VIRTUAL SCATTERING EFFECTS IN DIRECT NUCLEAR REACTIONS

V. A. KAMINSKIĬ, Yu. V. ORLOV and I. S. SHAPIRO

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Formulas for the cross sections for direct processes in which the initial and final state interactions are taken into account via the elastic scattering phase shifts in the physical region, are derived in detail. The derivation is based on dispersion theory. A method for taking simultaneous account of the nuclear and Coulomb interactions of charged particles is considered within the framework of this theory. The analytic structure of the reaction amplitude is investigated in a potential model for which dispersion relations can be written down. The contribution of nonphysical singularities of the nuclear scattering amplitude is considered. It is shown on the example of the direct reaction $C^{13}(\gamma, n)C^{12}$ that the contribution of nonphysical singularities can be neglected.

1. INTRODUCTION

In the description of direct processes one must, in general, take account of the effects of virtual scattering of the initial and final particles on each other, which are commonly called the effects of initial and final state interactions. This problem can be considered within the framework of dispersion theory.[1,2] In the treatment of the virtual scattering effects by dispersion methods the reaction amplitude is expressed through the amplitude describing the reaction mechanism without initial and final state interactions (called Born amplitude in the following) and the amplitudes for scattering in the initial and final states. In the language of graphs the account of the virtual scattering of the final products of the reaction $A + x \rightarrow B + y$ means the addition to the graph (or sum of graphs) representing the Born amplitude, of graphs of the loop type the left vertex of which is the Born amplitude while the right vertex is given by the amplitude for the virtual scattering of particles B and y (Fig. 1). The corresponding graphs are considered in[3].

Since both the Born amplitude and the scattering amplitude depend on the momentum transfer, the graph of Fig. 1 describing the virtual scattering contains singularities in the momentum transfer as well as in the energy. If we go over to partial wave amplitudes which depend only on the energy E, the singularities in the momentum transfer lead to

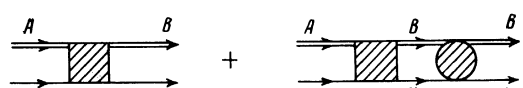

FIG. 1

the appearance of left-hand singularities in the complex plane of the variable E. Thus the amplitude with account of the virtual scattering of the final particles contains, besides the Born singularities, also left-hand singularities in E which arise out of the singularities of the Born amplitude, and which are connected with nonphysical singularities of the scattering amplitude. These singularities will be called quasi-Born singularities in the following.

It should be emphasized that the scattering amplitude entering in the graph under consideration is not on the mass shell and cannot be deduced directly from experiment. Scattering experiments determine the scattering amplitude only in the physical region. Within a concrete model one can obtain analytic expressions for the amplitudes which allow one to judge the behavior of the scattering amplitude for nonphysical values of the variables. However, the choice of a correct model for the scattering is no less problematic than the determination of the mechanism of the direct reaction, the more so as models with different analytic properties may lead to identical results for the scattering amplitude in the physical region; this is known, for example, from investigations of the optical model (cf., e.g.,[4,5]). To this we must add that strictly speaking the elastic scattering cannot be reduced to a potential problem in the presence of inelastic channels.

The deviation of the potential model scattering amplitude from the actual one can be particularly large in the unphysical region. However, if the quasi-Born singularities give a small contribution to the amplitude for the direct process, the effects of the virtual scattering can be included without the use of doubtful models. In the following we shall therefore neglect the contribution of the quasi-Born singularities in the account of the virtual scattering effects by the dispersion method. If contrary to our expectation the quasi-Born singularities turn out to be essential and depend moreover on the model used for the scattering, then the problem of including the initial and final state interactions can hardly be solved satisfactorily.

The results obtained in the present paper show that in some cases at least, the amplitude for a direct process with account of the virtual scattering can be expressed through the scattering phases in the physical region. It will be shown on the example of the direct photoeffect (Sec. 5) that the quasi-Born singularities can be neglected in the account of the nuclear interaction.

In the case of the Coulomb interaction the quasi-Born singularities coalesce with the Born singularities and their contribution becomes important. The problem is in this case made easier by the circumstance that the Coulomb interaction is known exactly, so that there is no need to use a model for the determination of the scattering amplitude in the nonphysical region. This allows us to indicate a practical method for the simultaneous account of the Coulomb and nuclear interactions (Sec. 3).

In the investigation of the analytic properties of the amplitude and in the estimate of the quasi-Born singularities (Sec. 4) we shall use a model with an analytic potential which does not lead to an essential singularity in the reaction amplitude. Potentials which are cut off at a finite distance have been studied in[4,6]; in these papers numerical calculations are presented which demonstrate that the integral over the large circle makes a small contribution to the amplitude.

Numerical calculations (Sec. 5) with a potential with diffuse boundary for the determination of the scattering phases were carried out for the partial amplitude for the direct photo-effect $C^{13}(\gamma, n)C^{12}$ corresponding to the transition of a neutron from a bound p state to an s state in the continuum.

2. THE OMNÈS-MUSKHELISHVILI EQUATION AND ITS SOLUTION

The use of the Omnès-Muskhelishvili equation (OM) in the theory of direct nuclear reactions has been considered earlier in[2,4] (cf. also[1]).[1)] It is however useful to reproduce here the derivation and solution of the OM equation for the inclusion of the virtual scattering in the initial and final states in order to indicate some essential details which have partly been considered in[4]. To this we must add that there exist some inaccuracies in the literature concerning this problem to which we shall address ourselves below.

