RELATIVISTIC RECONSTRUCTION OF THE N-N SCATTERING MATRIX

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We show how the nucleon-nucleon scattering matrix can be reconstructed from the experimental data at fixed values of the angle and energy in the relativistic case. Different reconstruction procedures are considered. Relativistic formulae are obtained for reconstructing the nucleon-nucleon scattering matrix in a state with total zero isospin from the np and pp scattering data.

PROGRESS in the creation of polarized beams of nucleons and polarized proton targets makes the problem of extracting maximum information on nucleon-nucleon scattering—the reconstruction of the scattering amplitude—more amenable to solution. In the particle-energy region below the pion production threshold, such a reconstruction was realized by the method of the so-called "modified" phase-shift analysis, the reconstruction being furthermore unique for a number of energy values.

The phase-shift analysis method is effective in the region of not very large energies, when the number of phases to be determined is small and the inelastic processes do not play an essential role. With increasing energy, the number of phases to be determined from experiment increases. In addition, at colliding-particle energies much higher than the pion production threshold, the scattering phase shifts and the mixing parameters become complex. Detailed information on the pion production mechanism is essential for a reliable phase shift analysis in this energy region.

The general method for reconstructing the scattering matrix, free from additional hypotheses of the phase shift analysis, consists of determining the scalar functions that characterize the scattering matrix directly from the experimental data on elastic scattering at fixed values of the energy and scattering angle. This method makes it possible to reconstruct the scattering matrix accurate to a common phase shift. Within the region below threshold of pion production, the common phase is determined by the unitarity relation.

The method of direct reconstruction is apparently used to best advantage at energies greatly exceeding the pion production threshold.[1)] Since the nucleon-nucleon scattering matrix is characterized in the general case by five scalar amplitudes, it is necessary to measure for their determination at least nine quantities. Since the observed quantities are expressed in terms of scalar amplitudes bilinearly, additional measurements must be used for their unique determination.

The first to call attention to the possibility of direct reconstruction were Puzikov, Ryndin, and Smorodinskiĭ,[3] Smorodinskiĭ,[4] and Schumacher and Bethe.[5] Schumacher and Bethe have shown how, to determine uniquely all five amplitudes of the scattering matrix (accurate to a common phase) by measuring, at a given scattering angle and energy, the differential cross section, the depolarization, the depolarization-tensor components, the polarization transfer, and the polarization correlations (a total of 11 quantities). Simplifications of this procedure, in measurements of the polarization tensors of third and fourth orders, were recently considered by the authors[6] and by Winternitz, Lehar, and Janout.[7]

In this paper we show how to reconstruct the nucleon-nucleon scattering matrix in the relativistic case. Direct relativistic relations are obtained between the measured quantities and the components of the polarization tensors in the c.m.s. The formulas needed for a unique reconstruction of the nucleon-nucleon scattering matrix from measurements of the components of polarization tensors of rank not higher than second are given. Different variants of the reconstruction procedure are considered. We consider the scattering of neutrons by protons. Formulas are obtained for the reconstruction of the scattering matrix of nucleons by nucleons in a state with total isotopic spin equal to zero. It is assumed here that the scattering matrix in the state with total isospin equal to unity is reconstructed from data on the proton-proton scattering. In the nonrelativistic case the question of reconstruction of

1)The reconstruction of the moduli of the scalar amplitudes was carried out for 640-MeV protons in [1, 2]. The relative phases of the complex amplitudes were determined ambiguously, since the required number of experiments was not performed.

the nucleon-nucleon scattering matrix from data on np and pp scattering at fixed values of the angle and energy were first considered by Golovin, Dzhelepov, Nadezhdin, and Saratov[8] and by Kazarinov.[9]

1. SCATTERING MATRIX

The nucleon-nucleon scattering matrix $M(\mathbf{p}', \mathbf{p})$ ($\mathbf{p}$ and $\mathbf{p}'$ are the c.m.s. momenta of the incident and scattered particles), satisfying the requirements of invariance against rotations, space reflections, and time reversal has, as is well known,[10] the following general form:

$$M(\mathbf{p}', \mathbf{p}) = (u+v) + (u-v)(\sigma_1\mathbf{n})(\sigma_2\mathbf{n}) + c[(\sigma_1\mathbf{n}) + (\sigma_2\mathbf{n})] + (g-h)(\sigma_1\mathbf{m})(\sigma_2\mathbf{m}) + (g+h)(\sigma_1\mathbf{l})(\sigma_2\mathbf{l}). \quad (1)$$

Here u, v, c, g, and h are complex functions of the energy and of the scattering angle θ in the c.m.s., and the vectors $\mathbf{l}$, $\mathbf{m}$, and $\mathbf{n}$ are defined as follows:

$$\mathbf{l} = \frac{\mathbf{p}'+\mathbf{p}}{|\mathbf{p}'+\mathbf{p}|}, \quad \mathbf{m} = \frac{\mathbf{p}'-\mathbf{p}}{|\mathbf{p}'-\mathbf{p}|}, \quad \mathbf{n} = [\mathbf{lm}] = \frac{[\mathbf{pp}']}{|[\mathbf{pp}']|}. \quad (2)^*$$

From the Pauli principle, the matrix for proton-proton scattering satisfies the conditions

$$M(\mathbf{p}', \mathbf{p}) = -P(1,2)M(-\mathbf{p}', \mathbf{p}) = -M(\mathbf{p}', -\mathbf{p})P(1,2), \quad (3)$$

where $P(1, 2) = \frac{1}{2}(1 + \sigma_1 \cdot \sigma_2)$ is the operator of spin-variable permutation. It follows from (3) that in the case of pp scattering the scalar amplitudes have the following symmetry properties:

$$u(\pi-\theta) = -u(\theta), \quad h(\pi-\theta) = h(\theta),$$
$$c(\pi-\theta) = c(\theta), \quad v(\pi-\theta) = -g(\theta). \quad (4)$$

By virtue of the isotopic invariance, the neutron-proton scattering matrix is

$$M_{np}(\mathbf{p}', \mathbf{p}) = \tfrac{1}{2}M_1(\mathbf{p}', \mathbf{p}) + \tfrac{1}{2}M_0(\mathbf{p}', \mathbf{p}), \quad (5)$$

where $M_1(\mathbf{p}', \mathbf{p})$ and $M_0(\mathbf{p}', \mathbf{p})$ are the scattering matrices in states with total isospin I = 1 and I = 0, respectively, and $\mathbf{p}$ and $\mathbf{p}'$ are the momenta of the initial and final neutron in the c.m.s. From the generalized Pauli principle it follows that (i = 1, 0)

$$M_i(\mathbf{p}', \mathbf{p}) = (-1)^i P(1,2)M_i(-\mathbf{p}', \mathbf{p}) = (-1)^i M_i(\mathbf{p}', -\mathbf{p})P(1,2). \quad (6)$$

The matrices $M_i(\mathbf{p}', \mathbf{p})$ have the general form (1). From (6) we obtain the following symmetry conditions:

$$u_i(\pi-\theta) = (-1)^i u_i(\theta), \quad h_i(\pi-\theta) = (-1)^{i+1}h_i(\theta),$$
$$c_i(\pi-\theta) = (-1)^{i+1}c_i(\theta), \quad v_i(\pi-\theta) = (-1)^i g_i(\theta). \quad (7)$$

2. MEASURABLE QUANTITIES

We determine here the quantities that are measurable in the relativistic case and obtain the connection between these quantities and the components of the polarization tensors in the c.m.s. All the relations obtained in this section are valid for both pp and np scattering.

The spin density matrix of the final state in the c.m.s., as is well known, is

$$\rho = M(\mathbf{p}', \mathbf{p})\rho_0 M^+(\mathbf{p}', \mathbf{p}). \quad (8)$$

Here ρ_0 is the initial density matrix:

$$\rho_0 = \tfrac{1}{2}(1+\sigma_1\mathbf{P}_1)\tfrac{1}{2}(1+\sigma_2\mathbf{P}_2) \quad (9)$$

($\mathbf{P}_1$ and $\mathbf{P}_2$ are the polarization vectors of the incident nucleons and target nucleons, respectively). We shall assume that the density matrix (8) is normalized such that

$$\mathrm{Sp}\,\rho = \sigma, \quad (10)$$

where σ is the differential cross section of the scattering in the c.m.s.

