THE NATURE OF THE CENTRAL BODY IN THE SCHWARZSCHILD SOLUTION

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When spherical symmetry is assumed the requirement $ds^2 \geq 0$ for the matter of the central body leads to an oscillatory character of the motion, independently of the equation of state. We consider the motion of dustlike matter with zero pressure. In the nonrelativistic case the motion is oscillatory, and in the transition from contraction to expansion particles pass through the center. It is shown that when quantum effects are not taken into account at the singular point the oscillatory character of the motion is present also in the relativistic case, and the maximum radius r_{max} must be larger than the gravitational radius r_0. The possibility of observation of the oscillatory motion by an external observer is discussed. With the pulsating solution the boundary between the R and T regions is not stationary and does not pass through vacuum ($g_{00} \neq -g_{11}^{-1}$), and therefore the integral for t—the time of motion of a light ray to the R—T boundary—converges, $t < \infty$, and consequently the central body can be observed from the R region, and loss of energy by radiation is not forbidden. As energy is radiated away $r_{max} \to r_0$; the time for the energy to be radiated away can be larger than the characteristic time r_{max}/c by several orders of magnitude.

1. THE CAUSALITY PRINCIPLE IN GENERAL RELATIVITY THEORY

THE metric for a spherically symmetrical gravitational field in empty space is described by the Schwarzschild solution[1]:

$$ds^2 = (1 - r_0/r)c^2dt^2 - (1 - r_0/r)^{-1}dr^2 - r^2d\sigma^2,$$
$$r_0 = 2kM/c^2, \quad d\sigma^2 = d\theta^2 + \sin^2\theta d\varphi^2,$$
$$0 < r < \infty, \quad 0 < \theta < \pi, \quad 0 < \varphi < 2\pi. \qquad (1)$$

The coordinate r is defined so that the length of the circumference is $2\pi r$ and the area of the sphere is $4\pi r^2$. This metric has two singularities: at $r = r_0$ and at $r = 0$. The singularity at $r = r_0$ is nonphysical and can be removed by a transformation of coordinates. There are a number of papers devoted to the study of the singularity at $r = r_0$.[1-5]

The location of the light cones for the metric (1) in the Schwarzschild coordinates is shown in Fig. 1. Timelike intervals are shaded. For $r < r_0$ the only possible motion is toward the center or away from it, since from the condition that the interval be positive it follows that

$$(u^r)^2 = (dr/ds)^2 \neq 0. \qquad (2)$$

Let us now consider the central singularity $r = 0$. As was pointed out by Synge,[6] for a central body at rest

$$dt \neq 0, \quad dr = 0, \quad ds^2 \to -\infty. \qquad (3)$$

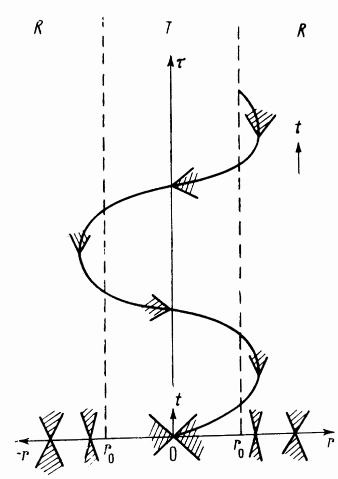


FIG. 1

The interval of the central singularity is spacelike. It can be seen directly from Fig. 1 that the world line of the central body lies outside the light cone. Owing to the invariance of the interval the condition $ds^2 < 0$ is of a physical nature and cannot be removed by a coordinate transformation.

We recall that in special relativity we have for any material body[1]

$$ds^2 \geqslant 0. \qquad (4)$$

Violation of the inequality (4) in special relativity theory leads to ultralight velocities and to violation of the causality principle.

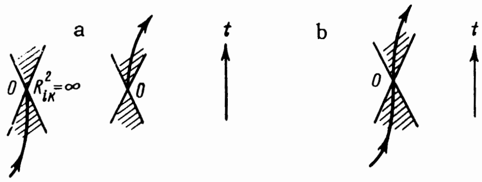


FIG. 2

Physical objects with spacelike intervals contradict the fundamentals of both general and special relativity theory. By means of them one can realize a signal into the past.[1,5] The fact that such an object is known in only one case—the central body of the Schwarzschild solution—and that despite the small lower limit on the mass set by quantum theory, 10^{-5}g, it cannot be produced under laboratory conditions, does not remove the logical contradictions of the theory. Therefore in general relativity theory (GRT) there are two possibilities.