Following the considerations in the introduction, we shall consider only the following of all the singularities associated with the virtual scattering effects in the amplitude for the reaction A + x → B + y: the right-hand cut along the real E axis ($E = E_y + E_B$) from the point E = 0 (E = Q) for Q > 0) (Q < 0), where $Q = m_x + m_A - m_y - m_B$ (i.e., from the normal threshold of the reaction), and the poles corresponding to the bound states of the system. For simplicity we shall not consider anomalous singularities in the energy; they can be taken into account by including particular graphs (in some reactions they do not occur at all). We assume that the reaction mechanism determining the Born amplitude B is known and corresponds to Feynman graphs with singularities in t or u:

$$t = -(\mathbf{p}_x - \mathbf{p}_y)^2 + 2(m_x - m_y)(E_x - E_y),$$

$$u = -(\mathbf{p}_x - \mathbf{p}_B)^2 + 2(m_x - m_B)(E_x - E_B). \quad (2.1)$$

Let us go over to the partial amplitudes $M_{ll_0}(E)$ using the following expansion:

$$M^{\beta\sigma\mu}_{\alpha sm}(\mathbf{p}_x, \mathbf{p}_y) = 4\pi \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \sum_{l_0=|J-s|}^{J+s} \sum_{l=|J-\sigma|}^{J+\sigma} i^{l_0-l},$$

$$M_{ll_0}(E) C^{JM}_{lM-\mu,\,\sigma\mu} C^{JM}_{l_0M-m,\,sm} Y^{*}_{l_0M-m}\left(\frac{\mathbf{p}_x}{p_x}\right) Y_{lM-\mu}\left(\frac{\mathbf{p}_y}{p_y}\right), \quad (2.2)$$

where sm and $\sigma\mu$ are the channel spin and its projection on the z axis for the initial (α) and final (β) channels; l_0 and l are the orbital angular momenta in channels α and β. The amplitude M is connected with the differential cross section by

$$\frac{d\sigma_{\alpha sm\to\beta\sigma\mu}}{d\Omega} = \frac{m_{xA} m_{yB}}{4\pi^2} \frac{p_y}{p_x} |M|^2, \quad (2.3)$$

where m_{xA} and m_{yB} are the reduced masses in the initial and final channels. The partial amplitude $M_{ll_0}(E)$ contains the factor $[E^l(E-Q)^{l_0}]^{1/2}$ which describes the known threshold behavior

$$M_{ll_0}(E) = [E^l(E-Q)^{l_0}]^{1/2} M_{ll_0}(E). \quad (2.4)$$

For neutral particles

$$M_{ll_0}(E) \to \text{const} \neq 0, \quad E \to 0, Q.$$

The factor $[E^l(E-Q)^{l_0}]^{1/2}$ is separated out in order to get rid of the kinematical singularities in M_{ll_0} (the corresponding Born amplitude $B_{ll_0}(E)$

[1)]These papers give references to earlier works.

has a cut which is absent in the amplitude B_{ll_0}).[2)] Using the analyticity of $M_{ll_0}(E)$, we write down a dispersion relation for the difference $M_{ll_0} - B_{ll_0}$ on the physical sheet of E (Im E > 0). We shall not make any subtractions, assuming that $M_{ll_0} - B_{ll_0}$ vanishes sufficiently rapidly for $|E| \to \infty$. Then the integral over the large circle can be neglected and we find

$$M_{ll_0}(E) = B_{ll_0}(E) + \frac{1}{\pi}\int_{E_0}^{\infty}\frac{A_{ll_0}(E')}{E'-E-i\varepsilon}dE', \quad \varepsilon \to +0. \quad (2.5)$$

Let us use the fact that $M_{ll_0}(E)$ is real on the real axis outside the cuts. Taking account of the fact that $B_{ll_0}(E)$ has only left-hand singularities connected with the singularities of the reaction amplitude in t or u, we obtain then

$$2iA_{ll_0}(E) = M_{ll_0}(E_+) - M_{ll_0}(E_-), \quad (2.6)$$

where the indices + and − denote the upper and lower branches of the right-hand cut, respectively, and

$$\operatorname{Im} M_{ll_0}(E_\pm) = \pm A_{ll_0}(E). \quad (2.7)$$

Thus (5) and (6) give

$$M_{ll_0}(E) = B_{ll_0}(E) + \frac{1}{\pi}\int_{E_0}^{\infty}\frac{\operatorname{Im} M_{ll_0}(E')}{E'-E-i\varepsilon}dE', \quad \varepsilon \to +0, \quad (2.8)$$

where the integral is taken along the upper branch of the right-hand cut and $E_0 = 0$ for $Q > 0$ and $E_0 = Q$ for $Q < 0$. Singling out those terms in the unitarity relation which correspond to the virtual scattering of the initial and final particles, we easily obtain

$$\operatorname{Im} M_{ll_0}(E) = M_{ll_0}(E)h_l^*(E) + M_{ll_0}^*(E)f_{l_0}(E-Q) + \tilde{A}_{ll_0}(E),$$

$$h_l(E) = \begin{cases} e^{i\delta_l}\sin\delta_l, & E > 0 \\ 0, & E < 0 \end{cases};$$

$$f_l(E-Q) = \begin{cases} e^{i\varphi_l}\sin\varphi_l, & E > Q \\ 0, & E < Q \end{cases} \quad (2.9)$$

(δ_l and φ_{l_0} are the scattering phases of the final and initial particles). In the derivation of (2.9) we have assumed that the scattering amplitude is diagonal in the orbital angular momentum.

