SOVIET PHYSICS JETP VOLUME 24, NUMBER 4 APRIL, 1967

PROPAGATION OF WAVE PACKETS AND SPACE-TIME SELF-FOCUSING IN A NONLINEAR MEDIUM

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Submitted to JETP editor April 23, 1966

J. Exptl. Theoret. Phys. (U.S.S.R.) 51, 1189–1194 (October, 1966)

Specific features of the propagation of modulated waves in a transparent nonlinear dispersive medium are investigated. In particular, it is shown that a monochromatic wave can be unstable with respect to traveling perturbations of its amplitude and frequency. In other cases, these perturbations undergo cumulative deformation instead of growing (two types of simple envelope waves are possible). Stationary modulated waves (periodic and single) were also found.

WAVE propagation in a dispersive medium with cubic nonlinearity gives rise to so-called self-action effects due to the dependence of the velocity of a wave upon its amplitude. The best currently known example is the self-focusing of stationary wave beams.[1]

The present paper deals with self-action of another kind manifested in the nonstationary case as nonlinear space-time deformation of the modulated-wave envelopes (wave packet).1) Typical of such processes is the relation between the variations of the amplitude and frequency envelopes caused by a simultaneous action of two factors: nonlinearity and dispersion.

A suitable method of analyzing such problems is coordinate and time averaging, which yields nonstationary equations for the amplitude and phase (frequency) wave envelopes. Equations for the one-dimensional case and their typical solutions are obtained below. Since nonstationary processes in a dispersive medium cannot be fully described by postulating a nonlinear permittivity for the harmonic field, we use for the nonlinear polarization a simple dynamic model that is valid for isotropic dielectrics (Eq. 1b). Incidentally, as expected from the derivation of averaged equations (terms with higher-than-second derivatives of slow variables were neglected), such equations are invariant for any type of time-dependent dispersion of the medium, preserving the qualitative character of the obtained results.

1. ENVELOPE EQUATIONS

The propagation of a plane wave in an isotropic weakly-nonlinear medium may be described without accounting for losses by the following equations for effective field E and polarization P:

$$c^2E_{zz} = E_{tt} + 4\pi P_{tt}, \tag{1a}$$

$$P_{tt} + \omega_0^2 P - \alpha P^3 = (\omega_p^2/4\pi)E, \tag{1b}$$

where ω_0, ω_p, and α are constants; the indices z and t denote differentiation.

A solution of (1) will be sought in the form of a traveling wave with slowly varying amplitude and phase:2)

$$E = A(z, t)\{\exp[i(\omega t - kz + \varphi(z, t))] + \exp[-i(\omega t - kz + \varphi(z, t))]\}. \tag{2}$$

Here, A and φ are real functions, and ω and k are constants related by the dispersion equation for a linear medium: $ck = \omega\sqrt{\epsilon}$, where $\epsilon = 1 + \omega_p^2/(\omega_0^2 - \omega^2)$.

Substituting (2) into (1b), we can find P to any degree of accuracy, obtaining the following averaged equations derived from (1a)3)

1)Examples of such nonstationary self-action were considered (apparently for the first time) by the author[2] and later by Whitham [3] (see also [4]). In particular, these papers fail to take dispersive blurring (the second derivatives of amplitude in (3)) into account; this must be considered in order to obtain the majority of effects discussed below.

2)The third harmonic generated in this process is usually small and can be found separately by the perturbation method. An exception is the case of weak dispersion ($\omega \to 0$) when the fundamental and third harmonic are synchronized and the field of the third harmonic can grow in a resonant manner.

3)The analysis is limited to the transparency regions; the resonant effects occurring near $\omega = \omega$, and strongly dependent on the losses in the medium are not considered.

$$A_t + vA_z = \frac{c^2 A_z \varphi_z - sA_t\varphi_t}{\omega} + \frac{A(c^2\varphi_{zz} - s\varphi_{tt})}{2\omega}$$

$$- 2g\left(\frac{3\omega_0^2 + \omega^2}{\omega_0^2 - \omega^2}\right)A^2 A_t, \quad (3a)$$

$$\varphi_t + v\varphi_z + \omega g A^2 = \frac{c^2\varphi_z^2 - s\varphi_t^2}{2\omega} - \frac{c^2 A_{zz} - sA_{tt}}{2A\omega}$$

$$- 2g\left(\frac{\omega_0^2 + 3\omega^2}{\omega_0^2 - \omega^2}\right)A^2\varphi_t, \quad (3b)$$

where $v = k_\omega^{-1}$ is the group velocity at the frequency ω.

$$s = 1 + \frac{\omega_0^2\omega_p^2(3\omega^2 + \omega_0^2)}{(\omega_0^2 - \omega^2)^3},$$

$$g = \frac{3\alpha\omega_p^6}{2(4\pi)^2(\omega_0^2 - \omega^2)^4} \quad (gA^2 \ll 1).$$

The right-hand sides of (3) contain second derivatives of slowly varying quantities and first derivatives of the nonlinear terms (typical of the nonstationary problem only). All are of second order of smallness, i.e., Eqs. (3) fall outside the limits of the geometrical optics approximation.