1. The introduction of a new class of physical objects with spacelike intervals (3), which exist at singular points of the gravitational field, where $R^2_{ik} = \infty$. The question of the equation of state of the matter at these singular points remains open.[7] The principle of causality is secured with a supplementary condition: the geodesic world lines of ordinary bodies can only either end at a singularity (collapse), or else begin at one (anticollapse) (Fig. 2, a). Passage of geodesic lines through a singularity O is forbidden,[7] and therefore the "singularity" cannot serve for the transmission of signals between ordinary bodies.

2. The logical contradiction (3) is a consequence of the assumption that the solution is a static one. Therefore we assume that the source of the Schwarzschild field is not a static object, but a nonstationary one composed of ordinary matter moving in such a way that the condition (4) is satisfied for it. Along with this the external field is still a static one.[1] Termination of geodesic lines is not permitted, as in special relativity theory (Fig. 2, b). In the present paper it will be shown that this point of view is free from contradiction.

The condition (4) is necessary but not sufficient. To satisfy the causality principle all velocity four-vectors must lie inside the upper sheet of the light cone (there must be no trajectories going into the past). In the special theory of relativity, in which there is absolute parallelism, it is easy to formulate this condition in an invariant way; in the general theory of relativity there does not exist an invariant definition of the interior of the upper sheet.[7]

Let us consider the structure of the light cones near the stationary boundary between the R and T regions (Fig. 3). For radial motion near the boundary we have

$$ds^2 = (\varepsilon / r_0)\, c^2 dt^2 - (r_0 / \varepsilon)\, dr^2; \quad \varepsilon = r - r_0, \tag{5}$$

$\varepsilon > 0$, R region;
$\varepsilon < 0$, T region;
$\varepsilon = 0$, boundary between R and T regions.

The light cone is determined by the equations $dt/d\epsilon = \pm r_0/\epsilon$. In Fig. 3 the timelike regions are shaded.

Let A and A_1 be the upper and lower regions of the light cone in the R region, and let B and B_1 be the two timelike regions in the T region. For the case of collapse we have the correspondence

$$A \leftrightarrow B, \quad A_1 \leftrightarrow B_1, \tag{6}$$

and for anticollapse[7]

$$A \leftrightarrow B_1, \quad A_1 \leftrightarrow B. \tag{7}$$

We shall show that it is impossible to give an unambiguous definition of interior of the upper sheet of the light cone in the entire space, including the T region. Let a test particle emerge from the singular point (anticollapse[7]). For it the interior of the upper light cone is a region directed away from the center, according to (7). If the energy is insufficient for the particle to go off to infinity, in the R region the particle turns back toward the center. After the return to the T region there is collapse, and the correspondence is established in accordance with (6). The scheme of motion is shown in Fig. 4. The "upper" regions of the light cone are shaded.

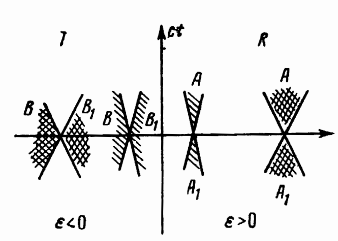
To satisfy the causality principle we require that in the R region the condition

$$d\tau / dt \geqslant 0, \tag{8}$$

hold, where τ is the proper time and t is the time of an observer at infinity. We also require that at nonsingular points of the T region the velocity vectors of particles lie inside only one of the sheets of the light cone. This requirement is evoked by the wish that the properties of a freely falling system in the R region and the T region should be the same; there is no more rigorous basis for this requirement.

The stationary boundary $\epsilon = 0$ is a singular point of the coordinate system, a caustic.[8] All of the tangents are parallel to the boundary, and it would

FIG. 3



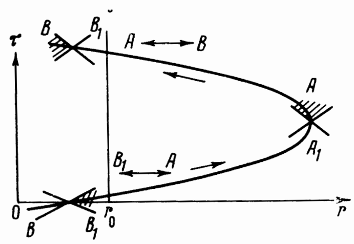


FIG. 4

seem that passage through the boundary is impossible. This assertion, however, is due to the apparent (nonphysical) singularity of the coordinate system. As is shown by a treatment in the freely falling (contracting or expanding) system of Lemaître,[7] passage is possible, and the observer does not even notice the R − T boundary.[4] If the boundary between the R and T regions is not stationary and is not located in vacuum, passage is also possible according to the time of an observer at infinity (see below, Sec. 3).

2. PULSATION OF THE SCHWARZSCHILD CENTRAL BODY

We shall construct the world lines of the Schwarzschild central body in such a way that the conditions (4) and (8) are satisfied. For this purpose we shall have to define a coordinate system.

