STUDY OF MAGNETIC PROPERTIES OF TWO-BAND SUPERCONDUCTORS IN THE VICINITY OF THE UPPER CRITICAL FIELD

V. A. MOSKALENKO

Mathematics Institute and Computer Center, Academy of Sciences, Moldavian S.S.R.

Submitted to JETP editor April 14, 1966

J. Exptl. Theoret. Phys. (U.S.S.R.) 51, 1163–1175 (October, 1966)

We formulate the basic equations for the electrodynamics of two-band superconductors. In the nearly free electron approximation we study the magnetic properties of pure two-band superconductors in the neighborhood of H_{c2} ($H_e \lesssim H_{c2}$). We obtain an expression for the free energy of the system in the second approximation in the difference between H_{c2} and the magnetic induction B, and using this we study other magnetic characteristics. We show that two-band superconductors of the second kind are characterized by two constants κ_1 and κ_2 which are larger than $1/\sqrt{2}$ and that as a consequence the free energy is, in general, not a monotonic function of the parameter σ of the theory of the mixed state, and that hence the minimum value of σ ($\sigma > 1$) is not always realized in such superconductors, in contradistinction from the single-band case. We show for the case of superconductors with a large concentration of non-magnetic impurities that the basic equations of the theory acquire a single-band character with an appropriate definition on a two-band basis of the critical temperature T_c and the Ginzburg-Landau parameter κ.

1. INTRODUCTION

In [1] (see also [2]) the thermodynamic properties of pure two-band superconductors were considered on the basis of the BCS model[3] and the Bogolyubov u-v transformation. An attempt to consider the electrodynamic properties of pure two-band superconductors is contained in a paper by Tilley,[5] in which the upper critical magnetic field for the superconductor was determined.

In the present paper we formulate the basic equations for the electrodynamics of two-band superconductors in a form valid for both pure superconductors and superconductors with impurities. We study the properties of two-band superconductors of the second kind. We show that the electromagnetic properties of pure two-band substances are determined by two parameters κ (larger than $1/\sqrt{2}$) and that two-band superconductors with a large concentration of non-magnetic impurities in the vicinity of the critical temperature T_c can be described by equations of the single-band type, provided the quantity T_c and the appropriate parameter κ are defined from the two-band scheme. The study is carried out on the basis of Abrikosov's theory[6] of type II superconductors and the development of that theory in the papers by Kleiner et al.[7] and Lasher.[8]

2. BASIC EQUATIONS

Using the Hamiltonian for two-band superconductors from [1] supplemented by an electron-impurity interaction, we get the equations for the temperature-dependent Green functions[9]

$$G_{mn}(\mathbf{x}\sigma\mathbf{x}'\sigma'|\tau-\tau') = \langle T\tilde{\psi}_m(\mathbf{x}\sigma\tau)\tilde{\bar{\psi}}_n(\mathbf{x}'\sigma'\tau')\rangle,$$
$$R_{mn}(\mathbf{x}\sigma\mathbf{x}'\sigma'|\tau-\tau') = \langle T\tilde{\psi}_m(\mathbf{x}\sigma\tau)\tilde{\psi}_n(\mathbf{x}'\sigma'\tau')\rangle,$$
$$P_{mn}(\mathbf{x}\sigma\mathbf{x}'\sigma'|\tau-\tau') = \langle T\tilde{\bar{\psi}}_m(\mathbf{x}\sigma\tau)\tilde{\bar{\psi}}_n(\mathbf{x}'\sigma'\tau')\rangle \quad (1)$$

in the form

$$\Big[-i\Omega_n + H_0\Big(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\Big)\Big] \times G_{mn}(\mathbf{x}\sigma\mathbf{x}'\sigma'|\Omega) + \sum_{n'ks}\int d\mathbf{y}\,\psi^*_{mk}(\mathbf{y}) \times \psi_{mk}(\mathbf{x})V_{\sigma s}(\mathbf{y})G_{n'n}(\mathbf{y}s\mathbf{x}'\sigma'|\Omega) + \sum_{n's}V_{mn'}\Delta_{n'\sigma s}(\mathbf{x}) \times P_{mn}(\mathbf{x}s\mathbf{x}'\sigma'|\Omega) = \delta_{\sigma\sigma'}\delta_{mn}\sum_k \psi_{mk}(\mathbf{x})\psi^*_{mk}(\mathbf{x}'), \quad (2)$$

$$\Big[-i\Omega_n - H_0\Big(i\hbar\nabla - \frac{e}{c}\mathbf{A}\Big)\Big]P_{mn}(\mathbf{x}\sigma\mathbf{x}'\sigma'|\Omega) - \sum_{skn'}\int d\mathbf{y}\,\psi_{mk}(\mathbf{y})\psi^*_{mk}(\mathbf{x})V_{s\sigma}(\mathbf{y})P_{n'n}(\mathbf{y}s\mathbf{x}'\sigma'|\Omega)$$

$$-\sum_{n's'} V_{n'm}\Delta^{*}_{n's\sigma}(\mathbf{x})G_{mn}(\mathbf{x}s\mathbf{x}'\sigma'|\Omega)=0; \qquad (3)$$

$$\Delta_{n\sigma s}(\mathbf{x})=R_{nn}(\mathbf{x}\sigma\mathbf{x}s|0)=\frac{1}{\beta}\sum_{\Omega}R_{nn}(\mathbf{x}\sigma\mathbf{x}s|\Omega),$$

$$\Delta^{*}_{ns\sigma}(\mathbf{x})=P_{nn}(\mathbf{x}s\mathbf{x}\sigma|0)=\frac{1}{\beta}\sum_{\Omega}P_{nn}(\mathbf{x}s\mathbf{x}\sigma|\Omega) \qquad (4)$$

A similar equation holds for the R functions. This set of equations generalizes Gor'kov's equations[10] to the two-band case (n,m = 1,2 are the band indices). Here $\psi_{mk}(\mathbf{x})$ are Bloch functions while the remainder of the notation is the usual one.

