"QUASILOCAL" HEISENBERG OPERATORS AND THE PROBLEM OF RENORMALIZABILITY IN QUANTUM FIELD THEORY

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By combining the dynamic formalism of the axiomatic approach of Bogolyubov, Medvedev, and Polivanov with the previously formulated Lagrangian formalism in the Heisenberg picture[6], explicit expressions are derived for the "current-like" operators Λ_ν and L_ν in terms of Heisenberg field operators. The operators Λ_ν and L_ν differ from each other, since they are related to the Lagrangian or the Hamiltonian of the theory, respectively. A class of theories is found for which the derived results coincide with those implied by the dynamical formalism of the axiomatic approach of Lehmann, Symanzik, and Zimmermann. This class turns out to be larger than the class of renormalizable theories (renormalizability being understood in terms of perturbation theory).

1. INTRODUCTION

FOR the past ten years the development of quantum field theory was essentially tied to the successes of the axiomatic approach which was based on two systems of fundamental axioms: those of Bogolyubov-Medvedev-Polivanov (BMP)[1] and those of Lehmann-Symanzik-Zimmermann (LSZ)[2]. The authors of these two systems have had in mind the development of a dynamical formalism which would obviate the well-known difficulties of perturbation theory[3]. The fundamental difference between the axiomatic approaches of BMP and LSZ consists in the formulation of the causality requirement, which has as a consequence a difference in the selection of the dynamical variables forming the object of the theory[1)].

Within the framework of the BMP axiomatics, the corresponding dynamical formalism has been developed in the papers of Medvedev and Polivanov[4] (cf. also[5]). This approach is based on the so-called "chronological" (time-ordered) representation for the S-matrix (cf. infra) and the current-like Heisenberg operators Λ_ν involved in this expression, which are the dynamical quantities that determine the theory. It was shown that, in distinction from the situation encountered in perturbation theory, in the theory under discussion the role of the higher-order "current-like" operators is just as important as the role of the current $j(x) \equiv \Lambda_1(x)$. At the same time it has become clear that the form of the operators Λ_ν is hard to determine even if one makes use of the Lagrangian formalism.

The connection of the dynamical formalism of the BMP-axiomatics with the apparatus of the Lagrangian and Hamiltonian formalisms was developed in[6]. The basis of this approach was the development of a clear distinction between the two types of time-ordered products: the Dyson product (T_D) and the Wick product (T_W), a fact which had already been stressed in[7]. In particular, Ref.[8] already contains, albeit implicitly, concrete expressions for the operators Λ_ν.

Recently, a new direction has been developed, in which the authors[9] start out from the LSZ axioms, but in view of the insufficiency of the causality requirements in this formalism, add to it a "dynamical principle." The subsequent development of these ideas becomes more and more a repetition of the results obtained long ago in[4,5] within the BMP framework. However, the inconsistency of these derivations, and the desire somehow to avoid the Bogolyubov causality requirement[1], led to some imprecisions.

It should be pointed out first that, as shown in[10] with the Zachariasen model[11] as an example, the local commutativity of Heisenberg field operators

[1)]Naturally, the two axiomatic approaches should be compared either on or off the energy shell. In talking about a dynamicel formalism based on the causality requirement we have in mind a formulation of the theory in one or the other of the two approaches, off the energy shell.

is a stronger condition than the causality requirement, since the first contains the latter and the condition of crossing symmetry. In theories which exhibit crossing symmetry it was established long ago (cf. e.g.[12]) that local commutativity (off the energy shell) is a consequence of the Bogolyubov causality condition[1].

Further, Refs.[9] contained a lot of confusions due to the fact that no clearcut distinction was made between the T_D product, which is more appropriate for the LSZ formalism, and the T_W product, which is typical of the BMP approach. This influenced, for instance, the definition of the Heisenberg vector field, the distinction between the Hamiltonian and the Lagrangian, the concrete expression for the operators Λ_ν, the relation of these operators to the Lagrangian, etc. Finally, these papers do not pay sufficient attention to the operators Λ_ν (especially to those of higher order), although these operators are introduced. The use of some "dynamical" principle instead of the Bogolyubov causality condition[1] leads to a restriction of the class of theories that are considered admissible.

The authors of[13] also make use of the framework of the BMP axiomatics, concentrating mainly on the development of the theory on the energy shell with minimal departures from this shell. In order to derive the required results these authors supplement the axioms of BMP with a principle of "minimal singularity," which requires a different formulation for each concrete theory, thus replacing the Lagrangian. Since they consider the determination of the operators Λ_ν as a very difficult problem, the authors of[13] usually tend to formulate the theory in such a manner as to eliminate the operators Λ_ν altogether.

All this forces us, in order to complete the synthesis of the dynamical formalism[4,5] of the BMP approach with the Lagrangian and Hamiltonian formalisms[6,8], to consider in more detail the problem of "quasilocal" Heisenberg operators. For the example of pseudoscalar meson theory we derive in Sec. 2 explicit expressions for the "current-like" operators Λ_ν. In Sec. 3 we consider a different set of "quasilocal" Heisenberg operators L_ν, which are more closely related with the Lagrangian formalism. Finally, in Sec. 4 we consider the problem of determining the class of theories in which the consequences of the axioms of BMP coincide with those of the axioms of LSZ. In the conclusion we consider the problem of renormalizability in quantum field theory. As a rule, we shall use without further reference the notations introduced in[4,6] and[6,8].