We shall see later that in a nonstationary spherically symmetrical solution "the matter pulsates," and at the greatest contraction particles pass through the center of symmetry, which is the origin of the coordinates.[9] In ordinary flat three-dimensional space the spherical coordinate system is inconvenient because the polar angles change discontinuously when a particle passes through the origin. Therefore it is convenient to define the spherical coordinates by the condition

$$-\infty < r < +\infty, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \varphi \leqslant \pi. \quad (9)$$

Then in passage through the center the radius changes sign and the polar angles are continuous functions. With this condition the metric (1) is

$$ds^2 = \left(1 - \frac{r_0}{|r|}\right) c^2\, dt^2 - \left(1 - \frac{r_0}{|r|}\right)^{-1} dr^2 - r^2 d\sigma^2. \quad (10)$$

What exactly is a nonstationary spherically symmetrical object? The metric outside the spherical ball is always of the Schwarzschild form, and therefore the particles of a spherical shell of the ball move in a space with the metric (10). Starting from the positions of the light cones, it is easy to construct the world lines of a spherical shell of the ball. The world lines constructed in the Schwarzschild coordinates (Fig. 1) describe spherically symmetrical pulsations of the ball, with the maximum radius of the ball larger than or equal to the gravitational radius r_0. The minimum radius is equal to zero, and all of the particles pass through the center simultaneously. The reason for this last fact is that at every nonsingular point of space the velocity four-vectors of the particles must lie inside only one sheet of the light cone. In Fig. 3 the future is directed upward and (8) holds in the R region. By what was said at the end of Sec. 1, Fig. 1 describes the pulsating central body only qualitatively; the scale on the t axis is nonuniform. The quantitative relations will be presented in a separate paper. It follows from the condition (4) that the matter in the T region cannot come to a stop, independently of whether or not it reaches the singularity.[8,10]

We shall now show that the picture drawn in Fig. 1 does not contradict the equations of motion. It is natural, following[7] and[11], to begin the discussion with the nonrelativistic case of dustlike matter. We imagine a spherical dust cloud whose particles attract each other according to Newton's law. The cloud as a whole begins to contract, and does so as long as there is no effect of other forces. If there are no other forces, the cloud will contract without limit, with the potential energy in the selfconsistent gravitational field of the particles changing into kinetic energy of the motion. If the initial density of the matter was everywhere the same, then all of the particles reach the center simultaneously. In this state the system has minimum potential energy and maximum kinetic energy, like a pendulum in its lowest position. Remaining in the framework of nonrelativistic mechanics, we can easily see that the contraction is succeeded by an expansion—the potential energy will increase at the expense of the kinetic energy.

We have been obliged to present this elementary argument, since one usually encounters in the literature the assertion that the contraction of a dust cloud to a point is irreversible in the nonrelativistic case.[11] We emphasize that in passing, in the case of ideal symmetry, from contraction to expansion the particles do not stop at the center, but fly through it with infinite speed ($v = dr/dt = \infty$).[9] If the matter is charged, then in the relativistic case it is possible to have a transition from contraction to expansion with the particles brought to a stop ($v = dr/dt = 0$).[12] The observed pattern does not depend, however, on the manner in which the passage from contraction to expansion has occurred, in particular in noncentral motion, and therefore the transition from contraction to expansion is the same for neutral and for weakly charged matter.

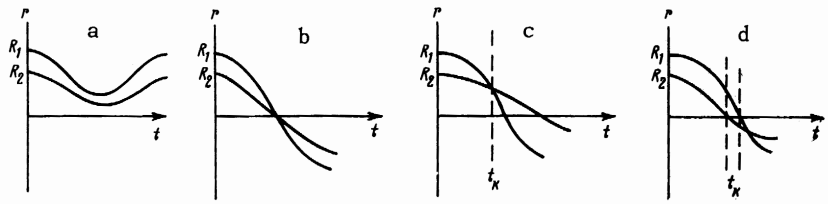


FIG. 5

Let us consider the equation of spherically symmetrical motion. We confine ourselves to the case of spherical symmetry only because in GRT only the spherically symmetrical motion of matter has been studied in detail,[4] so that it is easy to compare the results. Actually a spherically symmetrical contraction is unstable both in GRT and in classical mechanics.

The equation of motion of a layer of radius r(R), where R is the radius at the initial time, is

$$d^2r / dt^2 = -kM / r^2, \tag{11}$$

where M is the mass inside the layer. If the mass does not change in the process of motion—and only in this case—we arrive at the energy integral

$$\dot{r}^2 = 2\left[E(R) + \frac{kM(R)}{|r|}\right]; \quad \frac{dM}{dt} = 0, \tag{12}$$

where M depends only on R.