We introduce the Green function $g_{nm}(x\sigma x'\sigma'|\Omega)$ of the normal metal with impurities. We can then write Eqs. (2) and (3) in the form

$$G_{mn}(\mathbf{x}\sigma\mathbf{x}'\sigma'|\Omega)+\sum_{lpss'}\int d\mathbf{y}\, V_{pl}\, g_{mp}(\mathbf{x}\sigma\mathbf{y}\sigma'|\Omega)\Delta_{ls's}(\mathbf{y})$$
$$\times P_{pn}(\mathbf{y}s\mathbf{x}'\sigma'|\Omega)=g_{mn}(\mathbf{x}\sigma\mathbf{x}'\sigma'|\Omega), \qquad (5)$$

$$P_{nm}(\mathbf{x}'\sigma'\mathbf{x}\sigma|\Omega)-$$
$$\times -\sum_{lpss'}\int d\mathbf{y}\, V_{lp}\Delta^{*}_{lss'}(\mathbf{y})G_{pn}(\mathbf{y}s\mathbf{x}'\sigma'|-\Omega)$$
$$\times g_{pm}(\mathbf{y}s'\mathbf{x}\sigma|\Omega)=0. \qquad (6)$$

We introduce new quantities

$$\Gamma^{p}_{\alpha'\alpha}(\mathbf{y})=\sum_{l}V_{pl}\,\Delta_{l\alpha'\alpha}(\mathbf{y}),$$

$$\Gamma^{*p}_{\alpha'\alpha}(\mathbf{y})=\sum_{l}V_{lp}\,\Delta^{*}_{l\alpha'\alpha}(\mathbf{y}). \qquad (7)$$

It is well known that at temperatures T close to the critical temperature and also in the case of large concentrations of paramagnetic impurities the quantities $\Delta_{n\sigma\sigma'}$ are small; we can then start from the integral equations (5) and (6) to obtain the following set:

$$\Gamma^{*m}_{\sigma'\sigma}(\mathbf{x})=\frac{1}{\beta}\sum_{\Omega}\sum_{nn'ss'}\int d\mathbf{y}\, V_{nm}\, g_{n'n}(\mathbf{y}s'\mathbf{x}\sigma|\Omega)\Gamma^{*n'}_{ss'}(\mathbf{y})$$
$$\times g_{n'n}(\mathbf{y}s\mathbf{x}\sigma'|-\Omega)-\frac{1}{\beta}\sum_{\Omega}\sum_{nn'pp'}\sum_{\alpha\alpha'\beta\beta'ss'}$$
$$\times\int d\mathbf{y}\,d\mathbf{y}'\,d\mathbf{y}''\, V_{nm}\,\Gamma^{*n'}_{ss'}(\mathbf{y})$$
$$\times g_{n'n}(\mathbf{y}s'\mathbf{x}\sigma|\Omega)\Gamma^{p}_{\alpha'\alpha}(\mathbf{y}')g_{n'p}(\mathbf{y}s\mathbf{y}'\alpha'|-\Omega)\Gamma^{*p'}_{\beta\beta'}(\mathbf{y}'')$$
$$\times g_{p'p}(\mathbf{y}''\beta\mathbf{y}'\alpha|\Omega)g_{p'n}(\mathbf{y}''\beta'\mathbf{x}\sigma'|-\Omega). \qquad (8)$$

All quantities occurring in these formulae depend on the electromagnetic potential $\mathbf{A}(\mathbf{x})$. According to [10], the dependence of the Green functions on the magnetic field is in the form of phase factors. Expanding these factors in a power series in the quantity A and assuming that the quantities $\Gamma(\mathbf{x})$ depend weakly on their argument $\mathbf{x}$, we obtain the following set of Ginzburg-Landau[11] equations for two-band superconductors:

$$\Gamma^{*m}_{\sigma'\sigma}(\mathbf{x})=\sum_{nn'ss'}V_{nm}\,Q^{n'n}_{s'\sigma s\sigma'}(\mathbf{x})\Gamma^{*n}_{ss'}(\mathbf{x})$$
$$+\sum_{nn'}V_{nm}\sum_{ss'j}Q^{n'n}_{s'\sigma s\sigma'}(\mathbf{x};j)\left(\nabla_j+\frac{2ie}{c\hbar}A_j(\mathbf{x})\right)\Gamma^{*n'}_{ss'}(\mathbf{x})$$
$$+\frac{1}{2}\sum_{nn'}\sum_{ss'}\sum_{jl}V_{nm}\,Q^{n'n}_{s'\sigma s\sigma'}(\mathbf{x};jl)\left(\nabla_j+\frac{2ie}{c\hbar}A_j(\mathbf{x})\right)$$
$$\times\left(\nabla_l+\frac{2ie}{c\hbar}A_l(\mathbf{x})\right)\Gamma^{*n'}_{ss'}(\mathbf{x})-\sum_{nn'm'p}\sum_{\substack{\alpha\alpha'\beta\beta'\\ ss'}}B^{n'n\,n'pm'pm'n}_{s'\sigma\, s\alpha'\,\beta\alpha\,\beta'\sigma'}(\mathbf{x})$$
$$\times\Gamma^{*n'}_{ss'}(\mathbf{x})\Gamma^{p}_{\alpha'\alpha}(\mathbf{x})\Gamma^{*m'}_{\beta\beta'}(\mathbf{x}), \qquad (9)$$

where

$$Q^{n'n}_{s'\sigma s\sigma'}(\mathbf{x})=\int d\mathbf{y}\,Q^{n'n}_{s'\sigma s\sigma'}(\mathbf{x};\mathbf{y}),$$
$$Q^{n'n}_{s'\sigma s\sigma'}(\mathbf{x};j)=\int d\mathbf{y}\,(y-x)_j Q^{n'n}_{s'\sigma s\sigma'}(\mathbf{x};\mathbf{y}),$$
$$Q^{n'n}_{s'\sigma s\sigma'}(\mathbf{x};jl)=\int d\mathbf{y}\,(y-x)_j(y-x)_l Q^{n'n}_{s'\sigma s\sigma'}(\mathbf{x};\mathbf{y}),$$
$$Q^{n'n}_{s'\sigma s\sigma'}(\mathbf{x};\mathbf{y})=\frac{1}{\beta}\sum_{\Omega}g_{n'n}(\mathbf{y}s'\mathbf{x}\sigma|\Omega)g_{n'n}(\mathbf{y}s\mathbf{x}\sigma'|-\Omega),$$

$$B^{n'nn'pm'pm'n}_{s'\sigma s\alpha'\beta\alpha\beta'\sigma'}(\mathbf{x})=\frac{1}{\beta}\sum_{\Omega}\int d\mathbf{y}\,d\mathbf{y}'\,d\mathbf{y}''\,g_{n'n}(\mathbf{y}s'\mathbf{x}\sigma|\Omega)$$
$$\times g_{n'p}(\mathbf{y}s\mathbf{y}'\alpha'|-\Omega)g_{m'p}(\mathbf{y}''\beta\mathbf{y}'\alpha|\Omega)g_{m'n}(\mathbf{y}''\beta'\mathbf{x}\sigma'|-\Omega) \qquad (10)$$