Relation (2.9) has been written in the physical energy region, where the factor $[E^l(E-Q)^{l_0}]^{1/2}$ is real for arbitrary l and l_0 so that the same relation holds for the amplitude $M_{ll_0}(E)$ of (2.4):

$$\operatorname{Im} M_{ll_0}(E) = M_{ll_0}(E)h_l^*(E) + M_{ll_0}^*(E)f_{l_0}(E-Q) + \tilde{A}_{ll_0}(E). \quad (2.10)$$

Relation (2.10) can be continued analytically from the physical into the unphysical region between the points E = 0 and Q, where only $h_l(f_{l_0})$ is different from zero for Q > 0 (< 0).

Substituting (2.10) in (2.8), we obtain the singular OM integral equation:

$$M_{ll_0}(E) = B_{ll_0}(E) + \frac{1}{\pi}\int_{E_0}^{\infty}\frac{M_{ll_0}(E')h_l^*(E') + M_{ll_0}^*(E')f_{l_0}(E'-Q) + \tilde{A}_{ll_0}(E')}{E'-E-i\varepsilon}dE', \quad \varepsilon \to +0. \quad (2.11)$$

Equation (2.11) differs from (2.2) of[2] by the presence of the term $\tilde{A}_{ll_0}$ under the integral sign.

The solution of (2.11) can be obtained as in[2], and we arrive at the expression

$$M_{ll_0}(E) = B_{ll_0}(E) + [\pi F_l(q)F_{l_0}(k)]^{-1} \times \int_{E_0}^{\infty}\frac{F_l(q')F_{l_0}(k')[B_{ll_0}(E')H_{ll_0}(E') + \tilde{A}_{ll_0}(E')]}{E'-E-i\varepsilon}dE', \quad \varepsilon \to +0. \quad (2.12)$$

In (2.12) we have used the notation

$$H_{ll_0}(E) = \begin{cases} \exp[i(\delta_l^* + \varphi_{l_0})]\sin(\delta_l^* + \varphi_{l_0}), & E > 0, Q \\ \exp(i\delta_l^*)\sin\delta_l^*, & 0 \leqslant E \leqslant Q,\ Q > 0; \\ \exp(i\varphi_{l_0})\sin\varphi_{l_0}, & 0 \leqslant E \leqslant Q,\ Q < 0 \end{cases}$$

$$k = [2m_{xA}(E-Q)]^{1/2}/\hbar, \qquad q = (2m_{yB}E)^{1/2}/\hbar. \quad (2.13)$$

$F_l(q)$ is the "dynamical" Jost function connected with the scattering phase $\delta_l(E)$ by the integral relation

$$F_l(\pm q) = \left[\prod_{\nu=1}^{n}\left(1+\frac{E_\nu}{E}\right)\right] \Big/ \exp\left[\frac{1}{\pi}\int_0^{\infty}\frac{\delta_l(E')dE'}{E'-E\mp i\varepsilon}\right], \quad \varepsilon \to +0, \quad (2.14)$$

where the E_ν (> 0) are the energies of the bound states with given l (the zeroes of $F_l(q)$ for Im q > 0). The function $F_{l_0}(\pm k)$ is defined by (2.14) with E replaced by E − Q and δ_l by φ_{l_0}.

We note that (2.12) has been obtained under the condition that the system has no bound states. The bound states can be included by a method indicated in[2]. However, there is a simpler solution[4] based on the fact that (2.12) can also be used in the presence of bound states, regarding it as an analytic continuation in the interaction strength.

If we want to express the amplitude M_{ll_0} solely in terms of the Born amplitude B_{ll_0} and the scattering phases of the initial and final particles, then we must drop the term $\tilde{A}_{ll_0}$ in (2.12). The quantity

2)We note that in [7] this kinematical factor was not separated out, so that the discontinuity of the partial amplitude on the right-hand cut was not determined correctly.

$\tilde{A}_{ll_0}$ is the discontinuity of the amplitude due to the effects of all inelastic channels in the unitarity relation (2.10). As already noted,[8] the assumption that the main contribution to the mechanism of the direct process comes from the graphs (with singularities in t or u) included in the amplitude B_{ll_0} implies that the integral term in (2.12) containing $\tilde{A}_{ll_0}$ is small in comparison with the remaining terms in (2.12) and can therefore be neglected. The assertion of Dar and Tobocman[9] that the initial and final state interactions can be included via the solution of the OM equation (or by the distorted wave method) only if the reaction under study is the dominating inelastic channel, is in general incorrect. Only comparison with experiment can show whether the virtual elastic scattering does indeed play the main role in the amplitude for the direct process. We note however, that if the Born amplitude receives important contributions from graphs which also have singularities in E, then we must take for $\tilde{A}_{ll_0}$ the discontinuities of the corresponding partial amplitudes on the right-hand cut.

