\mu_0 = 0$, since it is in our theory connected with the presence of a derivative of the meson field in the interaction Lagrangian. In this case there may indeed be no renormalization of the meson mass.

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