2. THE "CURRENT-LIKE" HEISENBERG OPERATORS

As was shown in[4], the coefficient functions of the S-matrix in the BMP formalism can have a "chronologic" (time-ordered) representation of the form

$$S=\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_1\ldots dx_n\,\langle 0|S^{(n)}(x_1,\ldots,x_n)$$
$$\times\,|0\rangle : \varphi_{in}(x_1)\ldots\varphi_{in}(x_n):, \tag{1}$$

where

$$S^{(n)}(x_1,\ \ldots\ x_n)=S^{+}\frac{\delta^n S}{\delta\varphi_{in}(x_1)\ldots\delta\varphi_{in}(x_n)}$$
$$=(-i)^n T_D(\Lambda_1(x_1)\ldots\Lambda_1(x_n))$$
$$+\sum_m\frac{(-i)^m}{m!}P(x_1,\ldots,x_{\nu_1}$$
$$\times\,|x_{\nu_1+1},\ldots,x_{\nu_1+\nu_2}|\ldots|x_{\nu_1+\ldots+\nu_{m-1}+1},\ldots,x_n)$$
$$\times\,T_D(\Lambda_{\nu_1}(x_1,\ldots,x_{\nu_1})\ldots\Lambda_{\nu_m}(x_{\nu_1+\ldots+\nu_{m-1}+1},\ldots,x_n))$$
$$-\,i\Lambda_n(x_1,\ldots,x_n). \tag{2}$$

Here $\Lambda_1(x)\equiv j(x)$; the summation is over all ν such that $2\le m\le n-1$, $\nu_1+\ldots+\nu_m=n$, $\nu_i\ge 1$ and p denotes the symmetrization operator as defined in[3].

The "current-like" operators Λ_ν in (2) are hermitian Heisenberg operators, which are quasi-local and symmetric in their explicit arguments[4]. These operators satisfy the equations of motion given in[4]. The solution of these equations obtained in[8] allowed to express the current-like operators in terms of the field operators in the interaction picture. In this section we shall use a different approach to obtain directly the expressions of the operators Λ_ν in terms of the Heisenberg field operators.

This is based on the fact that we have established in[4] a second expression for the functions $S^{(n)}(x_1,\ \ldots,\ x_n)$, namely

$$S^{(n)}(x_1,\ldots,x_n)=(-i)^n K_{x_1}\ldots K_{x_n}N_Q(\mathbf{A}(x_1)\ldots\mathbf{A}(x_n)), \tag{3}$$

where N_Q is the "quasinormal" product of Heisenberg field operators A(x)[4,6]. This allows one, in principle, to determine successively the Λ_ν of higher order if one knows the expression of the current j(x), at least, in terms of the field operators A(x)[6]. Indeed, by definition the N_Q-product of the fields A(x) is related to their "quasi-Wick" product (T_{QW}) by means of a Wick theorem with the contraction $(-i)D^c(x-y)$. It was also shown in[6] that

$$T_{QW}(\mathbf{A}(x_1)\ldots\mathbf{A}(x_n)) = T_D(\mathbf{A}(x_1)\ldots\mathbf{A}(x_n)). \quad (4)$$

at least in renormalizable theories. Therefore, in order to be able to compare the two expressions (2) and (3), one must only differentiate the expressions which arise in this manner. We start this procedure with the radiative operator $S^{(2)}(x, y)$;

$$S^{(2)}(x,y) = (-i)^2 T_D(\mathbf{j}(x)\mathbf{j}(y)) - i\Lambda_2(x,y)$$
$$= (-i)^2 K_x K_y N_Q(\mathbf{A}(x)\mathbf{A}(y))$$
$$= (-i)^2 K_x K_y T_{QW}(\mathbf{A}(x)\mathbf{A}(y)) + (-i)^3 K_x \delta(x-y). \quad (5)$$

It remains only to transform the term containing the T_{QW}-product, making use of (4) and differentiating the T_D-products of fields A(x), taking into account their definition and the equation of motion $K_x A(x) = j(x)$. Thus

$$K_x K_y T_{QW}(\mathbf{A}(x)\mathbf{A}(y)) = K_x K_y T_D(\mathbf{A}(x)\mathbf{A}(y))$$
$$= T_D(\mathbf{j}(x)\mathbf{j}(y)) - \delta(x^0-y^0)[\dot{\mathbf{A}}(x), \mathbf{j}(y)]$$
$$-\frac{\partial}{\partial x^0}\{\delta(x^0-y^0)[\mathbf{A}(x), \mathbf{j}(y)]\}$$
$$-K_x\{\delta(x^0-y^0)[\dot{\mathbf{A}}(y), \mathbf{A}(x)]\}$$
$$-K_x\frac{\partial}{\partial x^0}\{\delta(x^0-y^0)[\mathbf{A}(x), \mathbf{A}(y)]\}. \quad (6)$$