We denote by $\mathbf{a}_l'$ and $\mathbf{b}_l''$ arbitrary unit vectors in the l.s. The projection of the polarization vector of the scattered particles (particle with a momentum $\mathbf{p}'$) on the direction $\mathbf{a}_l'$, measured in the l.s., is[11-14]

$$\langle\sigma_1\rangle_l \mathbf{a}_l' = \sigma^{-1}\,\mathrm{Sp}\,\sigma_1(\mathbf{a}_l')_R\rho. \quad (11)$$

Here $(\mathbf{a}_l')_R = R_n(\Omega')\mathbf{a}_l'$, and $R_n(\Omega')$ is the operator of rotation about the normal n through an angle $\Omega' = \theta - 2\theta_l$ (θ_l is the scattering angle in the l.s.). Similarly, the projection of polarization of the recoil particles on the direction $\mathbf{b}_l''$, measured in the l.s., is

$$\langle\sigma_2\rangle_l \mathbf{b}_l'' = \sigma^{-1}\,\mathrm{Sp}\,\sigma_2(\mathbf{b}_l'')_R\rho, \quad (12)$$

where $(\mathbf{b}_l'')_R = R_n(\Omega'')\mathbf{b}_l''$, $R_n(\Omega'')$ is the operator of rotation about the normal $\mathbf{n}$ through an angle $\Omega'' = 2\varphi_l - \varphi$ (φ_l is the recoil angle in the l.s., and $\varphi = \pi - \theta$ is the recoil angle in the c.m.s.). We note also that the experimentally measured (a_l', b_l'')-component of the polarization correlation is equal to

$$\langle(\sigma_1\mathbf{a}_l')(\sigma_2\mathbf{b}_l'')\rangle_l = \sigma^{-1}\,\mathrm{Sp}\,(\sigma_1(\mathbf{a}_l')_R)(\sigma_2(\mathbf{b}_l'')_R)\rho. \quad (13)$$

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*$[\mathbf{pp}'] = \mathbf{p}\times\mathbf{p}'$.

We introduce in the l.s. the following three orthonormal vector systems:

$$\mathbf{n}_l, \quad \mathbf{k}_l, \quad \mathbf{s}_l = [\mathbf{n}_l\mathbf{k}_l]; \tag{14}$$

$$\mathbf{n}_l, \quad \mathbf{k}_l', \quad \mathbf{s}_l' = [\mathbf{n}_l\mathbf{k}_l']; \tag{15}$$

$$\mathbf{n}_l, \quad \mathbf{k}_l'', \quad \mathbf{s}_l'' = [\mathbf{n}_l\mathbf{k}_l'']. \tag{16}$$

Here $\mathbf{k}_l$, $\mathbf{k}_l'$, and $\mathbf{k}_l''$ are unit vectors in the directions of the momenta of the incident nucleon, scattered nucleon, and recoil nucleon, respectively, and $\mathbf{n}_l = [\mathbf{k}_l \times \mathbf{k}_l']/|[\mathbf{k}_l \times \mathbf{k}_l']|$ is a normal to the scattering plane ($\mathbf{n}_l = \mathbf{n}$). The experimentally measured polarization vector of the scattered particle (recoil particle) will be characterized by the components in the system (15) (the system (16)). The polarization of the incident nucleons and the polarization of the target nucleons will be characterized by components in the system (14). The polarization tensors of first and second rank are defined as follows:[3, 15]

$$P_i = \frac{1}{4\sigma_0}\operatorname{Sp}\sigma_{1i}MM^+ = \frac{1}{4\sigma_0}\operatorname{Sp}\sigma_{2i}MM^+, \tag{17}$$

$$A_i = \frac{1}{4\sigma_0}\operatorname{Sp}M\sigma_{1i}M^+ = \frac{1}{4\sigma_0}\operatorname{Sp}M\sigma_{2i}M^+, \tag{18}$$

$$D_{ik} = \frac{1}{4\sigma_0}\operatorname{Sp}\sigma_{1i}M\sigma_{1k}M^+ = \frac{1}{4\sigma_0}\operatorname{Sp}\sigma_{2i}M\sigma_{2k}M^+, \tag{19}$$

$$K_{ik} = \frac{1}{4\sigma_0}\operatorname{Sp}\sigma_{2i}M\sigma_{1k}M^+ = \frac{1}{4\sigma_0}\operatorname{Sp}\sigma_{1i}M\sigma_{2k}M^+, \tag{20}$$

$$C_{ik} = \frac{1}{4\sigma_0}\operatorname{Sp}\sigma_{1i}\sigma_{2k}MM^+ = \frac{1}{4\sigma_0}\operatorname{Sp}\sigma_{1k}\sigma_{2i}MM^+, \tag{21}$$

$$A_{ik} = \frac{1}{4\sigma_0}\operatorname{Sp}M\sigma_{1i}\sigma_{2k}M^+ = \frac{1}{4\sigma_0}\operatorname{Sp}M\sigma_{1k}\sigma_{2i}M^+. \tag{22}$$

These quantities are connected by the well known relations

$$P_i = A_i = Pn_i, \qquad C_{ik}(\mathbf{p}', \mathbf{p}) = A_{ik}(-\mathbf{p}, -\mathbf{p}').$$

In the foregoing expressions $\sigma_0 = {}^1\!/_4\operatorname{Sp}MM^+$ as the differential cross section for the scattering of polarized particles, P_i and A_i are the polarization and asymmetry vectors, D_{ik} and K_{ik} are the depolarization and polarization-transfer tensors, and C_{ik} and A_{ik} are the polarization-correlation and asymmetry tensors.

We proceed now to determine the experimentally measurable parameters. Using invariance considerations, we obtain the following expression for the differential cross section for the scattering of a polarized beam by a polarized target

$$\begin{aligned}\sigma_{\mathbf{P}_1\mathbf{P}_2} = \sigma_0[1 &+ P(\mathbf{P}_1\mathbf{n}_l + \mathbf{P}_2\mathbf{n}_l) + A_{nn}(\mathbf{P}_1\mathbf{n}_l)(\mathbf{P}_2\mathbf{n}_l)\\ &+ A_{ss}(\mathbf{P}_1\mathbf{s}_l)(\mathbf{P}_2\mathbf{s}_l) + A_{kk}(\mathbf{P}_1\mathbf{k}_l)(\mathbf{P}_2\mathbf{k}_l)\\ &+ A_{sk}((\mathbf{P}_1\mathbf{s}_l)(\mathbf{P}_2\mathbf{k}_l) + (\mathbf{P}_1\mathbf{k}_l)(\mathbf{P}_2\mathbf{s}_l))],\end{aligned}$$

$$A_{ab} = (\mathbf{a}_l)_i A_{ik}(\mathbf{b}_l)_k. \tag{23}$$

The quantities A_{nn}, A_{ss}, etc. can be measured directly in experiment. To determine A_{ss}, for example, it is necessary to measure the cross section for the scattering of a beam with polarization $\mathbf{P}_1 = P_1\mathbf{s}_l$ by a target polarized in the direction $\mathbf{s}_l$ ($\mathbf{P}_2 = P_2\mathbf{s}_l$). It is easy to relate these quantities to the components of the tensor C_{ik} in the c.m.s. Recognizing that

$$\mathbf{n}_l = \mathbf{n}, \qquad \mathbf{k}_l = \mathbf{k}, \qquad \mathbf{s}_l = \mathbf{s}, \tag{24}$$

where $\mathbf{k} = \mathbf{p}/|\mathbf{p}|$ is a unit vector in the direction of the c.m.s. momentum of the incident nucleon and $\mathbf{s} = \mathbf{n} \times \mathbf{k}$, we obtain

$$\begin{aligned}A_{ss} &= C_+ - C_{lm}\sin\theta - C_-\cos\theta,\\ A_{kk} &= C_+ + C_{lm}\sin\theta + C_-\cos\theta,\\ A_{sk} &= -C_{lm}\cos\theta + C_-\sin\theta,\end{aligned} \tag{25}$$

where

$$C_+ = {}^1\!/_2(C_{ll} + C_{mm}), \qquad C_- = {}^1\!/_2(C_{ll} - C_{mm}).$$

It is obvious that the component C_{nn} is measured directly in the experiment, with

$$A_{nn} = C_{nn}. \tag{26}$$

The remaining components are

$$\begin{aligned}C_+ &= {}^1\!/_2(A_{ss} + A_{kk}),\\ C_{lm} &= -A_{sk}\cos\theta + {}^1\!/_2(A_{kk} - A_{ss})\sin\theta,\\ C_- &= A_{sk}\sin\theta + {}^1\!/_2(A_{kk} - A_{ss})\cos\theta.\end{aligned} \tag{27}$$