From now on it will be more convenient to consider the alternating component of the frequency rather than the phase correction. Differentiating (3b) with respect to t and denoting $\varphi_t/\omega = \delta \ll 1$, we obtain the following system in place of (3):

$$A_t + v(1 + \varkappa\delta + g\xi_1 A^2)A_z - {}^1\!/_2\varkappa A\delta_t = 0,$$

$$\delta_t + v(1 + \varkappa\delta - g\xi_2 A^2)\delta_z \quad (4a)$$

$$+ 2g[1 - c^2v^{-2}gA^2 + \xi_2\delta]AA_t$$

$$= -(\varkappa/2\omega^2)(A_{tt}/A)_t. \quad (4b)$$

Here, $\kappa = c^2/v^2 - s = -c\omega\sqrt{\epsilon}\, k_{\omega\omega}$, and

$$\xi_1 = 3\frac{c^2}{v^2} - 2\left(\frac{3\omega_0^2 + \omega^2}{\omega_0^2 - \omega^2}\right), \quad \xi_2 = \frac{c^2}{v^2} - 2\left(\frac{\omega_0^2 + 3\omega^2}{\omega_0^2 - \omega^2}\right).$$

We note that the sign of κ coincides with that of the difference $\omega^2 - \omega_0^2$.

2. NONSTATIONARY SOLUTIONS

A. We consider first the behavior of small deviations A' and δ' from the constant values of A and δ (we can obviously let $\delta = 0$ without loss of generality). Considering A' and δ' proportional to $\exp[i(\Omega t - Kz)]$, we get from (4) a dispersion equation of the type

$$vK = \Omega\{1 \pm |\varkappa|[-g'A^2 + (\Omega/\omega)^2]^{1/2}\}, \quad g' = g/\varkappa. \quad (5)$$

According to (5) small perturbations of the stationary plane wave produce different effects, depending upon the sign of g'. When $g' < 0$, the solution for the envelopes is a superposition of two undamped waves, fast and slow, whose velocities of propagation differ little from each other, $u(\Omega) = \Omega/K$. When $g' > 0$, the above statement remains valid only for sufficiently large Ω. If $\Omega^2 < \omega^2 g' A^2$, perturbations of the type defined in (5) can build up with an increment

$$K' = |\varkappa| v^{-1}\Omega[g'A^2 - (\Omega/\omega)^2]^{1/2}.$$

A maximum value of $K' = K'_m$ is reached at some frequency $\Omega = \Omega_m$; these values are defined as

$$\Omega_m = \omega(g'A^2/2)^{1/2}, \quad K_m' = \omega v^{-1}|g|A^2. \quad (6)$$

Thus when $g' > 0$, a monochromatic plane wave with a finite amplitude becomes unstable against perturbations of its amplitude and frequency at not too high a modulation frequency. The wave ultimately breaks up into sections with lengths of the order of $2\pi v/\Omega_m$, in which the wave energy is concentrated. The process represents space-time auto-modulation due to variation of both the amplitude and frequency. As shown in Sec. 3, stationary waves with a time scale of the order of $2\pi/\Omega_m$ are possible.

B. Equations (4) can be analyzed in greater detail if the variation of A and δ is sufficiently slow; in that case, third-derivative terms in the right-hand side of (4b) can be neglected. The remaining "abbreviated" second-order quasilinear system is hyperbolic when $g' < 0$, in which case it has two sets of characteristics. By setting $\delta = \delta(A)$ ($\delta(0) = 0$ to be specific), it is easy to find simple waves, i.e., particular solutions that are constant on the characteristics

$$A = F_\pm[t - z/u_\pm(A)], \quad \delta = \pm 2(|g'|A^2)^{1/2},$$

$$u_\pm = v[1 \pm 3|\varkappa|(|g'|A^2)^{1/2}]. \quad (7)$$

Here $F_\pm$ are arbitrary functions. Each of the waves (7) is deformed in the course of propagation until regions of ambiguity are produced, for which solutions (7) are no longer valid; at this point we must include the neglected term in (4b).