In the last expressions we have written the Green functions for A = 0. Apparently for crystalline systems with a center of inversion the term $Q^{n'n}(\mathbf{x},j)$ vanishes, as we shall assume in the following. If the impurities are randomly distributed we must average over their positions. Assuming that we can average independently the functions Γ and the factors Q and B we get again the set of Eqs. (9) but with functions $\bar{Q}$ and $\bar{B}$ instead of the functions given by (10).

In the present paper we restrict ourselves in the following to the case of pure substances and substances containing non-magnetic impurities.

We must supplement the set of Eqs. (9) with the Maxwell equation

$$\operatorname{rot}\operatorname{rot}\mathbf{A} = 4\pi c^{-1}\mathbf{j}, \tag{11*}$$

where we get easily for the current $\mathbf{j}(\mathbf{x})$ the expression

$$j^k(\mathbf{x}) = \frac{e}{2}\sum_{n'm'}\sum_{\sigma\alpha's s'}\sum_l \{I^{n'm'}_{s's\alpha\alpha'}(\mathbf{x}; kl)$$

$$\times \Gamma^{n'}_{s's}(\mathbf{x})\left(\nabla_l + \frac{2ie}{c\hbar}A_l(\mathbf{x})\right)\Gamma^{*m'}_{\alpha\alpha'}(\mathbf{x}) + \text{c.c.}\}, \tag{12}$$

$$I^{n'm'}_{s's\alpha\alpha'}(\mathbf{x}; kl) = \frac{1}{\beta}\sum_\Omega \sum_{mn\sigma}\int dy dy' (y'-x)$$

$$\times [(\hat{v}_k(x) g_{nn'}(\mathbf{x}\sigma\mathbf{y}s'|\Omega))]$$

$$\times g_{m'n'}(\mathbf{y}'\alpha\mathbf{y}s|-\Omega) g_{m'm}(\mathbf{y}'\alpha'\mathbf{x}\sigma|\Omega) - g_{nn'}(\mathbf{x}\sigma\mathbf{y}s'|\Omega)$$

$$\times g_{m'n'}(\mathbf{y}'\alpha\mathbf{y}s|-\Omega)\hat{v}_k(\mathbf{x}) g_{m'm}(\mathbf{y}'\alpha'\mathbf{x}\sigma|\Omega)], \tag{13}$$

where $\hat{v}_k(\mathbf{x})$ is the electronic velocity operator. Apart from expression (12) there is a second contribution to the current $\mathbf{j}$ which does not vanish for crystalline systems. For systems with an inversion center this term is apparently zero and we have therefore not written it down here.

We finally give an expression, which we need in the following, for the difference in the thermodynamic potentials for the superconducting and the normal states for small values of the quantities Γ. To do this we must use Eqs. (8) and differentiate with respect to the coupling constants $V_{nm}(\mathbf{x})$. We get

$$\Omega_s - \Omega_n = -\frac{1}{4\beta}\sum_{nmpp'\varepsilon\varepsilon'\alpha\alpha'\beta\beta'\sigma\sigma'}\sum \int d\mathbf{x}\,[B^{nm\ pnpp'mp'}_{\sigma\varepsilon\ \alpha'\varepsilon'\alpha\beta\ \sigma'\beta'}(\mathbf{x})]^*$$

$$\times \Gamma^p_{\alpha'\alpha}(\mathbf{x})\Gamma^m_{\sigma\sigma'}(\mathbf{x})\Gamma^{*p'}_{\beta\beta'}(\mathbf{x})\Gamma^{*n}_{\varepsilon'\varepsilon}(\mathbf{x}). \tag{14}$$

If there are impurities we must, as mentioned above, average over their positions in Eqs. (12)–(14). The formulae given by us will be used in the following for a study of systems with impurities.

In the present paper we shall first consider the case of pure superconductors. We can then introduce appreciable simplifications but even in this case we cannot completely finish the calculations because of the band character of the electronic energy spectrum. We can perform the calculations if we replace the rapidly oscillating functions $|\psi_{nk}(\mathbf{x})|^2$ under the summation sign by their average value $1/V$. If we make this approximation we can, of course, only obtain qualitative results for the two-band model. In this approximation Eqs. (8) become

$$\Gamma_m^*(\mathbf{x}) = \sum_n V_{nm}\left[Q_n + \frac{R_n}{6}\left(\nabla + \frac{2ie}{c\hbar}\mathbf{A}(\mathbf{x})\right)^2 - B_n|\Gamma_n(\mathbf{x})|^2\right]\Gamma_n^*(\mathbf{x}), \tag{15}$$

where

$$Q_n = N_n q_n = N_n \ln\left(\frac{2\gamma\beta\hbar\tilde{\omega}_n}{\pi}\right),$$

$$R_n = N_n\rho_n = \frac{7\zeta(3)}{8\pi^2}\beta^2 N_n v_n^2,$$

$$B_n = \frac{7\zeta(3)}{8\pi^2}\beta^2 N_n, \quad N_n = \frac{1}{2\pi^2}\left(\frac{k^2}{\nabla E_n(k)}\right)_{k_n^F}. \tag{16}$$

Moreover, we get instead of Eqs. (12) and (14)

$$\mathbf{j}(\mathbf{x}) = \frac{\hbar e}{3}\sum_n R_n\left[-\frac{4e}{c\hbar}\mathbf{A}(\mathbf{x})|\Gamma_n(\mathbf{x})|^2 - i(\Gamma_n^*(\mathbf{x})\nabla\Gamma_n(\mathbf{x}) - \Gamma_n(\mathbf{x})\nabla\Gamma_n^*(\mathbf{x}))\right], \tag{17}$$

$$\Omega_s - \Omega_n = -\frac{1}{2}\sum_n B_n\int|\Gamma_n(\mathbf{x})|^4 d\mathbf{x}. \tag{18}$$

3. STUDY OF THE VICINITY OF THE UPPER CRITICAL MAGNETIC FIELD OF PURE SUPERCONDUCTORS

Abrikosov[6] has shown that in magnetic fields above the lower critical field H_{c1} but below the upper critical field H_{c2} ($H_{c1} < H_e < H_{c2}$) there appears a mixed state characterized by a periodic structure of the magnetic field distribution and by the existence of maximum-field-strength filaments.