It should be stressed that in deriving (6) we have not yet made the theory concrete, except for condition (4), or more precisely, a similar condition obtained after K_{x_i} operates on all variables. For convenience we shall consider only the theory with the interaction Lagrangian:

$$\mathscr{L}_I^{in}(x) = gZ_1 : \varphi_{in}^4(x) : + \frac{Z_3-1}{2} : \varphi_{in}(x)$$
$$\times K_x\varphi_{in}(x) : - \frac{Z_3}{2}\,\delta m^2 : \varphi_{in}^2(x) :. \quad (7)$$

In the sequel we shall specify a whole class of theories for which one can carry out transformations of the type (6). As established in[6], an effective expression for the current j(x) in a theory with the Lagrangian $\mathscr{L}_I^{in}$ of the form (7) is[2)]:

$$\mathbf{j}(x) = -\frac{4gZ_1}{Z_3}N_Q(\mathbf{A}^3(x)) + \delta m^2 \mathbf{A}(x). \quad (8)$$

It is easy to see that the following relations hold in the theory under consideration[6,12]:

$$\delta(x^0-y^0)[\mathbf{A}(x), \mathbf{A}(y)] = \delta(x^0-y^0)[\mathbf{A}(x), \mathbf{j}(y)] = 0, \quad (9)$$

$$\delta(x^0-y^0)[\dot{\mathbf{A}}(y), \mathbf{A}(x)] = -\frac{i}{Z_3}\delta(x-y), \quad (10)$$

$$\delta(x^0-y^0)[\dot{\mathbf{A}}(x), \mathbf{j}(y)] = i\frac{12gZ_1}{Z_3^2}$$
$$\times N_Q(\mathbf{A}^2(x))\delta(x-y) - i\frac{\delta m^2}{Z_3}\delta(x-y). \quad (11)$$

We shall analyze Eqs. (9)—(11) in more detail in Sec. 4 below. Substituting now (9)—(11) into (6) and (6) into (5), we obtain

$$\Lambda_2(x,y) = -\frac{12gZ}{Z_3^2}N_Q(\mathbf{A}^2(x))\delta(x-y)$$
$$+\left(\frac{1}{Z_3}-1\right)K_x\delta(x-y) + \frac{\delta m^2}{Z_3}\delta(x-y). \quad (12)$$

The corresponding calculation for the radiative operator $S^{(3)}(x, y, z)$ is much more involved:

$$S^{(3)}(x,y,z) = iT_D(\mathbf{j}(x)\mathbf{j}(y)\mathbf{j}(z))$$
$$- T_D(\mathbf{j}(x)\Lambda_2(y,x)) - T_D(\mathbf{j}(y)\Lambda_2(x,z))$$
$$- T_D(\mathbf{j}(z)\Lambda_2(x,y)) - i\Lambda_3(x,y,z)$$
$$= iK_xK_yK_zN_Q(\mathbf{A}(x)\mathbf{A}(y)\mathbf{A}(z))$$
$$= iK_xK_yK_zT_D(\mathbf{A}(x)\mathbf{A}(y)\mathbf{A}(z))$$
$$+ \mathbf{j}(x)K_y\delta(y-z) + \mathbf{j}(y)K_x\delta(x-z)$$
$$+ \mathbf{j}(z)K_x\delta(x-y). \quad (13)$$

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2)The remainder of our discussion could be constructed starting directly from the definition of j(x), according to which [6]

$$\mathbf{j}(x) = -4gZ_1N_Q(\mathbf{A}^3(x)) + \frac{1-Z_3}{2}\mathbf{A}(x)K_x$$
$$+\frac{1-Z_3}{2}K_x\mathbf{A}(x) + Z_3\delta m^2\mathbf{A}(x).$$

However this would lead to considerable complications, leading in the end to the same result. Therefore we shall make use of the effective expression for j(x) of the form (8) which differs only insignificantly from the definition of j(x) [6].

It is not very convenient to differentiate the T_D-products of the fields A(x). It is therefore more convenient to use R_D-products[2] in the intermediate stages of the calculation. We have thus, for instance

$$T_D(\mathbf{A}(x)\mathbf{A}(y)) = R_D(\mathbf{A}(x); \mathbf{A}(y)) + \mathbf{A}(y)\mathbf{A}(x), \qquad (14)$$

$$T_D(\mathbf{A}(x)\mathbf{A}(y)\mathbf{A}(z)) = R_D(\mathbf{A}(x); \mathbf{A}(y)\mathbf{A}(z)) + \mathbf{A}(y)R_D(\mathbf{A}(x); \mathbf{A}(z)) + \mathbf{A}(z)R_D(\mathbf{A}(x); \mathbf{A}(y)) + R_D(\mathbf{A}(y); \mathbf{A}(z))\mathbf{A}(x) + \mathbf{A}(z)\mathbf{A}(y)\mathbf{A}(x). \qquad (15)$$