The components of the tensor C_{ik} can be determined also by measuring the nucleon-polarization correlation when unpolarized particles are scattered. In experiment one measures the components C_{nn} and the following quantities:

$$C_{s's''} = (\mathbf{s}_l')_{Ri}C_{ik}(\mathbf{s}_l'')_{Rk}, \quad C_{s'k''} = (\mathbf{s}_l')_{Ri}C_{ik}(\mathbf{k}_l'')_{Rk},$$

$$C_{k's''} = (\mathbf{k}_l')_{Ri}C_{ik}(\mathbf{s}_l'')_{Rk}, \quad C_{k'k''} = (\mathbf{k}_l')_{Ri}C_{ik}(\mathbf{k}_l'')_{Rk}. \tag{28}$$

Let $\mathbf{a}$ be an arbitrary vector. It is then obvious that

$$R_{\mathbf{n}}(\Omega)\mathbf{a} = (\mathbf{an})\mathbf{n}(1 - \cos\Omega) + \mathbf{a}\cos\Omega + [\mathbf{na}]\sin\Omega. \tag{29}$$

With the aid of (29) and (24) we easily obtain

$$\begin{aligned}(\mathbf{k}_l')_R &= R_{\mathbf{n}}(\Omega')\mathbf{k}_l' = \mathbf{l}\cos\alpha + \mathbf{m}\sin\alpha,\\ (\mathbf{s}_l')_R &= R_{\mathbf{n}}(\Omega')\mathbf{s}_l' = -\mathbf{l}\sin\alpha + \mathbf{m}\cos\alpha,\\ \alpha &= \theta/2 - \theta_l.\end{aligned} \tag{30}$$

Similarly we obtain

$$\begin{aligned}(\mathbf{k}_l'')_R &= R_{\mathbf{n}}(\Omega'')\mathbf{k}_l'' = -\mathbf{l}\sin\alpha' - \mathbf{m}\cos\alpha',\\ (\mathbf{s}_l'')_R &= R_{\mathbf{n}}(\Omega'')\mathbf{s}_l'' = \mathbf{l}\cos\alpha' - \mathbf{m}\sin\alpha',\\ \alpha' &= \varphi/2 - \varphi_l.\end{aligned} \tag{31}$$

We note that in the nonrelativistic limit ($\alpha = \alpha' = 0$)

$$(\mathbf{k}_l')_R = \mathbf{l}, \qquad (\mathbf{s}_l')_R = \mathbf{m}, \; (\mathbf{k}_l'')_R = -\mathbf{m}, \; (\mathbf{s}_l'')_R = \mathbf{l}. \tag{32}$$

Using (30), we obtain the following expressions for the experimentally measured quantities (28):

$$C_{s's''} = -C_+ \sin(\alpha + \alpha') + C_{lm} \cos(\alpha - \alpha') - C_- \sin(\alpha - \alpha'),$$
$$C_{s'k''} = -C_+ \cos(\alpha + \alpha') + C_{lm} \sin(\alpha - \alpha') + C_- \cos(\alpha - \alpha'),$$
$$C_{k's''} = C_+ \cos(\alpha + \alpha') + C_{lm} \sin(\alpha - \alpha') + C_- \cos(\alpha - \alpha'),$$
$$C_{k'k''} = -C_+ \sin(\alpha + \alpha') - C_{lm} \cos(\alpha - \alpha') + C_- \sin(\alpha - \alpha'). \quad (33)$$

It is obvious that the observed quantities (28) are connected by the following relation:

$$(C_{s's''} + C_{k'k''}) / (C_{s'k''} - C_{k's''}) = \mathrm{tg}\,(\alpha + \alpha'). \quad (34)^*$$

We express C_+, C_-, and C_{lm} in terms of the observable $C_{s's''}$, $C_{s'k''}$, and $C_{k's''}$. From (33) we get

$$C_+ = (C_{k's''} - C_{s'k''}) / 2\cos(\alpha + \alpha'),$$
$$C_- = {}^1/_2 (C_{s'k''} + C_{k's''}) \cos(\alpha - \alpha') - [C_{s's''} + {}^1/_2\, \mathrm{tg}\,(\alpha + \alpha')(C_{k's''} - C_{s'k''})] \sin(\alpha - \alpha'),$$
$$C_{lm} = {}^1/_2 (C_{s'k''} + C_{k's''}) \sin(\alpha - \alpha') + [C_{s's''} + {}^1/_2\, \mathrm{tg}(\alpha + \alpha')(C_{k's''} - C_{s'k''})] \cos(\alpha - \alpha'). \quad (35)$$

We now proceed to consider the polarization occurring when one of the initial particles is polarized. Let the polarization of the incident nucleon beam $\mathbf{P}_1$ differ from zero, and let $\mathbf{P}_2 = 0$. From invariance considerations it is obvious that the components of the polarization of the scattered particle, measured in the l.s., are

$$\sigma_{\mathbf{P}_1} \langle \sigma_1 \rangle_l \mathbf{n}_l = \sigma_0 (P + D_{nn}(\mathbf{P}_1 \mathbf{n})),$$
$$\sigma_{\mathbf{P}_1} \langle \sigma_1 \rangle_l \mathbf{k}_l' = \sigma_0 (D_{k'k}(\mathbf{P}_1 \mathbf{k}_l) + D_{k's}(\mathbf{P}_1 \mathbf{s}_l)),$$
$$\sigma_{\mathbf{P}_1} \langle \sigma_1 \rangle_l \mathbf{s}_l' = \sigma_0 (D_{s'k}(\mathbf{P}_1 \mathbf{k}_l) + D_{s's}(\mathbf{P}_1 \mathbf{s}_l)). \quad (36)$$

Here

$$D_{nn} = D = (\mathbf{n}_l)_i D_{ik} (\mathbf{n}_l)_k, \quad D_{s's} = R = (\mathbf{s}_l')_{Ri} D_{ik} (\mathbf{s}_l)_k,$$
$$D_{s'k} = A = (\mathbf{s}_l')_{Ri} D_{ik} (\mathbf{k}_l)_k, \quad D_{k's} = R' = (\mathbf{k}_l')_{Ri} D_{ik} (\mathbf{s}_l)_k,$$
$$D_{k'k} = A' = (\mathbf{k}_l')_{Ri} D_{ik} (\mathbf{k}_l)_k \quad (37)$$

are the known Wolfenstein triple-scattering parameters, determined with allowance for relativistic rotation,[11, 14] and $\sigma_{\mathbf{P}_1}$ is the differential cross section for the scattering of a polarized beam by an unpolarized target. From (23) we get

$$\sigma_{\mathbf{P}_1} = \sigma_0 (1 + P(\mathbf{P}_1 \mathbf{n})). \quad (38)$$

*tg ≡ tan.

Let us express the measurable quantities (37) in terms of the component of the tensor D_{ik} in the c.m.s. With the aid of (30) we get

$$D_{s's} = D_+ \cos\left(\alpha + \frac{\theta}{2}\right) - D_{lm} \sin\left(\alpha + \frac{\theta}{2}\right) - D_- \cos\left(\alpha - \frac{\theta}{2}\right),$$
$$D_{s'k} = -D_+ \sin\left(\alpha + \frac{\theta}{2}\right) - D_{lm} \cos\left(\alpha + \frac{\theta}{2}\right) - D_- \sin\left(\alpha - \frac{\theta}{2}\right),$$
$$D_{k's} = D_+ \sin\left(\alpha + \frac{\theta}{2}\right) + D_{lm} \cos\left(\alpha + \frac{\theta}{2}\right) - D_- \sin\left(\alpha - \frac{\theta}{2}\right),$$
$$D_{k'k} = D_+ \cos\left(\alpha + \frac{\theta}{2}\right) - D_{lm} \sin\left(\alpha + \frac{\theta}{2}\right) + D_- \cos\left(\alpha - \frac{\theta}{2}\right), \quad (39)$$

where

$$D_+ = {}^1/_2 (D_{ll} + D_{mm}), \qquad D_- = {}^1/_2 (D_{ll} - D_{mm}).$$

It is easy to show that the four measurable quantities (36) are related as follows:[14]

$$(D_{s'k} + D_{k's}) / (D_{k'k} - D_{s's}) = \mathrm{tg}\,\theta_l. \quad (40)$$