The particular feature of a simple wave is its exclusive property of having a common boundary with a region of constant A and δ.[5] This is no longer true, however, if $A = 0$ in this region and both velocities $u_\pm$ coincide (a case of "degeneracy" of the families of characteristics). On the other hand, when $A \to 0$ and $\delta \to 0$, the abbreviated system (4) is linear and one can readily find its general solution. A non-trivial case occurs when A and δ are small and $\delta \sim A^2$. The abbreviated system (4) is then linear with respect to the variables δ and A^2; its solution (valid for any sign of g') is

$$A^2 = F\left(t - \frac{z}{v}\right), \quad \delta = \frac{gz}{v}(A^2)_t + \Phi\left(t - \frac{z}{v}\right), \qquad (8)$$

where F and Φ are arbitrary functions. If when z = 0 only the amplitude is modulated, $\Phi \equiv 0$, and δ increases up to a value of the order of $(|g'|A^2)^{1/2}$, beyond which solution (7) is invalid.

Let us now consider two previously discussed cases,[2, 6] in which it is possible to integrate the abbreviated system (4). In a nondispersive nonlinear medium ($\kappa = 0$), the amplitude is deformed as a simple wave, and the frequency increases in some regions and decreases in others (it follows (7) in the initial stage).[2] In a linear medium ($\kappa = 0$ or $A \to 0$, $\delta \neq 0$) a simple wave is associated with frequency (δ) and the amplitude increases in the "compression" regions (such an effect is the basis of frequency-modulated radar-pulse compression[7]). The "modulation" approach is consequently useful also for the solution of linear problems.[6] In both cases the solutions are valid within limited intervals, beyond which one must consider terms with higher derivatives of A.

3. STATIONARY TRAVELING WAVES

One of the most interesting problems concerns the existence of solutions in the form of stationary "envelope waves" propagating without deformation. In fact, the complete system (4) has such solutions.4) Regarding them as functions of the variable $\eta = t - z/w$ (w = const), we find from (4a)

$$\varkappa\delta = \frac{N}{A^2} + \frac{w - v}{v} - \frac{1}{2}\xi_1 g A^2, \qquad (9)$$

where N is the integration constant.

Let us further consider a family of solutions that are finite for A = 0. For solutions of this type, N = 0 (an analysis of the general case presents no principal difficulties). Furthermore, one can as before consider without loss of generality that if $\delta = 0$ for A = 0, then w = v. Substituting (9) into (4b) and integrating once with respect to η, we obtain

$$A_{\eta\eta} + 2\omega^2 A(g'A^2 - Q) = 0 \qquad (10)$$

(Q is the constant of integration).

Equation (10) can be easily analyzed in the phase plane. The most interesting case is that of

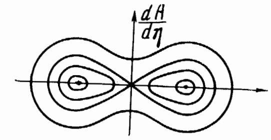

FIG. 1. Phase plane of equation (10) for g' > 0 and Q > 0.

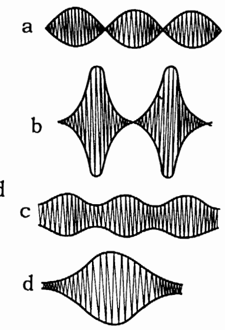

FIG. 2. Types of stationary modulated waves: (a) for g' < 0; (b, c, d) for g' > 0 and Q > 0 (the last three cases correspond to various trajectory types in Fig. 1).

$g' > 0$ and $Q > 0$; its phase plane is shown in Fig. 1. A stationary unit pulse (corresponding to a closed separatrix of a saddle at the origin of coordinates) is a solution in this case. The peak amplitude A_m of this pulse is $(2Q/g')^{1/2}$, and the characteristic pulse length T is of the order of $2\pi/\omega(g'A_m^2/2)^{1/2}$. These relations are also valid for periodic solutions. We note that the frequency $\Omega_m = 2\pi/\tau$ coincides with the frequency found above for the maximum instability of the plane wave; this gives us reason to assume that the plane-wave decay described here leads to our stationary solutions or to similar ones. Figure 2 illustrates the basic types of solutions in this and other cases.5)

Thus we have in the general case a four-parameter family of stationary solutions of the systems (4), including a two-parameter family of waves such as a single wave packet (defined, for example, by the amplitude and frequency at its maximum. The length of such a packet is inversely proportional to its amplitude, and its velocity is equal to the group velocity at the frequency corresponding to the "wings" of the pulse where the amplitude is low. The remaining solutions are periodic and their amplitude either passes through zero, or does not turn to zero at any point. The latter solutions are possible only when $g' > 0$ and $Q > 0$, and an unmodulated wave with an amplitude equal to $A_m/\sqrt{2}$ is a particular case of such a solution.

4)Stationary waves in an active medium[8] are of a different nature and can be considered in the approximation of geometric optics.

5)Similar solutions are also typical of instantaneous field values in a weakly dispersive medium without dissipation.[9] It is not clear if we can extend this analogy to shock waves [2,3]; the existence of stationary shock waves of envelopes apparently requires the presence of a fairly specific type of dissipation in the medium.