Practical calculations with (2.12) simplify considerably if the Jost function $F_l(q)$ is known. In[2] the corresponding expression for a square well was quoted. In numerical calculations using the optical model the wave function in the internal region is matched to the free solution in the external region at the point r = R, where the nuclear interaction can be neglected. In this case the Jost function can be defined by the expression[4]

$$F_l(\pm q) = \mp iW[xh_l^{(1,2)}(x), u_l(q, r)], \quad x = qr, \tag{2.15}$$

where $h_l(x)$ is the spherical Hankel function, $u_l(q, r) = r\psi_l(q, r)$ and $\psi_l(q, r)$ is the radial solution of the Schrödinger equation with an optical potential, normalized by the condition

$$u_l(q,r) \to xj_l(x), \quad r \to 0;$$

$$W[\varphi(x), \psi(r)] \equiv q^{-1}\left[\frac{d\varphi}{dr}\psi - \varphi\frac{d\psi}{dr}\right]_{r=R}. \tag{2.16}$$

Since we need to know the Jost function only in the physical region of E, we can use the numerical calculations of $u_l(q, r)$ with the optical model.

3. SIMULTANEOUS ACCOUNT OF COULOMB AND NUCLEAR INTERACTIONS

In the description of processes involving charged particles one must include the Coulomb interaction along with the nuclear interaction. For small energies, for example, the effects of the virtual Coulomb scattering are predominant (barrier effect). As noted in the introduction, in taking account of the Coulomb interaction it is impossible to neglect the quasi-Born singularities. Let us consider a method in which these singularities are taken into account rigorously while the effects of the virtual nuclear scattering are included in the approximation of the dispersion method discussed above. The possibility of a rigorous account of the quasi-Born Coulomb singularities is connected with the fact that in contrast to the nuclear interaction, the Coulomb force is known exactly. Since we regard the Born amplitude as known, we separate from the full reaction amplitude the terms which do not contain virtual nuclear scattering and can therefore be determined rigorously without use of any model for the scattering. Then we obtain the sum of graphs shown in Fig. 2 in which the first term is the Born amplitude, the second is a graph containing in the right vertex the amplitude for virtual Coulomb scattering, and the third is the analogous graph with "nuclear" scattering in the right vertex (the term "nuclear" is put between quotation marks, since we are talking about nuclear scattering in the presence of the Coulomb field).

The "nuclear" scattering amplitude is the difference between the full and the Coulomb scattering amplitudes. The phases $\tilde{\delta}_l$ corresponding to the "nuclear" scattering amplitude on the energy shell are defined by

$$\exp(2i\tilde{\delta}_l) - 1 = \exp(2i\delta_l^{(C)})[\exp(2i\delta_l) - 1]. \tag{3.1}$$

Here $\delta_l^{(C)}$ is the Coulomb phase with account of screening (the problem of the screening of the Coulomb potential and the definition of the phase $\delta_l^{(C)}$ are considered below), and δ_l is the nuclear phase equal to the difference between the full phase and the phase $\delta_l^{(C)}$.

After expansion of the desired reaction amplitude into the sum of graphs of Fig. 2, the problem reduces to the calculation of the last term by the dispersion method. Here we can use the solution of the OM equation (2.12) in which the scattering phase is to be replaced by $\tilde{\delta}_l$; we must add to this amplitude the amplitude corresponding to the second term in Fig. 2. However, the calculation of the Jost function for the phase $\tilde{\delta}_l$ by formula (2.14) is a complicated problem owing to the bad behavior

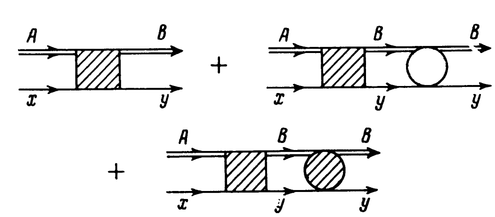

FIG. 2

of $\delta_l^{(C)}$ for small energies. Let us therefore consider a second equivalent but more convenient method for the inclusion of the nuclear-Coulomb interaction allowing us to use the expression for the Jost function with the phase δ_l from the optical model:

$$F_l(\pm q) = -W(G_l, u_l) \mp iW(F_l, u_l). \quad (3.2)$$

Formula (3.2) is a generalization of (2.15) for the case of a Coulomb asymptotic form in the scattering problem, where $F_l(q, r)$ and $G_l(q, r)$ are the regular and irregular solutions of the Schrödinger equation with a Coulomb field, respectively.