For a study of the immediate neighborhood of the upper critical field ($H \lesssim H_{c2}$) we follow Lasher[8] and change to dimensionless quantities:

$$\Gamma_n(x, y) = \left(\frac{eB}{6\pi c\hbar}\frac{R_n}{B_n}\right)^{1/2}\psi_n(u, v),$$

$$\mathbf{A}(x, y) = \frac{1}{2}\left(\frac{c\hbar B}{\pi e}\right)^{1/2}\mathbf{a}(u, v),$$

$$(x, y) = \left(\frac{\pi c\hbar}{eB}\right)^{1/2}(u, v). \tag{19}$$

Instead of the quantities V_{nm} we also introduce new coupling constants $\bar{V}_{nm}$:

$$\bar{V}_{nm} = V_{nm}\frac{eB}{6\pi c\hbar}R_n\left(\frac{R_n B_m}{R_m B_n}\right)^{1/2}, \tag{20}$$

and also

$$\bar{Q}_n = \frac{6\pi c\hbar}{eB}Q_n, \tag{21}$$

where B is the magnetic induction (directed along the z axis):

*rot ≡ curl.

$$B = \langle \mathrm{rot}_z A(x, y) \rangle = \frac{B}{2\pi} \langle \mathrm{rot}_z \mathbf{a}(u, v) \rangle,$$
$$\langle \mathrm{rot}_z \mathbf{a}(u, v) \rangle = 2\pi. \quad (22)$$

We write the vector **a** as a sum:

$$\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_p(u, v), \quad (23)$$

where

$$\mathbf{a}_0 = (0, 2\pi u, 0), \quad \mathrm{rot}\, \mathbf{a}_0 = 2\pi, \quad (23')$$

so that

$$\langle \mathrm{rot}\, \mathbf{a}_p \rangle = 0. \quad (24)$$

In the new notation the electrodynamic equations become

$$\psi_m^*(u, v) = \sum_n \bar{V}_{nm} [(\nabla + i\mathbf{a})^2 + \bar{Q}_n - |\psi_n|^2] \psi_n^*; \quad (25)$$

$$\mathrm{rot}\,\mathrm{rot}\, \mathbf{a} = \frac{4\pi}{9} \frac{e^2}{c^2} \sum_n \frac{R_n^2}{B_n} [\psi_n^*(-i\nabla - \mathbf{a})\psi_n + \mathrm{c.c.}],$$
$$\mathrm{div}\, \mathbf{a} = 0. \quad (26)$$

We introduce the zeroth approximation wave function φ_m^*:

$$\varphi_m^*(u, v) = \sum_n \bar{V}_{nm} [(\nabla + i\mathbf{a}_0)^2 + \bar{Q}_n + 2\pi(1-\lambda)] \varphi_n^*(u, v), \quad (27)$$

where λ is the dimensionless upper critical field:

$$\lambda = H_{c2} / B, \quad (28)$$

determined from Eq. (27):

$$(1 - \bar{V}_{11}(\bar{Q}_1 - 2\pi\lambda)) [1 - \bar{V}_{22}(\bar{Q}_2 - 2\pi\lambda)]$$
$$- \bar{V}_{12}\bar{V}_{21}(\bar{Q}_1 - 2\pi\lambda)(\bar{Q}_2 - 2\pi\lambda) = 0. \quad (28')$$

The general solution of this equation has the form

$$\varphi_m(u, v) = \bar{C}_m \varphi(u, v), \quad (29)$$

where

$$\varphi(u, v) = \sum_{s=-\infty}^{\infty} d_s e^{-iksv} \exp\{-\pi(u - ks/2\pi)^2\},$$

$$\left(\frac{\bar{C}_2}{\bar{C}_1}\right)^* = \frac{(\bar{Q}_1 - 2\pi\lambda)\bar{V}_{12}}{1 - \bar{V}_{22}(\bar{Q}_2 - 2\pi\lambda)}. \quad (29')$$

The quantity λ vanishes at the critical temperature. Retaining only terms linear in $\ln(T_c/T)$ we get from (28′) the equation

$$H_{c2} = 3\frac{c\hbar}{e} \ln\frac{\beta}{\beta_c}\left(q_1^c + q_2^c - \frac{b}{a}\right)$$
$$\times [\rho_1^c(V_{11}N_1/a - q_2^c) + \rho_2^c(V_{22}N_2/a - q_1^c)]^{-1}, \quad (30)$$

where the quantity T_c and the parameters a and b are defined as follows:

$$b = V_{11}N_1 + V_{22}N_2, \quad a = N_1N_2(V_{11}V_{22} - V_{12}V_{21}),$$
$$1 - V_{11}N_1q_1^c - V_{22}N_2q_2^c + q_1^c q_2^c a = 0,$$
$$q_i^c = q_i|_{T=T_c}, \quad \rho_i^c = \rho_i|_{T=T_c}. \quad (31)$$

Apart from the definition of H_{c2} we introduce an expression for the thermodynamic critical field H_c[1] which is necessary for the following:

$$H_c^2 = \frac{32\pi^3}{7\zeta(3)\beta_c^2}\left(\ln\frac{\beta}{\beta_c}\right)^2\left[N_1 + N_2\frac{(V_{21}N_1/a)^2}{(q_2^c - V_{11}N_1/a)^2}\right]^2$$
$$\times\left[N_1 + N_2\frac{(V_{21}N_1/a)^4}{(q_2^c - V_{11}N_1/a)^4}\right]^{-1}. \quad (32)$$