Differentiating in (13) with the use of (14), (15), and (9) we obtain

$$\begin{aligned} K_xK_yK_zT_D(\mathbf{A}(x)\mathbf{A}(y)\mathbf{A}(z)) &= T_D(\mathbf{j}(x)\mathbf{j}(y)\mathbf{j}(z)) - \\ &- \delta(x^0-y^0)\,T_D\{[\dot{\mathbf{A}}(x), \mathbf{j}(y)]\,\mathbf{j}(z)\} \\ &- \delta(x^0-z^0)\,T_D\{[\dot{\mathbf{A}}(x), \mathbf{j}(z)]\,\mathbf{j}(y)\} \\ &- \delta(y^0-z^0)\,T_D\{[\mathbf{A}(y), \mathbf{j}(z)]\,\mathbf{j}(x)\} \\ &- \mathbf{j}(y)K_x\{\delta(x^0-z^0)\,[\dot{\mathbf{A}}(z), \mathbf{A}(x)]\} \\ &- \mathbf{j}(z)K_x\{\delta(x^0-y^0)\,[\dot{\mathbf{A}}(y), \mathbf{A}(x)]\} \\ &- K_y\{\delta(y^0-z^0)\,[\dot{\mathbf{A}}(z), \mathbf{A}(y)]\}\,\mathbf{j}(x) \\ &+ \delta(x^0-z^0)\,\delta(y^0-z^0)\,[\dot{\mathbf{A}}(x), [\dot{\mathbf{A}}(y), \mathbf{j}(z)]]. \end{aligned} \qquad (16)$$

Substituting (16) into (13) and making use of (10) and (11) we obtain

$$\begin{aligned} \Lambda_3(x, y, z) &= i^2[\dot{\mathbf{A}}(x), [\dot{\mathbf{A}}(y), \mathbf{j}(z)]]\,\delta(x^0-z^0)\,\delta(y^0-z^0) \\ &= i[\dot{\mathbf{A}}(x), \Lambda_2(y, z)]\,\delta(x^0-z^0) \\ &- \frac{24gZ_1}{Z_3^3}\mathbf{A}(x)\,\delta(x-y)\,\delta(y-z). \end{aligned} \qquad (17)$$

A similar but even more involved calculation yields

$$\Lambda_4(x, y, z, u) = -\frac{24gZ_1}{Z_3^4}\delta(x-y)\,\delta(y-z)\,\delta(z-u). \quad (18)$$

Making use of the formalism developed in[6] for the expression of operators in the Heisenberg picture in terms of those in the interaction picture, it is easy to see (12), (17), and (18) imply for the interaction picture operators $\Lambda_\nu^{\mathrm{int}}$, defined by

$$\Lambda_\nu(x_1, \ldots, x_\nu) = S^+(x_i^0, -\infty)\,\Lambda_\nu^{\mathrm{int}}(x_1, \ldots, x_\nu)\,S(x_i^0, -\infty), \qquad (19)$$

expressions which coincide with those obtained in[8] by solving the equations of motion[4] for the operators Λ_ν. We see again that in the formalism introduced in[6] there is a correspondence between the Lagrangian formalism and the dynamical formalism of the BMP axiomatic approach[4,5].

We remark on the following circumstances. The Hamiltonian form of the equations of motion[6] in the theory under consideration implies that

$$\mathbf{j}(x) = \dot{\Lambda}_1(x) = i[\dot{\mathbf{A}}(x), \Lambda_0], \quad \Lambda_0 = \int d\mathbf{y}\,\mathbf{H}_I(y). \quad (20)$$

Joining this formula with (12), (17), and (18) we obtain, in general (taking into account the rules for variations with respect to A(x)[6])

$$\begin{aligned} \Lambda_n(x_1, \ldots, x_n) &= i\,[\dot{\mathbf{A}}(x_1), \Lambda_{n-1}(x_2, \ldots, x_n)] \\ &= i^n\,[\dot{\mathbf{A}}(x_1), [\dot{\mathbf{A}}(x_2), [\ldots, \Lambda_0]\ldots]|_{x_1^0=\ldots=x_n^0} \\ &= i^n\frac{1}{Z_3^n}\frac{\delta^n\Lambda_0}{\delta\mathbf{A}(x_1)\ldots\delta\mathbf{A}(x_n)} \\ &= \frac{i^n}{Z_3^n}[\pi_{\mathbf{A}}(x_1), [\pi_{\mathbf{A}}(x_2), [\ldots, \Lambda_0]\ldots]|_{x_1^0=\ldots=x_n^0}, \end{aligned} \qquad (21)$$

where[6] $\pi_{\mathbf{A}}(x) = Z_3\dot{\mathbf{A}}(x)$. Only the term with $\Lambda_2(x, y)$ disappears from these equations, due to the fact that it contains no operators. Thus (21) can be considered as an alternate form of the equation of motion for the operators Λ_ν. Obviously the higher-order operators Λ_ν (starting with $\nu = 5$) vanish in the theory considered here, confirming the conclusion reached in[4].

3. THE "CURRENT-LIKE" HEISENBERG OPERATORS L_ν

We have thus obtained explicit expressions for the operators Λ_ν which occur in the representation (2). It was necessary to make use of a round-about method of computation, since a direct calculation is inhibited by the fact that the in-transforms[4,6,8] of the operators Λ_ν are unknown. This circumstance is related to the fact that the operators Λ_ν occurring in the T_D-products in (2) are related to the Hamiltonian, rather than to the Lagrangian of the theory, as was shown in[8].