Let us express D_+, D_-, and D_{lm} in terms of the measurable parameters $D_{s's}$, $D_{s'k}$, and $D_{k's}$. From (39) we obtain

$$D_- = -{}^1/_2 (D_{s'k} + D_{k's}) / \sin(\alpha - \theta/2),$$
$$D_+ = -{}^1/_2 (D_{s'k} - D_{k's}) \sin(\alpha + \theta/2) + [D_{s's} - {}^1/_2 (D_{s'k} + D_{k's}) \mathrm{ctg}(\alpha - \theta/2)] \cos(\alpha + \theta/2),$$
$$D_{lm} = -{}^1/_2 (D_{s'k} - D_{k's}) \cos(\alpha + \theta/2) - [D_{s's} - {}^1/_2 (D_{s'k} + D_{k's}) \mathrm{ctg}(\alpha - \theta/2)] \times \sin(\alpha + \theta/2). \quad (41)$$

It is obvious from (19) that the components of the depolarization tensor D_{ik} can be determined also by measuring the polarization of the second particle (recoil particle), produced upon scattering of an unpolarized beam by a polarized target ($\mathbf{P}_1 = 0$, $\mathbf{P}_2 \neq 0$). Using considerations of invariance against rotations and reflections, we obtain

$$\sigma_{\mathbf{P}_2} \langle \sigma_2 \rangle_l \mathbf{n}_l = \sigma_0 (P + D_{nn}(\mathbf{P}_2 \mathbf{n}_l)),$$
$$\sigma_{\mathbf{P}_2} \langle \sigma_2 \rangle_l \mathbf{k}_l'' = \sigma_0 (D_{k''k}(\mathbf{P}_2 \mathbf{k}_l) + D_{k''s}(\mathbf{P}_2 \mathbf{s}_l)),$$
$$\sigma_{\mathbf{P}_2} \langle \sigma_2 \rangle_l \mathbf{s}_l'' = \sigma_0 (D_{s''k}(\mathbf{P}_2 \mathbf{k}_l) + D_{s''s}(\mathbf{P}_2 \mathbf{s}_l)). \quad (42)$$

Here

$$D_{k''k} = (\mathbf{k}_l'')_{Ri} D_{ik} (\mathbf{k}_l)_k, \qquad D_{k''s} = (\mathbf{k}_l'')_{Ri} D_{ik} (\mathbf{s}_l)_k,$$
$$D_{s''k} = (\mathbf{s}_l'')_{Ri} D_{ik} (\mathbf{k}_l)_k, \qquad D_{s''s} = (\mathbf{s}_l'')_{Ri} D_{ik} (\mathbf{s}_l)_k. \quad (43)$$

From (19) and (31) we obtain the following expressions for these quantities

$$D_{s''s} = -D_{+}\sin(\alpha' - \theta/2) + D_{lm}\cos(\alpha' - \theta/2) + D_{-}\sin(\alpha' + \theta/2),$$
$$D_{s''k} = D_{+}\cos(\alpha' - \theta/2) + D_{lm}\sin(\alpha' - \theta/2) + D_{-}\cos(\alpha' + \theta/2),$$
$$D_{k''s} = -D_{+}\cos(\alpha' - \theta/2) - D_{lm}\sin(\alpha' - \theta/2) + D_{-}\cos(\alpha' + \theta/2),$$
$$D_{k''k} = -D_{+}\sin(\alpha' - \theta/2) + D_{lm}\cos(\alpha' - \theta/2) - D_{-}\sin(\alpha' + \theta/2). \quad (44)$$

The quantities (43) are related by

$$(D_{s''k} + D_{k''s}) / (D_{s''s} - D_{k''k}) = \operatorname{tg}\varphi_l. \quad (45)$$

We note that in the nonrelativistic limit the measurable quantities (37) and (43) are connected by

$$D_{s's} = D_{ms} = -D_{k''s}, \qquad D_{s'k} = D_{mk} = -D_{k''k},$$
$$D_{k's} = D_{ls} = D_{s''s}, \qquad D_{k'k} = D_{lm} = D_{s''k}. \quad (46)$$

The components of the depolarization tensor in the c.m.s. are expressed in terms of the measurable quantities (43) as follows:

$$D_{-} = (D_{s''k} + D_{k''s}) / 2\cos(\alpha' + \theta/2),$$
$$D_{+} = {}^1/_2(D_{s''k} - D_{k''s})\cos(\alpha' - \theta/2) - [D_{s''s} - {}^1/_2(D_{s''k} + D_{k''s})\operatorname{tg}(\alpha' + \theta/2)] \times \sin(\alpha' - \theta/2),$$
$$D_{lm} = {}^1/_2(D_{s''k} - D_{k''s})\sin(\alpha' - \theta/2) + [D_{s''s} - {}^1/_2(D_{s''k} + D_{k''s})\operatorname{tg}(\alpha' + \theta/2)] \times \cos(\alpha' - \theta/2). \quad (47)$$

As is well known, measurement of the nucleon polarization in the energy interval from 20 to 100 MeV is made difficult by the lack of analyzers with sufficient analyzing ability. This means that when experiments with polarized beams are set up, the components of the tensor D_{ik} cannot be determined in the entire angle interval. It is obvious that this difficulty does not arise if we determine the depolarization tensor B_{ik} in the c.m.s. by means of experiments with both a polarized beam and a polarized target. In addition, as seen from (41), to determine any of the components of the tensor D_{ik} (except D_{nn}) in polarized-beam experiments it is necessary to carry out difficult measurements of the longitudinal polarization of the scattered particle (the parameter $D_{k's}$). The use of a polarized target would replace these experiments by simpler ones, in which the transverse polarization of the recoil particle would be measured. From (39) and (45) it is obvious that the components D_{ll}, D_{mm}, and D_{lm} can be determined by measuring, for example, $D_{s's}$, $D_{s'k}$, and $D_{s''s}$. We obtain

$$D_{+} = -\Delta^{-1}[(D_{s's}\sin\theta + D_{s'k}\cos\theta)\cos(\alpha + \alpha') + D_{s's}\sin(\alpha' - \alpha) + D_{s''s}\cos 2\alpha],$$
$$D_{lm} = \Delta^{-1}[(-D_{s's}\cos\theta + D_{s'k}\sin\theta)\cos(\alpha + \alpha') + D_{s's}\cos(\alpha' - \alpha) + D_{s''s}\sin 2\alpha], \quad (48)$$
$$D_{-} = -\Delta^{-1}[D_{s's}\sin(\alpha + \alpha') + D_{s'k}\cos(\alpha + \alpha') + D_{s''s}],$$

where

$$\Delta = 2\cos(\alpha + \alpha')\sin(\alpha - \theta/2).$$

We proceed to consider the polarization-transfer tensor K_{ik}. Let a polarized beam be scattered by an unpolarized target ($\mathbf{P}_1 \neq 0$, $\mathbf{P}_2 = 0$). For the experimentally-measured polarization components of the recoil particles we obtain the following expressions:

$$\sigma_{\mathbf{P}_1}\langle\sigma_2\rangle_l \mathbf{n}_l = \sigma_0(P + K_{nn}(\mathbf{P}_1\mathbf{n}_l)),$$
$$\sigma_{\mathbf{P}_1}\langle\sigma_2\rangle_l \mathbf{k}_l'' = \sigma_0(K_{k''k}(\mathbf{P}_1\mathbf{k}_l) + K_{k''s}(\mathbf{P}_1\mathbf{s}_l)),$$
$$\sigma_{\mathbf{P}_1}\langle\sigma_2\rangle_l \mathbf{s}_l'' = \sigma_0(K_{s''k}(\mathbf{P}_1\mathbf{k}_l) + K_{s''s}(\mathbf{P}_1\mathbf{s}_l)). \quad (49)$$

Here

$$K_{a''b} = (\mathbf{a}_l'')_R{}_i K_{ik}(\mathbf{b}_l)_k;$$

$\mathbf{a}_l''$ and $\mathbf{b}_l$ are the vectors made up of the triads (16) and (14) respectively. It is obvious that the relations between these quantities and the components of the tensor K_{ik}, and also the inverse relations, can be obtained from (44) and (47) with the aid of the substitution

$$D_{a''b} \rightarrow K_{a''b}, \qquad D_{ik} \rightarrow K_{ik}.$$

The components of the polarization-transfer tensor K_{ik} can be determined also by measuring the polarization of the scattered particle during the scattering of an unpolarized beam by a polarized target ($\mathbf{P}_1 = 0$, $\mathbf{P}_2 \neq 0$). From invariance considerations we obtain

$$\sigma_{\mathbf{P}_2}\langle\sigma_1\rangle_l \mathbf{n}_l = \sigma_0(P + K_{nn}(\mathbf{P}_2\mathbf{n}_l)),$$
$$\sigma_{\mathbf{P}_2}\langle\sigma_1\rangle_l \mathbf{k}_l' = \sigma_0(K_{k'k}(\mathbf{P}_2\mathbf{k}_l) + K_{k's}(\mathbf{P}_2\mathbf{s}_l)),$$
$$\sigma_{\mathbf{P}_2}\langle\sigma_1\rangle_l \mathbf{s}_l' = \sigma_0(K_{s'k}(\mathbf{P}_2\mathbf{k}_l) + K_{s's}(\mathbf{P}_2\mathbf{s}_l)), \quad (50)$$

where

$$K_{a'b} = (\mathbf{a}_l')_R{}_i K_{ik}(\mathbf{b}_l)_k.$$

Making the substitutions $D_{ik} \rightarrow K_{ik}$ and $D_{a'b} \rightarrow K_{a'b}$ in (39), we obtain relations between these quantities and K_+, K_-, and K_{lm}. By means of the same substitution we obtain from (41) the inverse relations.