Our method for including the nuclear interaction in the final state for charged particles consists in the following: We start from the amplitude $M_l^{(C)}$ which includes only the Coulomb interaction (first two terms in Fig. 2). The OM equation for the inclusion of the nuclear interaction is written down for the function $R_l(E)$ connected with the desired amplitude $M_l(E)$ by

$$M_l(E) = R_l(E)/F_l^{(C)}(q), \quad (3.3)$$

where $F_l^{(C)}(q)$ is the Coulomb Jost function. After this operation the function $R_l(E)$ contains no new singularities compared to the singularities of $M_l(E)$. At the same time the function $R_l(E)$ has no right-hand Coulomb cut, its discontinuity on the right-hand cut is determined solely by the behavior of the nuclear phase δ_l. The free term in the OM equation is given by the function $R_l^{(0)}(E)$ which is connected with the amplitude $M_l^{(C)}(E)$ by a relation analogous to (3.3):

$$M_l^{(C)}(E) = R_l^{(0)}(E)/F_l^{(C)}(q). \quad (3.4)$$

We recall that in the derivation of the OM equation we have constructed a function such that the free term in the equation has no discontinuity on the right-hand cut. To this end we have separated the kinematical factor $[E^l(E - Q)^l{}_0]^{1/2}$ in the partial amplitude. The separation of the factor $1/F_l^{(C)}(q)$ carried out above is done for the same purpose; the derivation of the OM equation for the function $R_l(E)$ is then carried through in the same way as for the pure nuclear interaction. If the partial amplitude with pure Coulomb virtual scattering is known (cf., for example,[10]) then the problem reduces to the determination of the Coulomb Jost function $F_l^{(C)}(q)$. We use the relation (for a repulsive potential)

$$F_l^{(C)}(q) = \exp\left\{-\frac{1}{\pi}\int_0^\infty \frac{\delta_l^{(C)}(E')}{E'-E-i\varepsilon}dE'\right\}, \quad \varepsilon \to +0, \quad (3.5)$$

where $\delta_l^{(C)}$ is the scattering phase for a screened Coulomb field, since it is impossible in the case of an unscreened Coulomb field to introduce rigorously the concept of a scattering phase because of the distortion of the plane wave at infinity.

Let us find $\delta_l^{(C)}$ for a screened Coulomb potential of the form

$$V(r) = \begin{cases} eZ/r, & r < R \\ 0, & r > R \end{cases}. \quad (3.6)$$

The screening radius R is sufficiently large compared to the range of the nuclear force. An approximate expression for the scattering phase for the potential (3.6) can be obtained by joining the asymptotic expression of the solution of the Schrödinger equation in the internal region to the solution in the external region at r = R:

$$\delta_l^{(C)} = \sigma_l - \eta \ln 2qR,$$
$$\sigma_l = \arg\Gamma(l+1+i\eta), \qquad \eta = Z_B Z_y e^2 m_{By}/\hbar q. \quad (3.7)$$

The second term in (3.7) is connected with the cut-off of the Coulomb potential. Expression (3.7) for the phase is a good approximation if

$$|(l+1+i\eta)(l-i\eta)|/2qR \ll 1, \quad (3.8)$$

we see from this relation for which values of l and q (3.7) can be used for fixed cut-off radius R. As we shall see in the following, using the approximate phase (3.7) has the effect that R enters in the reaction amplitude only as a phase factor which is independent of the orbital angular momentum l. Therefore the corresponding approximate cross section, expressed through the square modulus of the amplitude, does not contain the radius R.

Let us find the Jost function, using the relation

$$F_l^{(C)}(q) = \exp[-J_1(E) + J_2(E)], \quad (3.9)$$

where we have used the notation

$$J_1(E) = \frac{1}{\pi}\int_0^\infty \frac{\sigma_l(E')}{E'-E}dE',$$
$$J_2(E) = \frac{1}{\pi}\int_0^\infty \frac{\eta' \ln 2q'R}{E'-E}dE' \quad (3.10)$$

(E is a complex quantity). The first integral is easily calculated if we use

$$\sigma_l = \frac{1}{2i}\ln\frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)},$$

and consider the integral of the function

$$[2i(E'-E)]^{-1}\ln[\Gamma(l+1+i\eta)/\Gamma(l+1)]$$

over a closed contour in the E plane, circumventing all singularities of this function. As a result we obtain

$$J_1(E) = \ln\frac{\Gamma(l+1+i\eta)}{\Gamma(l+1)}. \quad (3.11)$$

The quantity $J_2(E)$ is calculated analogously, considering the integral of the function

$$(E'-E)^{-1}\eta'\ln 2q'R$$

over a closed contour in the E plane circumventing the singularities of this function. We have

$$J_2(E)={}^1/_2\pi\eta+i\eta\ln 2qR. \qquad (3.12)$$

Substituting (3.11) and (3.12) in (3.9), we obtain the Jost function for the screened Coulomb field in the form

$$F_l^{(C)}(q)=\frac{\Gamma(l+1)}{\Gamma(l+1+i\eta)}\exp\left(\frac{\pi\eta}{2}+i\eta\ln 2qR\right). \qquad (3.13)$$

We note that the factor $1/F_l^{(C)}(q)$ separated out in (3.3) coincides in absolute value with the known penetration factor for the Coulomb barrier. The Jost function (3.13) corresponding to the phase (3.7) (where R is arbitrarily large) is defined on the entire physical sheet of the energy including the lower branch of the right-hand cut. At the same time, the approximate S function corresponding to the phase (3.7),

$$S_l=\exp\left[2i(\sigma_l-\eta\ln 2qR)\right]$$

is given on the physical sheet of the energy and cannot be continued onto the lower branch of the right-hand cut.