From (30) and (32) we get

$$\left(\frac{H_{c2}}{H_c}\right)^2 = \frac{9c^2}{32\pi^3}\frac{7\zeta(3)\beta_c^2}{e^2}\left[N_1 + N_2\frac{(V_{21}N_1/a)^4}{(q_2^c - V_{11}N_1/a)^4}\right]$$
$$\times\left[R_1^c + R_2^c\frac{(V_{12}N_1/a)^2}{(q_2^c - V_{11}N_1/a)^2}\right]^{-1}. \quad (33)$$

Let us now turn to Eqs. (25) and (26) and let us consider the next approximations for the function ψ_n and for the vector potential **a**. To do this we introduce a small parameter $\epsilon = \lambda - 1 \geq 0$ in accordance with the fact that we are studying the immediate vicinity of H_{c2}. We introduce the notation:

$$\mathbf{p} = i\nabla + \mathbf{a}_0, \quad \mathbf{p}^* = -i\nabla + \mathbf{a}_0, \quad p_\pm = p_u + ip_v,$$

after which Eqs. (25) and (26) become

$$\psi_m^* + \sum_n \bar{V}_{nm}[\mathbf{p}^{*2} + 2\pi(\lambda - 1) - \bar{Q}_n]\psi_n^*$$
$$= \sum_n \bar{V}_{nm}[2\pi\varepsilon - |\psi_n|^2 - 2\mathbf{a}_p\mathbf{p}^* - \mathbf{a}_p^2]\psi_n^*,$$

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{a}_p = -\frac{4\pi}{9}\frac{e^2}{c^2}\sum_n\frac{R_n^2}{B_n}[\psi_n^*\mathbf{p}\psi_n + \mathrm{c.c.}]. \quad (34)$$

Moreover, we expand in terms of ϵ:

$$\psi_n^* = \varepsilon^{1/2}\varphi_n^* + \varepsilon^{3/2}\chi_n^* + \ldots,$$
$$\mathbf{a}_p = \varepsilon\mathbf{a}_1 + \varepsilon^2\mathbf{a}_2 + \ldots, \quad \langle \mathrm{rot}\,\mathbf{a}_i \rangle = 0. \quad (35)$$

In the first approximation we then get

$$\chi_m^* + \sum_n \bar{V}_{nm}[\mathbf{p}^{*2} + 2\pi(\lambda - 1) - \bar{Q}_n]\chi_n^*$$
$$= \sum_n \bar{V}_{nm}[2\pi - |\varphi_n|^2 - 2\mathbf{a}_1\mathbf{p}^*]\varphi_n^*,$$

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{a}_1 = -\frac{4\pi}{9}\frac{e^2}{c^2}\sum_n\frac{R_n^2}{B_n}[\varphi_n^*\mathbf{p}\varphi_n + \mathrm{c.c.}]. \quad (36)$$

Using the property that

$$p_+\varphi_n(u,v) = p_-^*\varphi_n^*(u,v) = 0, \qquad (37)$$

we get from the second of Eqs. (34):

$$\mathrm{rot}_z\,\mathbf{a}_1 = -\frac{4\pi}{9}\frac{e^2}{c^2}\sum_n \frac{R_n^2}{B_n}\left(|\varphi_n|^2 - \langle|\varphi_n|^2\rangle\right). \qquad (38)$$

Using (27) and (35) we get from the first Eq. (34) the condition

$$\sum_n \langle 2\pi|\varphi_n|^2 - |\varphi_n|^4 - \mathbf{a}_1(\varphi_n\mathbf{p}^*\varphi_n^* + \varphi_n^*\mathbf{p}\varphi_n)\rangle = 0. \qquad (39)$$

Using the equation

$$\langle \mathbf{a}_1(\varphi_n\mathbf{p}^*\varphi_n^* + \varphi_n^*\mathbf{p}\varphi_n)\rangle = \langle|\varphi_n|^2\,\mathrm{rot}\,\mathbf{a}_1\rangle$$

and Eq. (38) we can write (39) in the form

$$\sum_n \Big\langle 2\pi|\varphi_n|^2 - |\varphi_n|^4 + \frac{4\pi}{9}\frac{e^2}{c^2}\sum_m \frac{R_m^2}{B_m} \times \left(|\varphi_m|^2 - \langle|\varphi_m|^2\rangle\right)\Big\rangle = 0. \qquad (40)$$

We introduce the notation

$$|\bar{C}_1|^2\langle|\varphi|^2\rangle = N_0, \qquad |\bar{C}_1|^4\langle|\varphi|^4\rangle = N_0^2\sigma,$$

$$\sum_n \frac{R_n^2}{B_n}|\bar{C}_n|^{2r} = |\bar{C}_1|^{2r}\Xi_r \quad (r = 1, 2),$$

$$\sum_n |\bar{C}_n|^{2r} = |\bar{C}_1|^{2r}\Xi_{r+2} \quad (r = 1, 2),$$

$$2\varkappa_1^2 = \Xi_4 \Big/ \frac{4\pi}{9}\frac{e^2}{c^2}\Xi_1\Xi_3, \qquad \frac{\varkappa_2^2}{\varkappa_1^2} = \Xi_2\Xi_3/\Xi_1\Xi_4. \qquad (41)$$

Condition (40) leads to the following expression for the number of superconducting electrons N_0 in the first band:

$$N_0^{-1} = \frac{2}{9}\frac{e^2}{c^2}\Xi_1[1 + \sigma(2\varkappa_1^2 - 1)]. \qquad (42)$$

We introduce the following expression for the free energy of the system:

$$\Delta f = \frac{F_{SH} - F_n}{V} = \Big\langle \frac{H^2(xy)}{8\pi} - \frac{1}{2}\sum_n B_n|\Gamma_n(xy)|^4\Big\rangle, \qquad (43)$$

where

$$H_z(xy) = B + \frac{B\varepsilon}{2\pi}\mathrm{rot}_z\,\mathbf{a}_1 = B\Big[1 + \frac{2}{9}\frac{e^2}{c^2}\varepsilon\sum_n \frac{R_n^2}{B_n}\left(\langle|\varphi_n|^2\rangle - |\varphi_n|^2\right)\Big]. \qquad (44)$$