In order to illustrate the essence of this problem we make use of yet another representation of the S-matrix, which was also introduced in[4]. This representation is in the most natural manner related to the Lagrangian formalism and has the form (2), but with all T_D-products replaced by T_{QW}-products and with the operators Λ_ν replaced by another set of quasilocal operators $\mathbf{L}_\nu$. For example

$$S^{(2)}(x, y) = (-i)^2T_{QW}(\mathbf{j}(x)\mathbf{j}(y)) - i\mathbf{L}_2(x, y). \quad (22)$$

Here[4]

$$\mathbf{L}_n(x_1, \ldots, x_n) = S^+T_W(\mathscr{L}_n^{in}(x_1, \ldots, x_n)S), \quad (23)$$

with

$$\mathscr{L}_n{}^{in}(x_1,\ldots,x_n) = -\frac{\delta^n \mathscr{L}_0{}^{in}}{\delta\varphi_{in}(x_1)\ldots\delta\varphi_{in}(x_n)},$$

$$\mathscr{L}_0{}^{in} = \int dy\, \mathscr{L}_I{}^{in}(y). \qquad (24)$$

Simple calculations in a theory defined by the Lagrangian $\mathscr{L}_I^{in}$ of Eq. (7) yield

$$L_2(x,y) = -12gZ_1N_Q(\mathbf{A}^2(x))\delta(x-y)$$
$$-(Z_3-1)K_x\delta(x-y) + Z_3\delta m^2\delta(x-y), \qquad (25)$$

$$\mathbf{L}_3(x,y,z) = -24gZ_1\mathbf{A}(x)\delta(x-y)\delta(y-z), \qquad (26)$$

$$\mathbf{L}_4(x,y,z,u) = -24gZ_1\delta(x-y)\delta(y-z)\delta(z-u). \qquad (27)$$

It is easy to see that these expressions differ from the corresponding expressions for the Λ_n only in their coefficients. By definition[4] only the expressions for the currents agree completely, i.e.,

$$\mathbf{j}(x) \equiv \Lambda_1(x) \equiv \mathbf{L}_1(x). \qquad (28)$$

Finally, in the theory under consideration one can write for $\mathbf{L}_n$ equations of motion of the form

$$\delta\mathbf{L}_n(x_1,\ldots,x_n)\,/\,\delta\mathbf{A}(y) = \mathbf{L}_{n+1}(y,x_1,\ldots,x_n). \qquad (29)$$

Thus, although the representation (2) is the most natural one for the dynamical formalism of the BMP approach, the in-transforms of the corresponding operators Λ_ν cannot be obtained by taking variations of $\int dy \mathscr{L}_I^{in}(y)$ (this is contrary to the assertion in[9]; it was shown in[8] that these operators cannot be obtained by variation of $\int dy H_I^{int}(y)$, either). This is possible however for the operators $\mathbf{L}_\nu$ which are directly related to the Lagrangian formalism, but do not coincide with the Λ_ν. This does not mean, as asserted in[13] that the operators Λ_ν are completely unknown objects. Their operator structure (at least in renormalizable theories) is essentially determined if $\mathscr{L}_I^{in}(y)$ is known, and the exact coefficients can be determined either by means of the methods of Sec. 2, or as was done in[8].

Since the operators Λ_ν and $\mathbf{L}_\nu$ are different, the T_{QW}-products of the currents $\mathbf{j}(x)$ will not coincide with the corresponding T_D-products. In particular, comparing (5) and (22), one obtains that in a theory with $\mathscr{L}_I^{in}(x)$ of the form (7):

$$T_{QW}(\mathbf{j}(x)\mathbf{j}(y)) = T_D(\mathbf{j}(x)\mathbf{j}(y))$$
$$+ i\{-12gZ_1(1/Z_3^2-1)N_Q(\mathbf{A}^2(x))\delta(x-y)$$
$$+ Z_3^{-1}(1-Z_3)^2K_x\delta(x-y)\}. \qquad (30)$$

In other words, the difference between the T_{QW}-products and the T_D-products of the currents $\mathbf{j}(x)$ vanishes only for $Z_3 = 1$ (i.e., in a theory without self-energy counterterms), whereas for the fields $\mathbf{A}(x)$ the difference vanishes even for $Z_3 \neq 1$ (cf. (4)). If one compares (22) with (3) (for n = 2) it is easy to see that the difference between $T_{QW}(\mathbf{j}(x)\mathbf{j}(y))$ and $K_xK_yT_{QW}(\mathbf{A}(x)\mathbf{A}(y))$ is non-zero even for $Z_3 = Z_1 = 1$, but it coincides with the difference between $T_D(\mathbf{j}(x)\mathbf{j}(y))$ and $K_xK_yT_D(\mathbf{A}(x)\mathbf{A}(y))$, i.e., the following relation is always true:

$$K_xK_yT_{QW}(\mathbf{A}(x)\mathbf{A}(y)) \equiv K_xK_yS^+T_W(\varphi_{in}(x)\varphi_{in}(y)S)$$
$$= S^+T_W(K_x\varphi_{in}(x)K_y\varphi_{in}(y)S) \neq T_{QW}(\mathbf{j}(x)\mathbf{j}(y))$$
$$\equiv S^+T_W(j^{in}(x)j^{in}(y)S) \qquad (31)$$

4. A COMPARISON OF THE DYNAMICAL FORMALISMS IN THE AXIOMATIC APPROACHES OF BMP AND LSZ

In the preceding sections we have established concrete expressions for the operators Λ_ν and $\mathbf{L}_\nu$ in a theory with $\mathscr{L}_I^{in}(x)$ of the form (7). Here we shall determine a whole class of theories in which transformations of the type (6) are valid.