Since (3.7) is approximate, it does not satisfy the relation

$$\delta_l^{(C)}(-q)=-\delta_l^{(C)}(q), \qquad (3.14)$$

which is a general consequence of the invariance of the Schrödinger equation under the interchange $q \leftrightarrow -q$. This is a consequence of the fact that in deriving (3.7) we have used the asymptotic expression of the Coulomb wave function containing the confluent hypergeometric function

$$F(l+1+i\eta,\ 2l+2,\ -2iqR).$$

Although this function is an entire function of q, it does not have a unique asymptotic representation in the entire q plane. However, the expression for the S function for potential (3.6) obtained from the phase (3.7) is a good approximation for real positive values of q, i.e., in the physical region of the energy; this is quite sufficient for the solution of the problem of the effects of the virtual scattering in nuclear reactions.

Coming back to our remark on the absence of a rigorous concept of a scattering phase for an unscreened Coulomb potential, let us clarify the meaning of the use of the phase σ_l in Coulomb scattering problems. Since the partial wave expansion converges for all nonzero scattering angles we may use a screened Coulomb potential with sufficiently large cut-off radius. The cut-off radius is determined by the condition that the phase (3.7) can be used for all terms making an appreciable contribution to the sum of partial amplitudes Then the introduction of a cut-off in the Coulomb potential only leads to the appearence of an unessential phase factor in the full amplitude, since the second term in (3.7) is independent of l In calculating the Jost function by (3.5) the cut-off in the Coulomb potential plays an essential role.

4. ANALYTIC PROPERTIES OF THE MATRIX ELEMENTS

As already noted, the quasi-Born singularities in the reaction amplitude, which are neglected in the dispersion method, are essentially determined by the choice of the model for the scattering. It is impossible to estimate their contribution to the reaction amplitude without recourse to a concrete model. Let us introduce a potential model for the elastic scattering. Moreover, let us for simplicity consider only the interaction in the final state. In this approximation the amplitude corresponding to the graph of Fig. 1 coincides formally with the matrix element obtained by the distorted wave method (DWM) which it is convenient to use for our methodological purposes.

Since we are interested in the analytic structure of the matrix elements of the DWM, we shall use nuclear potentials V(r) which are analytic in the right half-plane of the complex variable r. The amplitudes calculated with such potentials have no essential singularity at infinity. We note that the potentials used customarily in nuclear physics (square well, Saxon-Woods potential) lead to the appearance of an essential singularity at infinity, so that the investigation of the quasi-Born singularities is reduced solely to numerical estimates. For definiteness we shall consider only the effects of the interaction in the final state.

The radial matrix element of the DWM can be written in the form

$$M_l(E)=\int_0^\infty \psi_b(r)O(k,r)\psi_l(q,r)r^2dr, \qquad (4.1)$$

where $\psi_l(q, r)$ and $\psi_b(r)$ are the wave functions for the continuous spectrum and the bound state, respectively, O(k, r) is an operator determined by the reaction mechanism, and k and q are the wave numbers of the initial and final particles in the c.m.s., which we shall regard as independent variables (the connection between them required by the law of energy conservation can be taken into account in the final expressions).

Our problem consists in writing the wave functions in analytic form. The most convenient method for this purpose is the spectral method of Martin,[11,12] which makes use of the Laplace transformation for the wave function and the potential. This method may be used only for potentials which are analytic for Re r > 0, for which the Laplace transform exists everywhere for r > 0. In[13], where the three-prong vertex part was calculated and investigated as to its analytic properties, the following potentials were proposed which approximate closely the Saxon-Woods potential but are analytic for Re r > 0:

$$V_1(r) = -V_0 e^{-\mu r} \sum_{k=0}^{n} \frac{(\mu r)^{(C)}}{k!}, \qquad (4.2)$$

$$V_2(r) = -V_0[1 - (1 - e^{-\mu r})^n]. \qquad (4.3)$$

In potential V_1 the role of a radius is played by the quantity na ($a = 1/\mu$ is the diffuseness of the surface) and in the potential V_2 by the quantity $n \approx A$, where A is the number of nucleons in the nucleus ($A \lesssim 30$). The remaining parameters have the usual meaning.

The corresponding functions $\rho(\sigma)$ and their Laplace transforms

$$V(r) = V_0 \int_0^\infty \rho(\sigma) e^{-\sigma r}\, d\sigma \qquad (4.4)$$

have the form

$$\rho_1(\sigma) = -\sum_{k=0}^{n} \frac{\mu^k}{k!} \delta^{(k)}(\sigma - \mu), \qquad (4.5)$$

where $\delta^{(k)}(\sigma - \mu)$ is a function defined by the relation

$$\int_a^b f(x')\, \delta^{(k)}(x' - x)\, dx' = \begin{cases} (-1)^k \dfrac{d^k f(x)}{dx^k}, & x \text{ inside } (a, b) \\ 0, & x \text{ outside} (a, b) \end{cases}, \qquad (4.6)$$

$$\rho_2(\sigma) = \sum_{k=1}^{n} (-1)^k C_n^k \delta(\sigma - k\mu). \qquad (4.7)$$