Using the fact that $\langle \mathrm{curl}\,\mathbf{a}_z\rangle = 0$, the approximation (44) turns out to be sufficient to evaluate the quantity Δf up to terms of second order in ϵ:

$$\Delta f = \Delta f_0 + \varepsilon^2\Delta f_2, \quad \Delta f_0 = B^2/8\pi,$$

$$\Delta f_2 = -\frac{B^2}{8\pi}\frac{1 + \sigma(2\varkappa_2^2 - 1)}{[1 + \sigma(2\varkappa_1^2 - 1)]^2}. \qquad (45)$$

Using these results we get for the external magnetic field H_e

$$H_e = 4\pi\frac{\partial\Delta f}{\partial B} = B\Big[1 + \varepsilon\frac{1 + \sigma(2\varkappa_2^2 - 1)}{[1 + \sigma(2\varkappa_1^2 - 1)]^2}\Big]. \qquad (46)$$

Hence we have

$$B = H_e - \frac{(H_{c2} - H_e)[1 + \sigma(2\varkappa_2^2 - 1)]}{\sigma[2(2\varkappa_1^2 - 1) - (2\varkappa_2^2 - 1) + \sigma(2\varkappa_1^2 - 1)^2]}. \qquad (47)$$

The magnetic moment of the system has the form

$$M = \frac{B - H_e}{4\pi} = -\frac{B\varepsilon}{4\pi}\frac{1 + \sigma(2\varkappa_2^2 - 1)}{[1 + \sigma(2\varkappa_1^2 - 1)]^2} = -\frac{(H_{c2} - H_e)[1 + \sigma(2\varkappa_2^2 - 1)]}{4\pi\sigma[2(2\varkappa_1^2 - 1) - (2\varkappa_2^2 - 1) + \sigma(2\varkappa_1^2 - 1)^2]}. \qquad (48)$$

We finally write Eq. (44) for the acting magnetic field H in the most convenient form

$$H_z(xy) = H_e - \frac{H_{c2} - H_e}{2(2\varkappa_1^2 - 1) - (2\varkappa_2^2 - 1) + \sigma(2\varkappa_1^2 - 1)^2} \times \Big[\frac{|\varphi_1|^2}{N_0}\frac{1 + \sigma(2\varkappa_1^2 - 1)}{\sigma} + 2(\varkappa_2^2 - \varkappa_1^2)\Big]. \qquad (49)$$

Putting $T = T_c$ in Eq. (29) we get

$$\frac{\bar{C}_2}{\bar{C}_1} = \frac{v_1}{v_2}\frac{V_{12}N_1/a}{(q_2^c - V_{11}N_1/a)}, \qquad (50)$$

whence we get, using (41) and (33)

$$2\varkappa_2^2|_{T=T_c} = (H_{c2}/H_c)^2 > 1. \qquad (51)$$

This last inequality is the condition for type II superconductors. One sees easily that from the fact that N_0 must be positive, $\sigma > 1$, and M must be negative, and from condition (50) we also get the inequality ($H_{c2} > H_e$):

$$2\varkappa_1^2 > 1, \quad \sigma > \frac{2\varkappa_2^2 - 1}{2\varkappa_1^2 - 1}\sigma_0, \quad \sigma_0 = \frac{1}{2\varkappa_1^2 - 1}\Big[1 - 2\frac{2\varkappa_1^2 - 1}{2\varkappa_2^2 - 1}\Big]. \qquad (52)$$

Let us return to Eq. (45) for Δf and study the way the function

$$\chi(\sigma) = \frac{1 + \sigma(2\varkappa_2^2 - 1)}{[1 + \sigma(2\varkappa_1^2 - 1)]^2} \qquad (53)$$

depends on σ. Clearly, the quantity Δf will have its smallest value for that value of the parameter

σ for which the positive function χ has a maximum. This value of σ will be realized in the superconductor. When $\sigma = \sigma_0$ the derivative of the function χ vanishes, while $\chi' < 0$ for $\sigma > \sigma_0$ and $\chi' > 0$ for $\sigma < \sigma_0$. Thus, when $\sigma_0 < 1$, i.e., when $\kappa_1^2 > 1$ for any $\kappa_2^2 > 1/2$ or, when

$$1/2 < \varkappa_1^2 < 1, \quad \varkappa_2^2 < \varkappa_1^2 / 2(1-\varkappa_1^2), \tag{54}$$

the function χ decreases in the region $\sigma > 1$ of interest to us, and the smallest value of the free energy corresponds thus to the minimum value of σ. According to [7] $\sigma \approx 1.16$ and the magnetic field distribution is characterized by a triangular lattice.

When $\sigma_0 > 1$, i.e., when the inequalities

$$1/2 < \varkappa_1^2 < 1, \quad \varkappa_2^2 > \varkappa_1^2/2(1-\varkappa_1^2), \quad \varkappa_2^2 > \varkappa_1^2 \tag{55}$$

are satisfied, σ must, according to (52) remain larger than the quantity $(2\kappa_2^2 - 1)\sigma_0/(2\kappa_1^2 - 1)$ (which is larger than unity). If this last quantity is, for instance, equal to 1.18, a square magnetic lattice will be realized in two-band superconductors satisfying conditions (55).

We have thus shown that in two-band superconductors the symmetry of the magnetic lattice depends on whether inequalities (54) or (55) are satisfied, i.e., on the parameters of the system.

Using (41) we can show that

$$\frac{\varkappa_2^2}{\varkappa_1^2} = 1 + \frac{(1-\tau)(1-|\bar{C}_2/\bar{C}_1|^2)|\bar{C}_2/\bar{C}_1|^2}{(1+\tau|\bar{C}_2/\bar{C}_1|^2)(1+|\bar{C}_2/\bar{C}_1|^4)}, \tag{56}$$

where

$$\tau = \left(\frac{R_2}{R_1}\right)^2 \frac{B_1}{B_2} = \frac{N_2}{N_1}\left(\frac{v_2}{v_1}\right)^4 = \frac{N_2}{N_1}\left(\frac{m_1}{m_2}\right)^4\left(\frac{k_2^F}{k_1^F}\right)^4. \tag{57}$$

In the case where the second band (for instance, a d band) is narrow ($N_2 \gg N_1$, $m_2 \gg m_1$), $\tau < 1$ and κ_2 can thus be larger than κ_1, if $\bar{C}_2/\bar{C}_1 < 1$, and less than κ_1 when $\bar{C}_2/\bar{C}_1 > 1$. A study of the properties of the quantity (50) shows that for well-determined relations between the constants V_{ij} and the densities N_i both these possibilities are admissible (so far their exact values are not yet known).