Let us consider in more detail the condition that the mass is constant inside the layer, Eq. (12). Let R_1 and R_2 be the initial coordinates of two particular layers. Figure 5 shows plots of the functions

$$r_1 = r(R_1, t), \quad r_2 = r(R_2, t), \quad R_1 > R_2. \tag{13}$$

Constancy of the mass means that particles do not enter the inside of the layer and do not emerge from it. Owing to the assumption of spherical symmetry it is necessary for constancy of the mass that we always have $|r_1| > |r_2|$. In other words, (12) is satisfied only for motions of types a and b in Fig. 5. For Fig. 5, c, d, $dM/dt \neq 0$, and the relation (10) (sic) ceases to hold for $t > t_k$.

In classical mechanics absolute space and absolute time exist and there is no question about separating the phenomenon and the coordinate system. In GRT the investigation is usually conducted in the comoving coordinate system.[1,13] It is easy to see that when the world lines intersect at r = 0 there is a singular point; when intersection occurs at $r \neq 0$ the comoving coordinate system is nonunique, and the formulas for going over to it must contain discontinuous functions.

In the framework of the spherically symmetrical problem, on passage through the singular point r = 0 the density ρ becomes infinite and the question arises as to the applicability and meaning of the condition (4) and of the equations of GRT themselves, owing to quantum effects for $\rho > 10^{93}$ g/cm^3. Nevertheless all of the subsequent treatment is based on the fundamental assumption that the quantum theory of gravitation does not qualitatively change the results.

Let us consider the relativistic problem. For centrally symmetrical motion of dustlike matter the interval in the comoving system is given by[1]

$$ds^2 = c^2 d\tau^2 - r^2(R, \tau)\, d\sigma^2 - e^{\omega} dR^2, \tag{14}$$

where r is the radius vector, defined so that the area of the sphere of radius r is $4\pi r^2$. The equations of GRT have the first integral[1]

$$\dot{r}^2 = f(R) + F(R) / |r|. \tag{15}$$

It follows from a comparison of (15) and (12) that the integral (15) is valid only when there is no self-intersection, i.e., for cases a, b of Fig. 5. For cases c and d the integral (15) does not hold. Case a is excluded because there are no forces that can stop the contraction.[7,9] For cases c and d the integral (15) and the solutions that follow from it are valid only up to the moment t_k at which the world lines intersect. For case d (15) is valid only up to the time the particles pass through the center. The use of the integral (15) in cases c and d after the passage through the center implicitly assumes that the matter stops at the center, but there is no justification for this assumption. Case b, in which all the particles pass through the center simultaneously, leads to an oscillating solution.[9]

Accordingly, when the second point of view is consistently developed the solution is a periodic function of the proper time; contraction passes over into expansion.

Since the maximum radius r_{max} of the pulsations exceeds r_0, the static Schwarzschild solution cannot be applied for $r < r_0$. The relativistic oscillatory collapse differs greatly from the small vibrations studied in mechanics—there is no position of equilibrium, stable or unstable, and the amplitude of the oscillations must exceed r_0.

3. PULSATING COLLAPSE FROM THE POINT OF VIEW OF AN EXTERNAL OBSERVER

Since in oscillatory collapse the matter goes out into the R region when it expands, it is to be expected that we shall observe oscillatory collapse from outside, and loss of energy by radiation is not forbidden. This was pointed out without proof at the end of[14], and therefore we think it appropriate to present a proof.

It is well known that from the point of view of an external observer a ray of light, or a freely falling body, reaches the Schwarzschild sphere at $t \to \infty$.[1,7] This result is easily derived from (5). If r_0 does not depend on t, that is if the boundary of the R and T regions is stationary, the equation of a ray of light will be

$$t = \int \frac{dr}{(-g_{00}g_{11}^{-1})^{1/2}} = \frac{r_0}{c}\int \frac{d\varepsilon}{\varepsilon} \sim \frac{r_0}{c} \ln \varepsilon \xrightarrow[\varepsilon\to\infty]{} \infty.$$