In conclusion we present expressions for the differential cross section σ_0 of the polarization P and the components of the tensors D_{ik}, K_{ik}, and C_{ik}, calculated with the aid of the scattering matrix (1):

$$\sigma_0 = 2(|u|^2 + |v|^2 + |c|^2 + |g|^2 + |h|^2), \tag{51}$$

$$\sigma_0 P = 4\,\mathrm{Re}\, cu^*, \tag{52}$$

$$\sigma_0 D_{nn} = 2(|u|^2 + |v|^2 + |c|^2 - |g|^2 - |h|^2), \tag{53}$$

$$\sigma_0 D_+ = 4\,\mathrm{Re}\, uv^*, \tag{54}$$

$$\sigma_0 D_- = 4\,\mathrm{Re}\, gh^*, \tag{55}$$

$$\sigma_0 D_{lm} = 4\,\mathrm{Im}\, cv^*, \tag{56}$$

$$\sigma_0 K_{nn} = 2(|u|^2 - |v|^2 + |c|^2 + |g|^2 - |h|^2), \tag{57}$$

$$\sigma_0 K_+ = 4\,\mathrm{Re}\, ug^*, \tag{58}$$

$$\sigma_0 K_- = 4\,\mathrm{Re}\, vh^*, \tag{59}$$

$$\sigma_0 K_{lm} = 4\,\mathrm{Im}\, cg^*, \tag{60}$$

$$\sigma_0 C_{nn} = 2(|u|^2 - |v|^2 + |c|^2 - |g|^2 + |h|^2), \tag{61}$$

$$\sigma_0 C_+ = 4\,\mathrm{Re}\, vg^*, \tag{62}$$

$$\sigma_0 C_- = 4\,\mathrm{Re}\, uh^*, \tag{63}$$

$$\sigma_0 C_{lm} = -4\,\mathrm{Im}\, ch^*. \tag{64}$$

3. DIRECT RECONSTRUCTION OF THE NUCLEON-NUCLEON SCATTERING MATRIX

With the aid of the formulas obtained in the preceding sections we can determine in the relativistic case, from the experimentally measured quantities, the components of the polarization tensors in the c.m.s. We can then use relations (51)–(64) to express the amplitudes u, v, c, g, and h in terms of the cross section σ_0, the polarization P, and the components of the polarization tensors D_{ik}, K_{ik}, and C_{ik}. As already noted above, the direct reconstruction of the nucleon-nucleon scattering matrix from the experimental data at a fixed angle and energy were considered by Schumacher and Bethe.[5] We generalize here the method proposed by them.

As seen from relations (51), (53), (57), and (61), the differential cross section σ_0 and the normal components D_{nn}, K_{nn}, and C_{nn} are expressed in terms of the squares of the moduli of the amplitudes u, v, c, g, and h. From this we get

$$|g|^2 = {}^1/_8\sigma_0(1 + K_{nn} - D_{nn} - C_{nn}),$$

$$|h|^2 = {}^1/_8\sigma_0(1 - K_{nn} - D_{nn} + C_{nn}),$$

$$|v|^2 = {}^1/_8\sigma_0(1 - K_{nn} + D_{nn} - C_{nn}),$$

$$|u|^2 + |c|^2 = {}^1/_8\sigma_0(1 + K_{nn} + D_{nn} + C_{nn}). \tag{65}$$

It is obvious that the scattering matrix can be reconstructed from (51)–(64) only accurate to a common phase factor. This means that one of the amplitudes can always be regarded, during the reconstruction of the scattering matrix, as real and positive. We shall assume that the amplitude c is real and positive (this means that we are reconstructing the matrix $\exp(-i\varphi_c)M(\mathbf{p}', \mathbf{p})$, where φ_c is the phase of the amplitude c). We then obtain from (52), (64), (56), and (60)

$$\mathrm{Re}\, u = \frac{1}{4c}\sigma_0 P, \quad \mathrm{Im}\, h = \frac{1}{4c}\sigma_0 C_{lm},$$

$$\mathrm{Im}\, v = -\frac{1}{4c}\sigma_0 D_{lm}, \quad \mathrm{Im}\, g = -\frac{1}{4c}\sigma_0 K_{lm}. \tag{66}$$

We now determine the amplitude c. To this end we use the relation

$$|x|^2|y|^2 - (\mathrm{Re}\, xy^*)^2 = |x|^2(\mathrm{Im}\, y)^2 + |y|^2(\mathrm{Im}\, x)^2 - 2\,\mathrm{Re}\, xy^*\,\mathrm{Im}\, x\,\mathrm{Im}\, y, \tag{67}$$

which is valid for any two complex numbers x and y. Choosing for x and y the amplitudes g and h, we obtain with the aid of (66) the following expression for c^2:

$$c^2 = \frac{|g|^2M^2 - |h|^2N^2 - 2\,\mathrm{Re}\, gh^*MN}{|g|^2|h|^2 - (\mathrm{Re}\, gh^*)^2}. \tag{68}$$

Here

$$M = {}^1/_4\sigma_0 C_{lm}, \quad N = -{}^1/_4\sigma_0 K_{lm},$$

and $|g|^2$, $|h|^2$, and Re gh* are given respectively by the expressions (65) and (55).

For a complete reconstruction of the scattering matrix it is necessary to determine only the signs of Im u, Re H, Re v, and Re g. The relative sign of Re g and Re h can be determined from

$$\mathrm{Re}\, gh^* = {}^1/_4\sigma_0 D_- = \mathrm{Re}\, g\,\mathrm{Re}\, h + \mathrm{Im}\, g\,\mathrm{Im}\, h. \tag{69}$$

With the aid of (54) we can determine the signs of Im u and Re v. Any of the remaining unused equations makes it possible to eliminate the remaining unambiguity. Other variants are also possible, namely, we determine with the aid of (58) or (63) the signs of Im u and Re g (or Im u and Re h), after which the sign of Re v can be determined with the aid of (54) or else (59), or else (62). Thus, for a unique reconstruction of the nucleon-nucleon scattering matrix it is necessary to know with sufficient accuracy 11 quantities in the c.m.s.

It is obvious that the accuracy with which the scattering matrix is reconstructed by our method depends on the accuracy with which the amplitude c is determined. In this connection, let us obtain other expressions for c^2. Choosing for x and y in (67) the amplitudes g and v, we obtain

$$c^2 = \frac{|g|^2 L^2 + |v|^2 N^2 - \mathrm{Re}\, gv^* LN}{|g|^2|v|^2 - (\mathrm{Re}\, gv^*)^2}, \tag{70}$$

where $L = -{}^1/_4 \sigma_0 D_{lm}$. We obtain similarly

$$c^2 = \frac{|h|^2 L^2 + |v|^2 M^2 - 2\mathrm{Re}\, hv^* LM}{|h|^2|v|^2 - (\mathrm{Re}\, hv^*)^2} \tag{71}$$

in the case when x and y in (67) are chosen to be h and v.