In conclusion we note that according to (49) the maximum value of the magnetic field $H(x, y)$ which is observed in points where the density of superconducting electrons vanishes, $|\varphi|^2 = 0$, is equal to

$$H_{z\,max} = H_e - \frac{2(H_{c2} - H_e)(\varkappa_2^2 - \varkappa_1^2)}{2(2\varkappa_1^2 - 1) - (2\varkappa_2^2 - 1) + \sigma(2\varkappa_1^2 - 1)}. \tag{58}$$

In the single-band case $\kappa_1 = \kappa_2$ and $H_{z\,max} = H_e$. In the two-band case κ_2 can only be equal to κ_1 if $\bar{C}_2 = \bar{C}_1$. If κ_1 is not equal to κ_2 the maximum magnetic field (the field of the normal substance filaments) differs from the external field H_e.

4. STUDY OF THE PROPERTIES OF SUPERCONDUCTORS WITH LARGE CONCENTRATIONS OF NONMAGNETIC IMPURITIES

In this case we get by averaging the basic quantities and the equations of Sec. 2 over the impurity positions and summing over the spin indices

$$\Gamma_n^*(x) = \sum_m V_{mn} \sum_{n'} \Big\{ Q_{n'm} \Gamma_{n'}^*(x) + \frac{R_{n'm}}{6} \times \left(\nabla + \frac{2ie}{c\hbar}A\right)^2 \Gamma_{n'}^*(x) - \sum_{m'p} \mathscr{B}(n'm' \mid np) \Gamma_{n'}^*(x) \Gamma_p(x) \Gamma_{m'}^*(x) \Big\},$$

$$j(x) = \frac{ie}{3} \sum_{nm} \left[R_{nm} \Gamma_n(x) \left(\nabla + \frac{2ie}{c\hbar}A\right) \Gamma_m^*(x) - \text{c.c.} \right], \tag{59}$$

where, for instance,

$$Q_{n'm} = \frac{1}{2} \sum_{ss'\sigma\sigma'} \langle \overline{Q^{n'mn'm}_{s'\sigma s\sigma'}}(x) \rangle g_{ss'} g_{\sigma'\sigma},$$

$$\mathscr{B}(n'm' \mid mp) = \frac{1}{2} \sum_{\substack{ss'\alpha\alpha' \\ \beta\beta'\sigma\sigma'}} \langle \overline{B^{n'm\,n'p\,m'p\,m'm}_{s'\sigma\ s\alpha'\ \beta\alpha\ \beta'\sigma'}} \rangle g_{ss'} g_{\alpha\alpha'} g_{\beta\beta'} g_{\sigma'\sigma}. \tag{60}$$

The coefficients R_{nm} are defined in an analogous way in terms of the $Q_{nmnm}(x, jl)$ and similarly we establish a connection between the averages of the quantities (13) and the R_{nm}. The bar over the quantities indicates an averaging over the impurities and the $\langle\ \rangle$ sign indicates averaging over the coordinates x.

In the limit of high concentrations of non-magnetic impurities ($\rho \gg 1$) calculations lead to the following results:

$$Q_{11} \approx -Q_{12} + N_1\xi_1, \quad \xi_i = \ln\left(\frac{2\gamma\beta\hbar\omega_i}{\pi}\right);$$

$$Q_{12} = \frac{N_1 N_2}{N_1 + N_2} \ln(2\gamma\rho), \quad \rho = \frac{\beta}{\pi}\frac{\hbar}{2}\left(\frac{1}{\tau_{12}} + \frac{1}{\tau_{21}}\right);$$

$$R_{nm} = R N_n N_m / (N_1 + N_2),$$

$$R = \frac{3\zeta(2)}{4}\left(\frac{\beta}{\pi}\right)^2 \frac{N_1 v_1^2 \nu_2 + N_2 v_2^2 \nu_1 + v_1 v_2 (N_1 \mu_{12} + N_2 \mu_{21})}{(N_1 + N_2)[\nu_1\nu_2 - \mu_{21}\mu_{12}]}$$

$$\times \quad \mathscr{B}(nm \mid n'm') = \frac{7\zeta(3)}{8}\left(\frac{\beta}{\pi}\right)^2 \frac{N_n N_m N_{n'} N_{m'}}{(N_1 + N_2)^3}, \tag{61}$$

where

$$\mu_{nm}=\frac{\beta}{\pi}\frac{\hbar}{2\theta_{nm}},\qquad \nu_i=\rho_{ii}-\mu_{ii}+\rho_i,$$

$$\rho_{ii}=\frac{\beta}{\pi}\frac{\hbar}{2\tau_{ii}},\quad \rho_1=\frac{\beta}{\pi}\frac{\hbar}{2\tau_{12}},\quad \rho_2=\frac{\beta}{\pi}\frac{\hbar}{2\tau_{21}}. \quad (62)$$

The relaxation times τ_{ij} have been determined before.[12] The quantities θ_{ij} differ from the τ_{ij} by the presence of cos θ under the integral sign, where θ is the angle between the vectors k_i and k_j. The coefficients Q_{nm} which we have not written out can be obtained from the coefficients given here by interchanging in them the indices 1 and 2.