The integral diverges logarithmically. The logarithmic divergence has appeared because for the Schwarzschild solution $g_{00} = g_{11}^{-1}$ and the root can be taken; the boundary of the R and T regions in the Schwarzschild solution is stationary and is located in vacuum. We shall show that if either of these assumptions is dropped the logarithmic divergence disappears. We consider first the case of a moving boundary:

$$r_0 = r_0(t), \quad \varepsilon = r - \overline{r_0(t)}, \quad \varepsilon_0(t) = r_0(t) - \overline{r_0(t)},$$
$$g_{00} = -g_{11}^{-1}, \quad \overline{\varepsilon_0(t)} = 0, \tag{16}$$

where the bar denotes averaging over the Schwarzschild time. Then, when we use (5), the equations of the world lines of light are

$$\frac{d\varepsilon}{cdt} = \frac{1}{c}(-g_{00}g_{11}^{-1})^{1/2} = -\frac{\varepsilon}{r_0} + \frac{\varepsilon_0(t)}{r_0}. \tag{17}$$

Solving (17), we get for the world line going to the Schwarzschild sphere

$$\varepsilon(t) = \varepsilon(0)e^{-t/cr_0} - \frac{1}{r_0}\int_0^t \varepsilon_0(\xi)e^{-(t-\xi)/cr_0}d\xi. \tag{18}$$

We shall show that this world line crosses the mean position of the boundary of the R and T regions, $r = \overline{r}_0$. The equation $\epsilon = 0$ reduces to the form

$$\varepsilon(0) - \frac{1}{r_0}\int \varepsilon_0(\xi)e^{-\xi/cr_0}d\xi = 0. \tag{19}$$

The periodic function $\epsilon_0(\xi)$ can be expanded in a Fourier series

$$\varepsilon_0(\xi) = \sum \varepsilon_{0k}e^{i\Omega_k\xi}, \quad \varepsilon_{0k}^* = \varepsilon_{0,-k}. \tag{20}$$

The integral of any one of the terms of (20) diverges at the upper limit, and consequently (19) has solutions for finite t.

If the boundary of the R and T regions is in a region filled with matter, then $-g_{11}^{-1} \neq g_{00}$,[1] and in the integral for t we get

$$t = \int^{r_0} \frac{dr}{(-g_{00}g_{11}^{-1})^{1/2}} < \infty, \quad g_{00}(r_0) = 0, \tag{21}$$

and the logarithmic divergence disappears. Accordingly, inability to observe the R − T boundary from the R region is due to special features of the Schwarzschild solution and does not occur in the general case. For irreversible collapse the presence of a quadrupole moment[15] also has the consequence that the time (21) is finite. When the integral (21) converges the world lines in Figs. 1 and 4 can cross the R − T boundary. If the boundaries between the R and T regions are interrupted, this also leads to a finite value of t. The proof that it is impossible to observe the collapse from the R region is based on the divergence of the integral for t. The convergence of the integral (21) means that periodic collapse with $r_{max} > r_0$ can be observed from the R region. In our opinion this result also follows from the treatment in[13].

Accordingly, in the pulsating model of the central body exchange of information between the R and T regions (R ↔ T) is possible, whereas it is usually stated[4] that only a one-way transfer of information is possible (R → T). We shall show that the necessity of a two-way exchange of information (R ↔ T) follows from the principle of equivalence. The coordinates of a collapsing star can be determined by an outside observer from the motion of planets around it, which are in the R region at large distances, where the field is Newtonian. Let an external uniform gravitational field act on a planetary system with a collapsing central star. This field penetrates from the R region into the T region of the collapsing mass (R → T) and changes its motion. The motion of the planets must occur around the new position of the central body, and therefore information about the new position must be transferred into the R region by the gravitational field (T → R). If any transformation of information (T → R) is forbidden, then the gravitational field of a collapsing star does not change when there is a change of its motion, and the planets move around the old position of the star without reacting to the external uniform gravitational field, which contradicts the principle of equivalence. This contradiction proves the necessity of two-way exchange of information (R ↔ T).

In[12], starting from the impossibility of two-way exchange of information (R ↔ T), it was shown that the R region around a collapsing body has more than one sheet. The meaning of a many-sheeted physical space, Euclidean at infinity, is unclear.

For each collapsing star there would have to be its own multiply connected space. Since the two-way exchange of information (R ↔ T) is indeed not impossible, there is no need to introduce a many-sheeted external space; an outside observer sees both the contraction and the expansion.

Accordingly, for a periodically collapsing body there is no closing-in on itself, and it is not impossible for large amounts of energy to be radiated away. As the energy is removed by radiation the total energy of the collapsing mass decreases and $r_{max} \rightarrow r_0$. Accordingly the collapsing mass slowly goes down to the Schwarzschild sphere, and the time for the radiation loss can be several orders of magnitude larger than the characteristic time r_0/c.

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