Using the explicit form of the function $\rho(\sigma)$, we can find the expression for the radial wave function. The spectral method is particularly simple in the case of s waves, to which we shall restrict ourselves in the following. In studying the analytic properties of the matrix element, the actual value of l is evidently not very important (for example, the location of the singularities of the partial amplitudes in the energy plane is generally independent of l). Omitting the index $l = 0$, we write the radial wave function in the continuous spectrum in the form

$$\psi(q, r) = \frac{1}{2iqrF(q)} [F(-q) f(-q, r) - F(q) f(q, r)]; \qquad (4.8)$$

$$f(\pm q, r) \to \exp(\mp iqr), \quad r \to \infty. \qquad (4.9)$$

Here F(q) is the Jost function, where

$$F(\pm q) = f(\mp q, 0). \qquad (4.10)$$

The Jost function (4.10) for the potential $V_2(r)$ has the form

$$F(\pm q) = 1 + \sum_{\nu=1}^{\infty} (K_0 a)^{2\nu} F_\nu(\pm q); \qquad (4.11)$$

$$F_\nu(\pm q) = \sum_{k_1, k_2, \ldots, k_\nu = 1}^{n} \prod_{l=1}^{\nu} \frac{(-1)^{k_l} C_n^{k_l}}{s_l(s_l \mp 2iaq)}, \qquad (4.12)$$

$$K_0 = \sqrt{2m_{By}V_0}/\hbar, \quad s_l = k_1 + k_2 + \ldots + k_l.$$

Substituting expression (4.8) for the wave function in the matrix element M of (4.1), we obtain

$$M(q) = \frac{1}{F(q)} \{F(-q)\Phi^{(1)}(q) + F(q)\Phi^{(2)}(q)\}, \qquad (4.13)$$

$$\Phi^{(1,2)}(q) = \mp \frac{1}{2iq} \int_0^\infty \psi_b(r) O(k, r) f(\mp q, r) r\, dr. \qquad (4.14)$$

For the potential $V_2(r)$ we find

$$\Phi^{(1,2)}(q) = B^{(1,2)}(\varkappa, q) + \sum_{\nu=1}^{\infty} (K_0 a)^{2\nu} B_\nu^{(1,2)}(q), \qquad (4.15)$$

$$B_\nu^{(1,2)}(q) = \sum_{k_1, k_2, \ldots, k_\nu = 1}^{n} B^{(1,2)}(\varkappa + \mu s_\nu; q) \prod_{l=1}^{\nu} \frac{(-1)^{k_l} C_n^{k_l}}{s_l(s_l \mp 2iaq)},$$

where

$$B^{(1,2)}(\varkappa, q) = \mp \frac{1}{2iq} \int_0^\infty \psi_b(r) O(k, r) e^{\pm iqr} r\, dr,$$

$$B(\varkappa, q) = B^{(1)}(\varkappa, q) + B^{(2)}(\varkappa, q),$$

$$r\psi_b(r) \to Ce^{-\varkappa r}, \quad r \to \infty. \qquad (4.16)$$

Here B is the Born amplitude of the reaction (for the s wave). Formulas (4.13), (4.11), (4.15) and (4.16) give an analytic expression for the partial amplitude of the DWM (for s waves) and can be used for the numerical calculation of the amplitude and for the analysis of its analytic properties.

Let us separate from the amplitude thus obtained the terms corresponding only to the quasi-Born singularities. To this end we write M in dispersion form, considering the Cauchy integral over a contour circumventing the singularities of the amplitude M. Using the fact that the discontinuity of the Born amplitude $B(\kappa, q)$ on the left-hand cut of the physical sheet of the q^2 plane is equal to

$$\Delta B(\varkappa, q) = \Delta B^{(2)}(\varkappa, q), \tag{4.17}$$

we obtain the following expression for the quasi-Born terms:

$$Q(q) = \sum_{\nu=1}^{\infty} (K_0 a)^{2\nu} Q_\nu(q),$$

$$Q_\nu(q) = \frac{1}{2\pi i} \sum_{k_1, k_2, \ldots, k_\nu = 1}^{n} \int_{L_\nu} dq'^2 \frac{\Delta B(\varkappa + \mu s_\nu; q')}{q'^2 - q^2} \cdot$$

$$\prod_{i=1}^{\nu} \frac{(-1)^{k_i} C_n^{k_i}}{s_i(s_i + 2iaq')} \tag{4.18}$$

The integral is taken along the upper branch (from left to right) of the cut, on which the function $B(\kappa + \mu s_\nu; q)$ has the discontinuity $\Delta B(\kappa + \mu s_\nu; q)$.

The separation of the terms containing only the quasi-Born singularities is a generalization of the analogous procedure used in[14] for the partial Born amplitude in the form of a pole term. It was shown in that paper that the contribution of the quasi-Born terms to the cross section for the photo-disintegration of the deuteron near threshold is negligibly small for a wide range of values of the parameter a.

We note that the usual argument for the neglect of the quasi-Born terms is based on the remoteness of the corresponding singularities in the physical sheet. However, as shown in[13] in connection with the calculation of the three-prong vertex part, the effect of the radius is determined by the contribution of just these remote singularities. It is of interest in this context to investigate the quasi-Born terms for the case of a finite radius. As will become clear in the following, an estimate of the contribution of the far quasi-Born singularities to the difference between M and M^{OM} requires the knowledge of the Jost function F(q) in the nonphysical region. However, for qualitative estimates of the relative role of the different quasi-Born singularities one may use directly formula (4.18).