We start from the definition of a field $\mathbf{A}(x)$ of the form[13,6,4]

$$\mathbf{A}(x) = S^+T_W(\varphi_{in}(x)\ S) = \varphi_{in}(x) - \int D^{ret}(x-y)\mathbf{j}(y)dy. \qquad (32)$$

It was shown in[12] that the commutator of such fields can be expressed in the form

$$[\mathbf{A}(x),\mathbf{A}(y)] = [\varphi_{in}(x),\varphi_{in}(y)]$$
$$-i\int D^{ret}(y-z)\frac{\delta\mathbf{j}(z)}{\delta\varphi_{in}(u)}D^{ret}(u-x)\,du\,dz$$
$$+i\int D^{adv}(y-z)\frac{\delta\mathbf{j}(u)}{\delta\varphi_{in}(z)}D^{adv}(u-x)\,du\,dz, \qquad (33)$$

and vanishes for $x \sim y$, due to the Bogolyubov causality condition[1,12]. By differentiation we obtain the other commutators occurring in (6):

$$[\dot{\mathbf{A}}(x),\mathbf{A}(y)] = [\dot{\varphi}_{in}(x),\varphi_{in}(y)] - i\int D^{ret}(y-z)\frac{\delta\mathbf{j}(z)}{\delta\varphi_{in}(u)}$$
$$\times\theta(u^0-x^0)\frac{\partial D(u-x)}{\partial x^0}\,du\,dz - i\int D^{adv}(y-z)\frac{\delta\mathbf{j}(u)}{\delta\varphi_{in}(z)}$$
$$\times\ \theta(x^0-u^0)\frac{\partial D(u-x)}{\partial x^0}\,du\,dz, \qquad (34)$$

$$[\mathbf{A}(x),\mathbf{j}(y)] = i\int\frac{\delta\mathbf{j}(y)}{\delta\varphi_{in}(u)}D^{ret}(u-x)\,du$$
$$-i\int\frac{\delta\mathbf{j}(u)}{\delta\varphi_{in}(y)}D^{adv}(u-x)\,du, \qquad (35)$$

$$[\dot{A}(x), j(y)] = i\int \frac{\delta j(y)}{\delta\varphi_{in}(u)}\theta(u^0-x^0)\frac{\partial D(u-x)}{\partial x^0}du + i\int\frac{\delta j(y)}{\delta\varphi_{in}(y)}\theta(x^0-u^0)\frac{\partial D(u-x)}{\partial x^0}du. \quad (36)$$

The last two expressions can also be obtained by means of direct commutation of A(x) with j(y) and of $\dot{A}(x)$ with j(y) if one takes into account (32). If one lets K_x and K_y operate simultaneously on (33), one obtains the well-known integrability condition[1,4].

We now take into account the fact that as a consequence of the equation of motion[4,5] we have for $\Lambda_2(z, u)$

$$\frac{\delta j(z)}{\delta\varphi_{in}(u)} = i\theta(z^0-u^0)[j(u), j(z)] + \Lambda_2(z, u). \quad (37)$$

Therefore if one substitutes Eq. (37) into (33)—(36) (and similarly for $\delta j(u)/\delta\varphi_{in}(z)$) it is easy to prove that the commutators of interest in (6) do not contain contributions from the terms containing θ-functions for $x^0 = y^0$. Therefore it follows from (33)—(36) that

$$\delta(x^0-y^0)[A(x), A(y)] - i\delta(x^0-y^0)\int du dz D(y-z)D(x-u) \times [\theta(z^0-y^0)-\theta(u^0-x^0)]\Lambda_2(u, z), \quad (38)$$

$$\delta(x^0-y^0)[\dot{A}(y), \dot{A}(x)] = -i\delta(x-y) - i\delta(x^0-y^0)\int du dz D(y-z)D(x-u)[\theta(z^0-y^0) - \theta(u^0-x^0)]\Lambda_2(u, z), \quad (39)$$

$$\delta(x^0-y^0)[A(x), j(y)] = -i\delta(x^0-y^0)\int D(x-u)\Lambda_2(u, y)du, \quad (40)$$

$$\delta(x^0-y^0)[\dot{A}(x), j(y)] = -i\delta(x^0-y^0)\int \dot{D}(x-u)\Lambda_2(u, y)du. \quad (41)$$

We note that the locality condition for A(x) at $x \equiv y$ implies

$$\int du dz D(x-u)D(x-z)[\theta(z^0-x^0) - \theta(u^0-x^0)]\Lambda_2(u, z) = 0, \quad (42)$$

which is of course valid for any theory, simply because of the symmetry of $\Lambda_2(u, z)$ in its arguments[4]. We further recall that the causality condition for a trivial "halved" S-matrix of the form[4,8]

$$i\delta S(x^0, -\infty)/\delta\varphi_{in}(y) = \theta(x^0-y^0)j(y) \quad (43)$$

is given by the requirement

$$[\theta(z^0-x^0) - \theta(u^0-x^0)]\Lambda_2(u, z) = 0. \quad (44)$$

It follows simply from (39) that the equal-time commutation relations for A(x) and $\dot{A}(x)$ are the same as the free-field relations. In a nontrivial theory with counterterms, where this is not true[6,8], Eq. (44) cannot hold, which invalidates Eq. (43)[8]. We finally call attention to the fact that (38) does not vanish identically, which does not contradict the locality of A(x) for $x \sim y$, since the condition $x^0 = y^0$ encompasses not only spacelike intervals, but also the vertex of the light cone.