The realization of these effects in various ranges depends upon two quantities: gA^2 (a nonlinearity parameter of the order of the nonlinear increment of ϵ) and κ (the dispersion parameter). In radio-frequency transmission lines containing the usual nonlinear elements (ferrites, semiconductors, and ferroelectrics) these parameters are sufficiently diverse in magnitude and sign to allow for any of the above processes. These processes may be of significance also in the propagation of radio pulses in the ionosphere, where the cubic nonlinearity due to collective phenomena can be considerable and the distances traveled by an unattenuated wave are large.

A similar estimate may also be of interest in optics. The values of E currently achievable in unfocused laser beams exceed 10^5 V/cm. In that case, the Kerr-effect nonlinearity in liquids (such as benzene) gives $gA^2 \sim 10^{-6}$ (the sign of g differs in various materials). The dispersion parameter κ in optics is of the order of 10^{-2}. As a result, (6) yields $\Omega_m \sim 10^{-2}\omega$ and $K' \gtrsim 10^{-6}$ k, whence for $\omega \sim 2 \times 10^{15}$ rad/sec and $\lambda = (2\pi/k)\ 10^{-4}$ cm we obtain

$$\Omega_m \sim 2\pi\cdot 3\cdot 10^{12} \text{ rad/sec}, \quad K' \gtrsim 10^{-2} \text{ cm}^{-1}.$$

Consequently, perturbations of an intense light pulse reach substantial magnitudes over distances of the order of 1 m; the modulation frequency corresponds to a wavelength of about 0.1 mm. These figures point to the possibility of observing the effect of increasing modulation of light (it persists even in multiple reflections between the resonator mirrors). We also note that the active medium of the laser has reactive nonlinearity for all modes that are detuned from the center of the spectral line of the active substance; it is therefore possible that laser emission pulsates with a very high modulation frequency (even in comparison to the frequency difference between neighboring axial modes).

Analogous estimates indicate the possibility of a marked broadening of the intense-light-pulse spectrum corresponding to (8).

Another possible area of application of the results is for problems in nonlinear quantum theory dealing with particle-like solutions of wave equations (see [10], for example). The stationary pulses derived in this paper represent a class of such particle-like solutions.

In conclusion, let us consider the analogy between the above effects and stationary spatial self-focusing. Equations (3) have much in common with two-dimensional equations of stationary quasi-optics,[11] in which the second derivatives with respect to the transverse coordinate are replaced by derivatives with respect to the longitudinal coordinate and time. In particular, along with the above-described plane-wave instability, the plane wave may also be unstable against stationary perturbations with a transverse field inhomogeneity.[12] However, there are also differences. Thus nonlinear terms with amplitude derivatives (proportional to $\xi_{1,2}$), causing a large variety of effects, are typical of the nonstationary case. Furthermore, whereas in our notation the condition for spatial self-focusing is $g > 0$ (positive nonlinear part of ϵ), the auto-modulation condition is $g' = g/\kappa > 0$. These differences are due to the time-dependent nature of dispersion.[6)] In general, the instability revealed here can be regarded as a mechanism of a longitudinal space-time self-focusing.

The author is indebted to A. V. Gaponov for reviewing the results of this work.

[1] G. A. Askar'yan, JETP **42**, 1567 (1962), Soviet Phys. JETP **15**, 1088 (1962).

[2] L. A. Ostrovskiĭ, ZhTF **33**, 905 (1963), Soviet Phys. Tech. Phys. **8**, 679 (1964).

[3] G. B. Whitham, Proc. Roy. Soc. **283**, 238 (1965).

[4] N. G. Basov and V. S. Letokhov, DAN SSSR **167**, 77 (1966), Soviet Phys. Doklady **11**, 246 (1966).

[5] L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Mechanics of Continuous Media), Gostekhizdat, 1953.

[6] L. A. Ostrovskiĭ, Radiotekhnika i élektronika **10**, 1176 (1965).

[7] C. E. Cook, Proc. IRE **48**, 310 (1960).

[8] N. G. Basov and V. S. Letokhov, Optika i spektroskopiya **18**, 1042 (1965).

[9] Yu. F. Filippov, Izv. Vuzov, Radiofizika **8**, 292 (1965).

[10] L. G. Zastavenko, PMM **29**, 430 (1965).

[11] V. I. Talanov, JETP Letters **2**, 218 (1965), transl. p. 138.

[12] V. I. Bespalov and V. I. Talanov, JETP Letters **3**, 471 (1966), transl. p. 307.

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[6)] In this sense, the quasi-optical analog would be a medium with spatial dispersion of a special type.