For simplicity and anschaulichkeit we consider the pole term Born amplitude

$$B(\varkappa, q) = (q^2 + \varkappa^2)^{-1}. \tag{4.19}$$

In this case the quasi-Born singularities are also simple poles. Collecting the terms with the same singularities, we arrive at the expression

$$Q(q) = \sum_{\nu=1}^{\infty} \tilde{Q}_\nu(q) = \sum_{\nu=1}^{\infty}{}' \frac{\beta_\nu}{q^2 + (\varkappa + \mu\nu)^2}. \tag{4.20}$$

To avoid unwieldy expressions we do not here quote the explicit forms of the terms β_ν, which can be easily found from (4.18).

The numerical estimates of the relative contribution of the closest poles of the function Q(q) near the reaction threshold (q = 0) for the potential $V_2(r)$ with the parameters V_0 = 40 MeV, n = 5, and κ = 0.5 F^{-1} are given below:

ν :	1	2	3	4
$\tilde{Q}_\nu/\tilde{Q}_1$:	1	0.05	−0.07	+0.07

We see from this that we can neglect all quasi-Born poles compared to the closest pole. This is in strong contrast to the situation which obtained in the consideration of the vertex part,[13] where the contributions of a number of poles turned out to be comparable.

We note that the account in the reaction amplitude, of the term which has only the closest quasi-Born singularity is equivalent to the inclusion of this singularity in the amplitude corresponding to the graph of Fig. 3. The wavy line in this graph corresponds to the Born amplitude for scattering on the potential

$$\tilde{V}_2(r) = -V_0 n e^{-\mu r}. \tag{4.21}$$

This model graph may be compared to the analogous graph in which the wavy line indicates the exchange of a "π meson." Then the physical meaning of the appearance of the factor n is simply that the number of possibilities of a "one-meson" exchange increases with increasing number of nucleons in the nucleus.

If the quasi-Born terms Q(q) are known, one can use the following expression for the calculation of the reaction amplitude:

$$M(q) = \frac{1}{2\pi i F(q)} \int dq'^2 \frac{F(q')}{q'^2 - q^2} [\Delta B(q') + \Delta Q(q')], \tag{4.22}$$

where the integrals are taken along the cuts of the functions B and Q. This expression is the analog

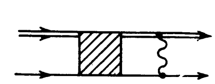

FIG. 3

of the corresponding formula for the amplitude obtained from the solution of the OM equation:

$$M^{\mathrm{OM}}(q)=\frac{1}{2\pi i F(q)}\int dq'^2 \frac{F(q')}{q'^2-q^2}\Delta B(q'). \quad (4.23)$$

It is seen from (4.22) that the contribution of the quasi-Born singularity depends on the magnitude of the Jost function on the corresponding cut. For example, in a realistic situation the relative contribution of the Born singularity located much closer

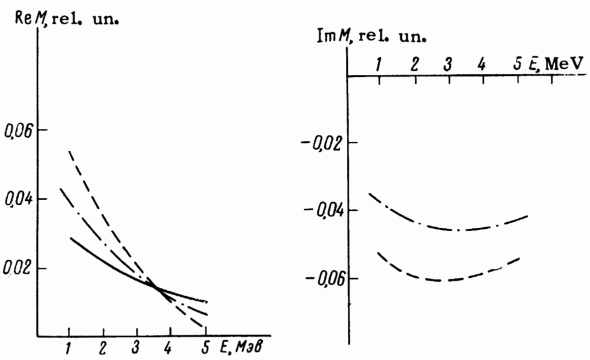

FIG. 4. Energy dependence of the partial s wave matrix element for the direct photo effect $C^{13}(\gamma, n)C^{12}$. Solid curve: Born approximation (× 0.1); dotted curve: M^{OM}; and dash-dotted curve: DWM. The parameters are given in the text.

to the physical region than the quasi-Born singularity, is enhanced by the weight factor F(q′) under the integral, which is different on the Born and quasi-Born cuts.

5. NUMERICAL CALCULATIONS

For an estimate of the total contribution of the quasi-Born singularities to M it suffices to know the Jost function in the physical energy region where it can be calculated with the help of formulas (2.15) or (3.2) (depending on whether the particles are charged or not). Formulas (2.15) and (3.2) contain quantities which are determined in an intermediate step in the calculation of the phases and the scattering cross sections by the optical model. The numerical calculations demonstrating the role of the quasi-Born effects were, as before,[4,6] carried out for the reaction $C^{13}(\gamma, n)C^{12}$. The bound state was again described by a model with a square-well potential (R = 2.86 F, V_0 = 33.65 MeV), which gives the correct value of the binding energy of the extra nucleon. The contribution of the region inside the nucleus was neglected. The new feature in the calculation is that the wave function of the continuous spectrum and the scattering phases were calculated by an optical model with a Saxon-Woods potential with parameters taken from[15], where the energy dependence of the depths of the real and imaginary parts was taken into account. The results for the energy dependence of the partial amplitude corresponding to an s wave of the neutron emitted from the p shell are shown in Fig. 4. The use of a Saxon-Woods potential for the determination of the scattering phases in the physical region is not inconsistent with the dispersion relations, since this potential can be approximated with sufficient accuracy by an analytically admissible potential of the type (4.2) or (4.3).

It is seen from a comparison of the corresponding curves for M^{OM} and M that the neglect of the quasi-Born singularities leads to an insignificant alteration of the value of the amplitude while completely preserving the character of its energy dependence. At the same time the effect as such of the virtual scattering of the products of the photoreaction is very important.

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