Substituting (38)—(41) into the general expression of $\Lambda_2(x, y)$ obtained as a result of the substitution of (6) into (5), we have

$$\Lambda_2(x, y) = \delta(x^0-y^0)\int \dot{D}(x-u)\Lambda_2(u, y)du + \frac{\partial}{\partial x^0}\left\{\delta(x^0-y^0)\int D(x-u)\Lambda_2(u, y)du\right\} + K_x\left\{\delta(x^0-y^0)\int du dz D(y-z)\dot{D}(x-u) \times [\theta(z^0-y^0)-\theta(u^0-x^0)]\Lambda_2(u, z)\right\} + K_x\frac{\partial}{\partial x^0}\left\{\delta(x^0-y^0)\int du dz D(y-z)D(x-u) \times [\theta(z^0-y^0)-\theta(u^0-x^0)]\Lambda_2(u, z)\right\}. \quad (45)$$

We thus obtain a condition imposed on the operator $\Lambda_2(x, y)$. Since the derivation of this condition assumed Eq. (4) (taking into account the action of K_xK_y) which is the condition that the dynamical formalisms of the BMP and LSZ approaches coincide[4], it follows that for all theories in which the radiative operators $S^{(n)}(x_1, \ldots, x_n)$ coincide in the two approaches, that Eq. (45) must turn into an identity.

Thus, in order to solve the proposed problem it remains to establish in which theories (45) becomes an identity. For this we use the following general representation of Λ_2:

$$\Lambda_2(x, y) = \sum_{i=0}^{\varkappa}\left[\frac{\partial^i}{\partial x_0{}^i}\delta(x_0-y_0)\right]\tilde{\Lambda}_2{}^i(\mathbf{x}-\mathbf{y}; x_0), \quad (46)$$

where $\tilde{\Lambda}_2^i$ is an operator which is quasilocal in the space coordinates. In the sequel we shall always select in the sum in (46) the term involving the highest-order derivative, since if the identity holds for this term, it will certainly hold for the other terms also. Finally, without loss of generality, we can assume that this highest-order term is of the form $\tilde{\Lambda}_2^{\kappa}(\mathbf{x}-\mathbf{y}; x^0 = C\delta(\mathbf{x}-\mathbf{y})$. If this were not so, the reduction of the derivatives from $\delta(x^0 - y^0)$ would transfer some of them onto $\tilde{\Lambda}_2^{\kappa}$ and thus the number of derivatives which has the possibility of being transferred onto the other factors of (45) would be diminished. But this corresponds to lower-order terms in the sum (46). In other words, it is sufficient to carry through the discussion for the (appropriately symmetrized) expression

$$\Lambda_2'(x, y) = C\frac{\partial^{\varkappa}}{\partial x_0^{\varkappa}}\delta(x-y). \qquad (47)$$

Obviously, the results we have derived will be valid even if the constant C in (47) is replaced by an arbitrary operator which does not contain derivatives with respect to the time. We should not be disturbed by the noncovariance of this expression, since the separation into the T_D-product of currents and Λ_ν is in general noncovariant, and derivatives with respect to time are essential for our discussion.

Substituting Eq. (47) into each of the terms of the right-hand side of (45) we can verify that the first term contributes for each even κ, the second term contributes for each odd κ, the third term starts contributing at $\kappa = 2$ for even κ, and the fourth term starts contributing for odd κ from $\kappa = 3$. In order to find the values of κ for which (45) becomes an identity, we consider in more detail the cases $\kappa = 2$ and $\kappa = 4$, to which the first and third terms in the right-hand side of (45) contribute, respectively.

For $\kappa = 2$ Eq. (45) has the form

$$C\frac{\partial^2}{\partial x_0^2}\delta(x-y) = C\delta(x^0-y^0)\dddot{D}(x-y) - CK_x\Big\{\delta(x^0-y^0) \times \int du\,dz\,D(y-z)\dot{D}(x-u)\delta(u^0-x^0)\left[\frac{\partial}{\partial z^0}\delta(z-u)\right]\Big\}$$
$$= C\left(\frac{\partial^2}{\partial x_0^2} + K_x\right)\delta(x-y) - CK_x\delta(x-y), \qquad (48)$$

i.e., becomes an identity.