Finally, we can make the amplitude u real and positive, and reconstruct from the experimental data the matrix $[\exp(-i\varphi_u)]\, M(\mathbf{p}', \mathbf{p})$ (φ_u is the phase of the amplitude u). From (52), (54), (58), (63), we then obtain

$$\mathrm{Re}\, c = \frac{1}{4u}\sigma_0 P, \qquad \mathrm{Re}\, v = \frac{1}{4u}\sigma_0 D_+,$$

$$\mathrm{Re}\, g = \frac{1}{4u}\sigma_0 K_+, \quad \mathrm{Re}\, h = \frac{1}{4u}\sigma_0 C_-. \tag{72}$$

To determine u we use the relation

$$|x|^2|y|^2 - (\mathrm{Re}\, xy^*)^2 = |x|^2(\mathrm{Re}\, y)^2 + |y|^2(\mathrm{Re}\, x)^2 - 2\,\mathrm{Re}\, x\, \mathrm{Re}\, y\, \mathrm{Re}\, xy^*, \tag{73}$$

where x and y are arbitrary complex numbers. With the aid of (72) and (73) we obtain the following expression for u^2:

$$u^2 = \frac{|g|^2 M_1^2 + |h|^2 N_1^2 - 2\,\mathrm{Re}\, gh^* M_1 N_1}{|g|^2|h|^2 - (\mathrm{Re}\, gh^*)^2}, \tag{74}$$

where

$$M_1 = {}^1/_4\sigma_0 C_-, \qquad N_1 = {}^1/_4\sigma_0 K_+.$$

For a unique reconstruction of the scattering matrix it remains to determine the signs of Im c, Im v, Im g, and Im h. The relative signs of Im g and Im h are obtained from (55). The signs of Im c and Im v can be determined from (56). The remaining uncertainty can be eliminated with the aid of any of the relations unused so far. (Other variants are: use (66) to determine the signs of Im c and Im g, or else (64) to determine the signs of Im c and Im h, after which use (56), or (59), or (62) to determine the sign of Im v.)

In conclusion we present other expressions for u^2, obtained with the aid of (72) and (73):

$$u^2 = \frac{|v|^2 M_1^2 + |h|^2 L_1^2 - 2\,\mathrm{Re}\, vh^* M_1 L_1}{|v|^2|h|^2 - (\mathrm{Re}\, vh^*)^2}, \tag{75}$$

$$u^2 = \frac{|v|^2 N_1^2 + |g|^2 L_1^2 - \mathrm{Re}\, vg^* N_1 L_1}{|v|^2|g|^2 - (\mathrm{Re}\, vg^*)^2}. \tag{76}$$

Here

$$L_1 = {}^1/_4\sigma_0 D_+.$$

It is clear that different variants of the reconstruction of the scattering matrix should be used in different energy and angle regions.

4. SCATTERING OF PROTONS BY PROTONS

In the case of proton-proton scattering, the amplitudes u, v, c, g, and h satisfy the symmetry conditions (4). The pp-scattering matrix is specified, consequently, by the amplitudes u, v, etc. in the interval $0 \le \theta \le \pi/2$. It is obvious that the identity of the particles leads also to symmetry of the observable quantities. Using the conditions (3), we find that the polarization tensors in the c.m.s. satisfy in the case of pp scattering the following relations:

$$P_i(\mathbf{p}', \mathbf{p}) = P_i(-\mathbf{p}', \mathbf{p}), \quad D_{ik}(\mathbf{p}', \mathbf{p}) = K_{ik}(-\mathbf{p}', \mathbf{p}),$$
$$C_{ik}(\mathbf{p}', \mathbf{p}) = C_{ik}(-\mathbf{p}', \mathbf{p}). \tag{77}$$

For the experimentally measured quantities in Sec. 2 we obtain

$$\begin{aligned}
&P(\theta) = -P(\pi-\theta), && D_{nn}(\theta) = K_{nn}(\pi-\theta),\\
&A_{nn}(\theta) = A_{nn}(\pi-\theta), && D_{s's}(\theta) = K_{s''s}(\pi-\theta),\\
&A_{ss}(\theta) = A_{ss}(\pi-\theta), && D_{s'k}(\theta) = -K_{s''k}(\pi-\theta),\\
&A_{sk}(\theta) = -A_{sk}(\pi-\theta), && D_{k's}(\theta) = -K_{k''s}(\pi-\theta),\\
&C_{s's''}(\theta) = C_{s's''}(\pi-\theta), && D_{s''s}(\theta) = K_{s's}(\pi-\theta),\\
&C_{s'k''}(\theta) = -C_{k's''}(\pi-\theta), && D_{s''k}(\theta) = -K_{s'k}(\pi-\theta),\\
&C_{k's''}(\theta) = -C_{s'k''}(\pi-\theta), && D_{k''s}(\theta) = -K_{k's}(\pi-\theta).
\end{aligned} \tag{78}$$

In connection with these relations, we make the following remark. If the incident photon beam is polarized, then the polarization of particles with c.m.s. momentum **p** differs in the general case from the polarization of particles with momentum −**p**. It is obvious that the measurement of the polarization of protons emitted in the c.m.s. at an angle θ ($0 \le \theta \le \pi/2$) makes it possible to determine $D_{a'b}(\theta)$, while measurement of the polarization of protons emitted in the c.m.s. at an angle $\pi - \theta$ makes it possible to determine $K_{a''b}$. In the case of an unpolarized beam and a polarized target, measurement of the polarization of the protons emitted at an angle θ in the c.m.s. ($0 \le \theta \le \pi/2$) makes it possible to determine $K_{a'b}(\theta)$, while measurement of the polarization of protons emitted at an angle $\pi - \theta$ makes it possible to determine $D_{a''b}$. The particles emitted at c.m.s. angles θ and $\pi - \theta$ have different energies in the l.s. The analyzing ability of the analyzer-targets depends on the energy of the particles incident on the target. This means that the determination of the components of the depolarization and polarization-transfer tensors can be greatly simplified if experiments are made not only with a polarized beam but also with a polarized target.

5. SCATTERING OF NEUTRONS BY PROTONS

By virtue of isotopic invariance, the neutron-proton scattering matrix is equal to half the sum of the scattering matrix in the states with total isotopic spin I equal to unity and zero (see (5)). While experiments on the scattering of protons by protons reconstruct the nuclear scattering matrix in the state with I = 1, $M_1(\mathbf{p}', \mathbf{p})$, experiments on np scattering must be used to reconstruct the matrix $M_0(\mathbf{p}', \mathbf{p})$. This matrix, owing to the symmetry conditions (7), is specified by the amplitudes u_0, v_0, c_0, g_0, and h_0 in the interval $0 \le \theta \le \pi/2$.

Let us consider observable quantities. The differential cross section for the scattering of unpolarized neutrons by unpolarized protons is

$$\sigma_{np}(\theta) = {}^1/_4\sigma_{I=1}(\theta) + {}^1/_4\sigma_{I=0}(\theta) + {}^1/_8\mathrm{Re\,Sp}\, M_1(\mathbf{p}', \mathbf{p}) M_0^+(\mathbf{p}', \mathbf{p}). \tag{79}$$

Here

$$\sigma_{I=1}(\theta) = {}^1/_4\mathrm{Sp}\, M_1(\mathbf{p}', \mathbf{p}) M_1^+(\mathbf{p}', \mathbf{p}),\quad \sigma_{I=0}(\theta) = {}^1/_4\mathrm{Sp}\, M_0(\mathbf{p}', \mathbf{p}) M_0^+(\mathbf{p}', \mathbf{p}) \tag{80}$$

are the differential cross sections for the scattering of unpolarized nucleons in states with I = 1 and I = 0 respectively (θ is the angle between the initial and final momenta of the neutron in the c.m.s.). From (6) we find that

$$\sigma_{I=1}(\theta) = \sigma_{I=1}(\pi-\theta), \quad \sigma_{I=0}(\theta) = \sigma_{I=0}(\pi-\theta),$$
$$\mathrm{Sp}\, M_1(\mathbf{p}', \mathbf{p}) M_0^+(\mathbf{p}', \mathbf{p}) = -\mathrm{Sp}\, M_1(-\mathbf{p}', \mathbf{p}) M_0^+(-\mathbf{p}', \mathbf{p}). \tag{81}$$

Let us consider the cross section for np scattering through angles θ and $\pi - \theta$. With the aid of (79) and (81) we obtain

$$2[\sigma_{np}(\theta) + \sigma_{np}(\pi-\theta)] = \sigma_{I=1}(\theta) + \sigma_{I=0}(\theta),$$
$$\sigma_{np}(\theta) - \sigma_{np}(\pi-\theta) = \sigma_{\mathrm{int}}(\theta) = {}^1/_4\mathrm{Re\,Sp}\, M_1(\mathbf{p}', \mathbf{p}) M_0^+(\mathbf{p}', \mathbf{p}). \tag{82}$$

Thus, measurement of the differential cross section for the scattering of unpolarized neutrons by unpolarized protons at angles θ and $\pi - \theta$ allows us to determine (if $\sigma_{pp}(\theta)$ is known) the scattering cross section $\sigma_{I=0}(\theta)$ in the state with zero isotopic spin, and the interference term Re Tr $M_1 \times (\mathbf{p}', \mathbf{p}) M_0^+(\mathbf{p}', \mathbf{p})$, in which the scattering amplitude in the state with I = 0, which is of interest to us, enters linearly.