In Eqs. (59) we use the equations

$$\Gamma_n=\Gamma+\vartheta\frac{N_2-N_1}{N_n}(-1)^n. \quad (63)$$

to change to new functions Γ and ϑ. We then get

$$\Gamma^*(x)=\sum_{nm}V_{mn}N_m\left[\frac{N_n}{N_1+N_2}+(-1)^n w\frac{N_1N_2}{N_2^2-N_1^2}\right]$$
$$\times\left[\xi_m{}^c+\ln\frac{\beta}{\beta_c}+\frac{R}{6}\left(\nabla+\frac{2ie}{c\hbar}A\right)^2-\frac{7\zeta(3)}{8}\left(\frac{\beta}{\pi}\right)^2|\Gamma(x)|^2\right]\Gamma^*(x), \quad (64)$$

$$\vartheta^*(x)=\frac{1}{1-w'}\frac{N_1N_2}{N_1^2-N_2^2}\sum_{nm}V_{mn}N_m(-1)^n$$
$$\times\left[\xi_m+\frac{R}{6}\left(\nabla+\frac{2ie}{c\hbar}A\right)^2-\frac{7\zeta(3)}{8}\left(\frac{\beta}{\pi}\right)^2|\Gamma(x)|^2\right]\Gamma^*(x), \quad (65)$$

$$j=\frac{ie}{3}R(N_1+N_2)\left[\Gamma(x)\left(\nabla+\frac{2ie}{c\hbar}A\right)\Gamma^*(x)-\text{c.c.}\right], \quad (66)$$

where

$$w=\frac{w''}{1-w'},\qquad w'=\frac{N_1N_2}{N_1+N_2}\sum_{nm}V_{mn}$$
$$\times\ (-1)^n(-1)^m\ln\left(\frac{2\omega_m}{1/\tau_{12}+1/\tau_{21}}\right),$$

$$w''=\frac{N_2-N_1}{N_2+N_1}\sum_{nm}V_{mn}(-1)^m N_n\ln\frac{2\omega_m}{(1/\tau_{12}+1/\tau_{21})},$$

$$\xi_m{}^c=\xi_m|_{T=T_c}.$$

Using the definition of the critical temperature T_c we get in this limiting case of high impurity concentration

$$1=\sum_{nm}V_{mn}N_m\left[\frac{N_n}{N_1+N_2}+(-1)^n w\frac{N_1N_2}{N_2^2-N_1^2}\right]\xi_m{}^c. \quad (67)$$

We change to dimensionless variables

$$x=x'\varkappa\left(\frac{6}{R}\ln\frac{\beta}{\beta_c}\right)^{1/2},\qquad A=A'\frac{c\hbar}{2e}\left(\frac{6}{R}\ln\frac{\beta}{\beta_c}\right)^{1/2},$$

$$\Gamma(x)=\frac{\pi}{\beta_c}\left(\frac{8}{7\zeta(3)}\right)^{1/2}\left(\ln\frac{\beta}{\beta_c}\right)^{1/2}\Phi(x'). \quad (68)$$

We obtain instead of Eqs. (64), (65) the following set:

$$\left(\frac{\nabla'}{\varkappa}+iA'\right)^2\Phi^*+(1-|\Phi|^2)\Phi^*=0,$$

$$\text{rot}'\,\text{rot}'\,A'=j'(x')=\frac{i}{2}\left(\Phi\frac{\nabla'}{\varkappa}\Phi^+-\Phi^+\frac{\nabla'}{\varkappa}\Phi\right)$$
$$-A'(x')|\Phi|^2, \quad (69)$$

where

$$\varkappa=\frac{3c}{8eR}\frac{\beta_c}{\pi}\left(\frac{7\zeta(3)}{\pi(N_1+N_2)}\right)^{1/2}. \quad (70)$$

In the case of a large concentration of non-magnetic impurities the Ginzburg-Landau equations of the two-band model reduce thus to Eqs. (69) of a single-band model, but the magnitude of the critical temperature T_c and the parameter κ depend in an essential way on the two-band character of the system.

We also give Eq. (14) in the limiting case of large impurity concentrations which we are considering:

$$\Omega_s-\Omega_n=-\frac{H_c^2}{8\pi}\int|\Phi(x)|^4dx, \quad (71)$$

where

$$\frac{H_c^2}{8\pi}=\frac{4}{7\zeta(3)}\left(\frac{\pi}{\beta_c}\right)^2(N_1+N_2)\left(\ln\frac{\beta}{\beta_c}\right)^2. \quad (72)$$

When there is a magnetic field present we must clearly add to expression (71) the energy of the field

$$\int\frac{H^2}{8\pi}dx=\frac{H_c^2}{4\pi}\int H'^2dx. \quad (73)$$

It is thus clear on the basis of the equations given in the foregoing that for high concentrations of non-magnetic impurities the thermal and electromagnetic properties of the two-band model of a superconductor is completely determined by the equations of the single-band model provided the quantities T_c and κ are appropriately defined by

Eqs. (69) and (70) and the density of states N is in the thermodynamic expressions replaced by $N_1 + N_2$.

The author expresses his deep gratitude to academician N. N. Bogolyubov for his interest in this paper and to L. Z. Kon and M. K. Kaplazhia for their assistance with the calculations.

[1] V. A. Moskalenko, FMM **8**, 503 (1959); Phys. Met. Metallogr. **8**, No. 4, 25 (1959).

[2] H. Suhl, B. T. Matthias, and L. R. Walker, Phys. Rev. Lett., **3**, 552 (1959).

[3] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[4] N. N. Bogolyubov, V. V. Tolmachev, and D. V. Shirkov, Novyĭ metod v teorii sverkhprovodimosti (New Method in the Theory of Superconductivity) AN SSSR, 1958; English translation published 1959 by Consultants Bureau, New York.

[5] D. R. Tilley, Proc. Phys. Soc. (London) **84**, 573 (1964).

[6] A. A. Abrikosov, JETP **32**, 1442 (1957), Soviet Phys. JETP **5**, 1174 (1957).

[7] W. H. Kleiner, L. M. Roth, and S. H. Autler, Phys. Rev. **133**, A1226 (1964).

[8] G. Lasher, Phys. Rev. **140**, A523 (1965).

[9] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fiziki (Quantum Field Theory Methods in Statistical Physics) Fizmatgiz, 1962; English translation published 1965 by Pergamon Press.

[10] L. P. Gor'kov, JETP **36**, 1918 (1959) and **37**, 1407 (1959), Soviet Phys. JETP **9**, 1364 (1959) and **10**, 998 (1960).

[11] V. L. Ginzburg and L. D. Landau, JETP **20**, 1064 (1950); English translation in "Collected Papers of L. D. Landau," Pergamon Press, 1965, p. 546.

[12] V. A. Moskalenko and M. E. Palistrant, JETP **49**, 770 (1965), Soviet Phys. JETP **22**, 536 (1966).

Translated by D. ter Haar
140