For $\kappa = 4$ we have similarly for the right-hand side of (45)

$$C\delta(x^0-y^0)\frac{\partial^5 D(x-y)}{\partial x_0^5} - CK_x\Big\{\delta(x^0-y^0)\int du\,dz\,D(y-z) \times \dot{D}(x-u)\delta(u^0-x^0)\left[\frac{\partial^3}{\partial z_0^3}\delta(z-u)\right]\Big\}$$
$$- CK_x\Big\{\delta(x^0-y^0)\int du\,dz\,D(y-z)\dot{D}(x-u) \times \frac{\partial^2}{\partial u_0^2}\left[\delta(u^0-x^0)\frac{\partial}{\partial z^0}\delta(z-u)\right]\Big\}$$
$$= C\left(\frac{\partial^2}{\partial x_0^2} + K_x\right)^2\delta(x-y) - 2CK_x\left(\frac{\partial^2}{\partial x_0^2} + K_x\right)\delta(x-y) = C\left(\frac{\partial^4}{\partial x_0^4} - K_xK_x\right)\delta(x-y), \qquad (49)$$

which is obviously different from $C\partial^4[\delta(x-y)]/\partial x_0^4$, i.e., it is not an identity. It is easy to see that for higher κ this happens because of the fact that the first term in (45) leads to powers of $(\partial^2/\partial x_0^2 + K_x)^{\kappa/2}\delta(x-y)$, which should cancel with parts of the third term, leaving only derivatives with respect to the time. In reality (except for $\kappa = 2$) this does not happen, since the third term contributes an expression of the form

$$\frac{\varkappa}{2}K_x\left(\frac{\partial^2}{\partial x_0^2} + K_x\right)^{(\varkappa-2)/2}\delta(x-y).$$

The cases $\kappa = 0$ and $\kappa = 1$ are trivial, and in the case $\kappa = 3$ part of the second term of the right-hand side of (45) (the part of the form $C\partial[\partial^2/\partial x_0^2 + K_x)\delta(x-y)]/\partial x^0$) cancels exactly with the fourth term $C\partial[K_x\delta(x-y)]/\partial x^0$, leading also to an identity. For $\kappa = 5$ etc. there is no identity for the same reasons as for the case $\kappa = 4$. The same expressions appear as in the neighboring terms with even κ, multiplied by a common first derivative with respect to x^0.

5. DISCUSSION OF THE RESULTS

We recall that in[4] the following condition was obtained for the validity of Eq. (4) (for n = 2):

$$\int dz\,du\,D(x-u)D(y-z)\Lambda_2(u,z)\{\theta(u^0-x^0)\theta(z^0-y^0) - \theta(u^0-x^0)\theta(x^0-y^0) - \theta(z^0-y^0)\theta(y^0-x^0)\} = 0 \qquad (50)$$

and similar conditions for higher Λ_n. It can be shown that, as expected, under the action of the operators K_xK_y this expression coincides with (45), rewritten as a condition on Λ_2.

Thus, if the problem is investigated by means of transforming Eq. (45) into an identity we have indeed specified the class of theories for which the dynamical formalisms of the approaches of BMP and LSZ coincide. It is interesting to note that the class of theories we have derived coincides with the class of theories[2,9] for which the LSZ axioms are at all valid. In other words, the BMP axioms allow, in principle, to treat a wider class of theories than the LSZ axioms.

Which are the theories which fall into the class we have determined? First of all (if we restrict our attention to scalar fields), this class contains all theories which are usually denoted as renormalizable, in particular, theories involving one first derivative (scalar electrodynamics), or the theory with the Lagrangian of the form (7). In addition, this case contains theories with

$$\mathcal{L}_I{}^{in}(x) = \lambda\colon \left(\frac{\partial \varphi_{in}}{\partial x^\nu}\right)^2 \left(\frac{\partial \varphi_{in}}{\partial x^\mu}\right)^2 \colon$$

or with

$$\mathcal{L}_I{}^{in}(x) = \lambda\colon \varphi_{in}(x) K_x \varphi_{in}(x) \left(\frac{\partial \varphi_{in}}{\partial x^\nu}\right)^2 \colon,$$

which are nonrenormalizable according to perturbation theory, since they involve counterterms with an increasing number of derivatives and field operators.

As is well-known[4], the maximal admissible number of field operators in the counterterms of the BMP approach is four. On the other hand, there are no explicit restrictions imposed on the number of derivatives involved. The restriction on the number of derivatives obtained above could be tentatively called the renormalizability condition in the axiomatic method (cf.[9]). Naturally, such a designation could be justified only by solvability of the corresponding axiomatic equations by means of nonperturbative methods (cf.[9]). At the same time it follows from[6,4] that even the imposition of such additional conditions as the hermiticity of the Lagrangian $\mathcal{L}_I^{in}(x)$ (with due account of the counterterms) considerably decreases the number of admissible derivatives. Thus, should it ever turn out that outside the framework of perturbation theory the class of theories obtained above is renormalizable, it could be that from the point of view of the Lagrangian formalism such theories should be described by non-hermitian Lagrangians $\mathcal{L}_I^{in}(x)$.

The preceding discussion of the quasilocal operators Λ_ν and L_ν and the fact that explicit expressions have been derived for these operators, makes it possible to turn seriously to the development of computational approaches to quantum field theory directly in the Heisenberg picture (cf. in particular[13]). This is important now in connection with the successes of a new direction in elementary particle theory—the so-called current algebra[14]. For a consequent development of this approach it is undoubtedly necessary to have at one's disposal correct expressions both for the current operators j(x) and for the "current-like" operators Λ_ν and L_ν in terms of the Heisenberg fields. Methods for obtaining such expressions have been formulated in[8] and in the present paper.

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