It is obvious that in the region where Coulomb interaction can be neglected we have

$$\sigma_{I=1}(\theta) = \sigma_{pp}(\theta). \tag{83}$$

Using (5) and (6) we obtain for the state with I = 0 the following expressions for the polarization and the second-rank polarization tensors:[2)]

$$2[\sigma_{np}(\theta)P_{np}(\theta) - \sigma_{np}(\pi-\theta)P_{np}(\pi-\theta)] - \sigma_{I=1}(\theta)P_{I=1}(\theta) = \sigma_{I=0}(\theta)P_{I=0}(\theta),$$
$$2[\sigma_{np}(\theta)D_{ik}^{np}(\mathbf{p}',\mathbf{p}) + \sigma_{np}(\pi-\theta)K_{ik}^{np}(-\mathbf{p}',\mathbf{p})] - \sigma_{I=1}(\theta)D_{ik}^{I=1}(\mathbf{p}',\mathbf{p}) = \sigma_{I=0}(\theta)D_{ik}^{I=0}(\mathbf{p}',\mathbf{p}),$$
$$2[\sigma_{np}(\theta)K_{ik}^{np}(\mathbf{p}',\mathbf{p}) + \sigma_{np}(\pi-\theta)D_{ik}^{np}(-\mathbf{p}',\mathbf{p})] - \sigma_{I=1}(\theta)K_{ik}^{I=1}(\mathbf{p}',\mathbf{p}) = \sigma_{I=0}(\theta)K_{ik}^{I=0}(\mathbf{p}'\ \mathbf{p}). \tag{84}$$

Similarly we find that the interference terms are equal to

$$\sigma_{np}(\theta)P_{np}(\theta) + \sigma_{np}(\pi-\theta)P_{np}(\pi-\theta) = (\sigma P)_{\mathrm{int}} = {}^1/_4\mathrm{Re\,Sp}\,(\boldsymbol{\sigma}_1\mathbf{n}) M_1(\mathbf{p}',\mathbf{p}) M_0^+(\mathbf{p}',\mathbf{p}),$$
$$\sigma_{np}(\theta)D_{ik}^{np}(\mathbf{p}',\mathbf{p}) - \sigma_{np}(\pi-\theta)K_{ik}^{np}(-\mathbf{p}',\mathbf{p}) = (\sigma D_{ik})_{\mathrm{int}} = {}^1/_4\mathrm{Sp\,Re}\,\sigma_{1i}M_1(\mathbf{p}',\mathbf{p})\sigma_{1k}M_0^+(\mathbf{p}',\mathbf{p}),$$
$$\sigma_{np}(\theta)K_{ik}^{np}(\mathbf{p}',\mathbf{p}) - \sigma_{np}(\pi-\theta)D_{ik}^{np}(-\mathbf{p}',\mathbf{p}) = (\sigma K_{ik})_{\mathrm{int}} = {}^1/_4\mathrm{Re\,Sp}\,\sigma_{2i}M_1(\mathbf{p}',\mathbf{p})\sigma_{1k}M_0^+(\mathbf{p}',\mathbf{p}),$$
$$\sigma_{np}(\theta)C_{ik}^{np}(\mathbf{p}',\mathbf{p}) - \sigma_{np}(\pi-\theta)C_{ik}^{np}(-\mathbf{p}',\mathbf{p}) = (\sigma C_{ik})_{\mathrm{int}} = {}^1/_4\mathrm{Re\,Sp}\,\sigma_{1i}\sigma_{2k}M_1(\mathbf{p}',\mathbf{p})M_0^+(\mathbf{p}',\mathbf{p}). \tag{85}$$

It is obvious that the cross section, polarization, and components of the tensors D_{ik}, K_{ik}, and C_{ik} in the state I = 0 are given by expressions (51)–(64) (in which u should be replaced by u_0, v by v_0, etc.). With the aid of (85) we obtain for the interference terms the expressions

$$^1/_4\,\mathrm{Re\,Sp}\, M_1M_0^+ = 2\,\mathrm{Re}\,(u_1u_0^* + v_1v_0^* + c_1c_0^* + g_1g_0^* + h_1h_0^*),$$
$$^1/_4\,\mathrm{Re\,Sp}\,(\boldsymbol{\sigma}_1\mathbf{n})M_1M_0^+ = 2\,\mathrm{Re}\,(u_1c_0^* + u_0^*c_1),$$
$$^1/_4\,\mathrm{Re\,Sp}\,\sigma_{1i}M_1\sigma_{1k}M_0^+ = 2[\mathrm{Re}\,(u_1u_0^* + v_1v_0^* + c_1c_0^* - g_1g_0^* - h_1h_0^*)n_in_k + \mathrm{Re}\,(u_1v_0^* + v_1u_0^* + g_1h_0^* + h_1g_0^*)l_il_k + \mathrm{Re}\,(u_1v_0^* + v_1u_0^* - g_1h_0^* - h_1g_0^*)m_im_k + \mathrm{Im}\,(c_1v_0^* - v_1c_0^*)(l_im_k - m_il_k)],$$

[2)] Relations (82)–(86) were first derived (in a different form) in[8].

$$^1/_4\,\mathrm{Re\,Sp}\,\sigma_{2i}M_1\sigma_{1k}M_0^+ = 2[\mathrm{Re}\,(u_1u_0^* - v_1v_0^* + c_1c_0^* + g_1g_0^* - h_1h_0^*)n_in_k + \mathrm{Re}\,(u_1g_0^* + g_1u_0^* + v_1h_0^* + h_1v_0^*)l_il_k + \mathrm{Re}\,(u_1g_0^* + g_1u_0^* - v_1h_0^* - h_1v_0^*)m_im_k + \mathrm{Im}\,(c_1g_0^* - g_1c_0^*)(l_im_k - m_il_k)],$$

$$^1/_4\,\mathrm{Re\,Sp}\,\sigma_{1i}\sigma_{2k}M_1M_0^* = 2[\mathrm{Re}\,(u_1u_0^* - v_1v_0^* + c_1c_0^* - g_1g_0^* + h_1h_0^*)n_in_k + \mathrm{Re}\,(u_1h_0^* + h_1u_0^* + v_1g_0^* + g_1v_0^*)l_il_k + \mathrm{Re}\,(-u_1h_0^* - h_1u_0^* + v_1g_0^* + g_1v_0^*)m_im_k - \mathrm{Im}\,(c_1h_0^* - h_1c_0^*)(l_im_k + m_il_k)]. \quad (86)$$

In conclusion we obtain formulas for the reconstruction of the nucleon-nucleon scattering matrix in the state with total isotopic spin equal to zero. The matrix $M_1(\mathbf{p}', \mathbf{p})$ is assumed known. If we measure at the angles θ and $\pi - \theta$ the differential cross section for the scattering of unpolarized neutrons by unpolarized protons, the polarization produced by collision of the unpolarized particles, and the normal components of the tensors D_{ik}, K_{ik}, and C_{ik}, then we obtain from (65) $|v_0|^2$, $|g_0|^2$, $|h_0|^2$, and $|u_0|^2 + |c_0|^2$ (the corresponding quantities for pp scattering at the angle θ are assumed known; it is obvious that it is necessary to make in (65) the substitutions $|v|^2 \to |v_0|^2$, $|g|^2 \to |g_0|^2$, etc., $K_{nn} \to K_{nn}^{I=0}$, $D_{nn} \to D_{nn}^{I=0}$, etc.).

It is convenient to introduce in lieu of the amplitudes u_0 and c_0

$$\alpha_0 = u_0 + c_0, \quad \beta_0 = u_0 - c_0. \quad (87)$$

We then get from (67) and (54)

$$|\alpha_0|^2 = {}^1/_8\sigma_{I=0}(1 + K_{nn}^{I=0} + D_{nn}^{I=0} + C_{nn}^{I=0} + 4P_{I=0}),$$

$$|\beta_0|^2 = {}^1/_8\sigma_{I=0}(1 + K_{nn}^{I=0} + D_{nn}^{I=0} + C_{nn}^{I=0} - 4P_{I=0}). \quad (88)$$

We proceed to consider the interference terms. From (86) and (87) we easily obtain

$$\mathrm{Re}\,v_0v_1^* = {}^1/_8(\sigma_{\mathrm{int}} + (\sigma D_{nn})_{\mathrm{int}} - (\sigma K_{nn})_{\mathrm{int}} - (\sigma C_{nn})_{\mathrm{int}}),$$

$$\mathrm{Re}\,h_0h_1^* = {}^1/_8(\sigma_{\mathrm{int}} - (\sigma D_{nn})_{\mathrm{int}} - (\sigma K_{nn})_{\mathrm{int}} + (\sigma C_{nn})_{\mathrm{int}}),$$

$$\mathrm{Re}\,g_0g_1^* = {}^1/_8(\sigma_{\mathrm{int}} - (\sigma D_{nn})_{\mathrm{int}} + (\sigma K_{nn})_{\mathrm{int}} - (\sigma C_{nn})_{\mathrm{int}}),$$

$$\mathrm{Re}\,\alpha_0\alpha_1^* = {}^1/_8(\sigma_{\mathrm{int}} + (\sigma D_{nn})_{\mathrm{int}} + (\sigma K_{nn})_{\mathrm{int}} + (\sigma C_{nn})_{\mathrm{int}} + 4(\sigma P)_{\mathrm{int}}),$$

$$\mathrm{Re}\,\beta_0\beta_1^* = {}^1/_8(\sigma_{\mathrm{int}} + (\sigma D_{nn})_{\mathrm{int}} + (\sigma K_{nn})_{\mathrm{int}} + (\sigma C_{nn})_{\mathrm{int}} - 4(\sigma P)_{\mathrm{int}}). \quad (89)$$

Here

$$\alpha_1 = u_1 + c_1, \quad \beta_1 = u_1 - c_1.$$

Thus, measurement of the differential cross section, the polarizations, and the normal components of the depolarization, polarization-transfer and polarization-correlation tensors at angles θ and $\pi - \theta$ makes it possible to determine at the angle θ the moduli of all five amplitudes in the state with I = 0, and the cosines of the phase differences of the amplitudes v_0 and v_1, h_0 and h_1, g_0 and g_1, α_0 and α_1, and β_0 and β_1.

In order to determine the signs of the corresponding phase differences, it is sufficient to find the signs of the imaginary parts $\mathrm{Im}\,v_0v_1^*$, $\mathrm{Im}\,h_0h_1^*$, $\mathrm{Im}\,g_0g_1^*$, $\mathrm{Im}\,\alpha_0\alpha_1^*$, and $\mathrm{Im}\,\beta_0\beta_1^*$. To this end, let us consider other observables. Assume that we have determined in the c.m.s. components of the polarization-correlation tensor in the state I = 0 and the corresponding interference terms (by measuring the cross section for the scattering of the polarized beam by a polarized target). With the aid of (85) and (86) we get

$$\sigma_{np}(\theta)C_+^{np}(\theta) - \sigma_{np}(\pi-\theta)C_+^{np}(\pi-\theta) = 2\,\mathrm{Re}\,(g_0v_1^* + v_0g_1^*) = \frac{2}{|g_1|^2}(\mathrm{Re}\,g_0g_1^*\,\mathrm{Re}\,g_1v_1^* - \mathrm{Im}\,g_0g_1^*\,\mathrm{Im}\,g_1v_1^*) + \frac{2}{|v_1|^2} \times (\mathrm{Re}\,v_0v_1^*\,\mathrm{Re}\,v_1g_1^* + \mathrm{Im}\,v_0v_1^*\,\mathrm{Im}\,g_1v_1^*). \quad (90)$$

It is obvious that the absolute values of $\mathrm{Im}\,g_0g_1^*$ and $\mathrm{Im}\,v_0v_1^*$ are

$$|\mathrm{Im}\,g_0g_1^*| = [|g_0|^2|g_1|^2 - (\mathrm{Re}\,g_0g_1^*)^2]^{1/2},$$

$$|\mathrm{Im}\,v_0v_1^*| = [|v_0|^2|v_1|^2 - (\mathrm{Re}\,v_0v_1^*)^2]^{1/2} \quad (91)$$

and consequently are known. Thus, relation (90) takes the form

$$A\varepsilon_g + B\varepsilon_v = C, \quad (92)$$

where

$$\varepsilon_g = \frac{\mathrm{Im}\,g_0g_1^*}{|\mathrm{Im}\,g_0g_1^*|}, \quad \varepsilon_v = \frac{\mathrm{Im}\,v_0v_1^*}{|\mathrm{Im}\,v_0v_1^*|},$$

and A, B, and C are quantities defined by the relations (90), (89), and (65) and the pp-scattering amplitudes. From this we can obtain uniquely ϵ_g and ϵ_v (exceptions are cases when one of the quantities A, B, or C vanishes).

The signs of the imaginary parts $\mathrm{Im}\,g_0g_1^*$ and $\mathrm{Im}\,v_0v_1^*$ can also be obtained from the relation

$$\sigma_{I=0}C_+^{I=0} = 4\,\mathrm{Re}\,v_0g_0^*. \quad (93)$$

Indeed, it is easy to show that (93) can be written in the form

$$A'\varepsilon_g + B'\varepsilon_v + C\varepsilon_g\varepsilon_v = F', \quad (94)$$

where A′, B′, C′, and F′ are known quantities.

With the aid of (94) we can determine uniquely ϵ_g and ϵ_v (with the exception of cases when $F' = C'$ and $A' = -B'$, or else $F' = -C'$ and $A' = B'$). Further, from the relation

$$\sigma_{np}(\theta) C_{lm}{}^{np}(\theta) - \sigma_{np}(\pi - \theta) C_{lm}{}^{np}(\pi - \theta) = 2\,\mathrm{Im}\,(h_0 c_1{}^* - \alpha_0 h_1{}^* + \beta_0 h_1{}^*) \quad (95)$$

we can determine in analogous fashion

$$\varepsilon_h = \frac{\mathrm{Im}\, h_0 h_1{}^*}{|\mathrm{Im}\, h_0 h_1{}^*|}, \quad \varepsilon_\alpha = \frac{\mathrm{Im}\, \alpha_0 \alpha_1{}^*}{|\mathrm{Im}\, \alpha_0 \alpha_1{}^*|}, \quad \varepsilon_\beta = \frac{\mathrm{Im}\, \beta_0 \beta_1{}^*}{|\mathrm{Im}\, \beta_0 \beta_1{}^*|}.$$

Indeed it is easy to see that expression (95) can be represented in the form

$$A'' \varepsilon_h + B'' \varepsilon_\alpha + C'' \varepsilon_\beta = F'', \quad (96)$$

where A'', B'', C'', and F'' are known quantities. From this we obtain ϵ_h, ϵ_α and ϵ_β.[3)] We note that ϵ_h, ϵ_α, and ϵ_β can be obtained from the relations

$$\sigma_{np}(\theta) C_-{}^{np}(\theta) + \sigma_{np}(\pi - \theta) C_-{}^{np}(\pi - \theta) = \mathrm{Re}\,(2 h_0 u_1{}^* + \alpha_0 h_1{}^* + \beta_0 h_1{}^*),$$

$$\sigma_{I=0} C_-{}^{I=0} = 4\,\mathrm{Re}\, u_0 h_0{}^*, \qquad \sigma_{I=0} C_{lm}^{I=0} = 4\,\mathrm{Im}\, c_0{}^* h_0. \quad (97)$$

It is obvious that ϵ_g, ϵ_v, etc. can also be determined if we know the components of the other polarization tensors.

Finally, ϵ_g, ϵ_v, ϵ_h, ϵ_α, and ϵ_β can also be obtained with the aid of the quantities that are measurable directly in the experiment. For example, from

$$\sigma_{np}(\theta) A_{ss}{}^{np}(\theta) - \sigma_{np}(\pi - \theta) A_{ss}{}^{np}(\pi - \theta) = 2(\mathrm{Re}\,(v_1 g_0{}^* + g_1 v_0{}^*) + \mathrm{Im}\,(c_1 h_0{}^* - h_1 c_0{}^*) \sin\theta - \mathrm{Re}\,(u_1 h_0{}^* + h_1 u_0{}^*) \cos\theta] \quad (98)$$

we obtain a relation that contains linearly all five quantities whose signs we must determine.

In conclusion we emphasize that owing to the interference of the amplitudes with I = 1 and I = 0 in the expressions for the observable np scattering, all 10 amplitudes are reconstructed accurate to one common phase.

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3) If F″ coincides in absolute magnitude with one of the coefficients in the left side of (98), and the remaining two coefficients also coincide in absolute magnitude, then ϵ_h, ϵ_α, and ϵ_β cannot be uniquely determined from (98).

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