PULSATING COLLAPSE FROM THE POINT OF VIEW OF AN EXTERNAL OBSERVER

M. E. GERTSENSHTEĬN and Yu. M. AĬVAZYAN

Institute of Physico-technical and Radio Measurements

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The picture of an oscillating collapse is considered in the coordinate system of an external observer located outside the pulsating sphere of constant density. It is shown that for an external observer the shell of the pulsating sphere is always in the R region, i.e., outside the Schwarzschild sphere. The period of oscillation for the external observer is given by Eq. (15). The time in the Schwarzschild coordinate system is real in the R region and has a constant imaginary part in the T region. The Schwarzschild coordinate system is unique in the vacuum outside the pulsating sphere. The ambiguity and the imaginary term in the time appear only inside the matter, where the Schwarzschild metric does not hold. The question of the picture which an external observer sees is discussed. It is shown that the symmetry of contraction and expansion is violated in the visible picture.

THE problem of gravitational collapse is one of the most interesting problems in relativistic astrophysics.[1] The oscillatory character of the collapse in the comoving coordinate system has been proved earlier in[2], and the possibility of observing it from the R region has been investigated in[3]. In the present paper we shall carry out a quantitative treatment of oscillatory collapse for an external observer. We confine ourselves to spherically symmetrical motion of dustlike matter (p = 0) and consider first the picture in a comoving coordinate system.[4] The interval in the comoving system is

$$ds^2 = c^2d\tau^2 - r^2d\sigma^2 - e^{\omega}dR^2, \quad r = r(\tau, R), \qquad d\sigma^2 = d\theta^2 + \sin^2\theta d\varphi^2. \tag{1}$$

The radius r is defined so that the area of the sphere is $4\pi r^2$. The equations of GTR (the general theory of relativity) have the first integral[4]:

$$\left(\frac{dr}{d\tau}\right)^2 = \dot{r}^2 = f(R) + \frac{F(R)}{r}, \quad F > 0. \tag{2}$$

This integral is valid for arbitrary τ only in the case in which all the particles pass through the center simultaneously.[2] If the particles do not pass through the center simultaneously, the integral (2) loses its validity and the solution is unknown. Therefore, in order to get a concrete result we confine ourselves to the case in which there is exact simultaneity of passage through the center, as is true for a sphere with a constant volume density.

For the initial time we take the position of greatest expansion:

$$\dot{r} = 0, \quad r = R_0, \quad f(R_0) = -F(R_0)/R_0 < 0. \tag{3}$$

The physical meaning of F(R) is the total energy per unit mass. For a sphere of constant density we have

$$F(R) = \begin{cases} 2kM(R/R_0)^3 & \text{for} \quad R \leqslant R_0 \\ 2kM & \text{for} \quad R \geqslant R_0 \end{cases}. \tag{4}$$

Equation (2) then takes the form

$$\dot{r}^2 = \left(\frac{1}{r} - \frac{1}{R}\right)F(R). \tag{5}$$

We shall show that the maximum radius R_0 is necessarily larger than the gravitational radius of the body, $r_0 = 2kM/c^2$. For the value of e^{ω} at the surface of the sphere we have[4]

$$e^{\omega} = \frac{(r')^2}{1 + fc^{-2}} = \frac{(r')^2}{1 - 2kM/R_0c^2} = \frac{(r')^2}{1 - r_0/R_0}, \qquad r' = \frac{\partial r}{\partial R}. \tag{6}$$

Since the metric form must contain three coefficients with the negative sign, $\alpha_0 = r_0/R_0 < 1$. This same result can also be obtained directly from the positions of the light cones in the Schwarzschild coordinates. Equation (5) can be easily integrated and leads to the following parametric representation for r and τ:

$$\tau = 2\sqrt{\frac{R_0^3}{F(R_0)}}\int \sin^2\beta d\beta = \frac{1}{2\sqrt{\alpha_0}}\frac{R_0}{c}(2\beta - \sin 2\beta),$$

$$r = R_0 \sin^2\beta, \quad \alpha_0 = r_0/R_0. \tag{7}$$

We note that the coefficient of the integral in (7) is independent of R and depends only on R_0; this assures that the passages of the particles through the center are simultaneous.

The curve of $\tau(r)$ is constructed in Fig. 1; r is a periodic function of τ. We recall that the curve of $r(\tau)$ is constructed by means of the classical equations of GTR, and the question as to whether quantum effects at points of infinite density $r = 0$ can be neglected is still open.[3] As the period we take the segment AB, which corresponds to $0 < \beta < 2\pi$. If we mark a particular particle, then after a time equal to the period it returns to its former position; after a time equal to the half-period it will be at the opposite point in the sphere. We emphasize that the particles are not reflected from the center, but pass through it.[2] If we use coordinates in which $-\infty < r < \infty$, we get the curve shown in Fig. 1 as a solid line. If, however, we make the other assumption, that $0 < r < \infty$, the second half of the period is given by the dashed line in Fig. 1. The period in the comoving coordinate system is given by

$$T_\tau = \frac{2\pi}{\sqrt{\alpha_0}}\frac{R_0}{c}. \tag{8}$$

We emphasize once more that this formula for the period is for the case of dustlike matter with the pressure equal to zero. In Fig. 1 the region inside the curves is shaded; it is filled with matter. Outside these curves is vacuum, and the Schwarzschild metric is valid for a stationary observer. Since $\alpha_0 < 1$, the boundary of the R and T regions, which lies in vacuum, necessarily ends on the boundaries of the shaded regions.[3]

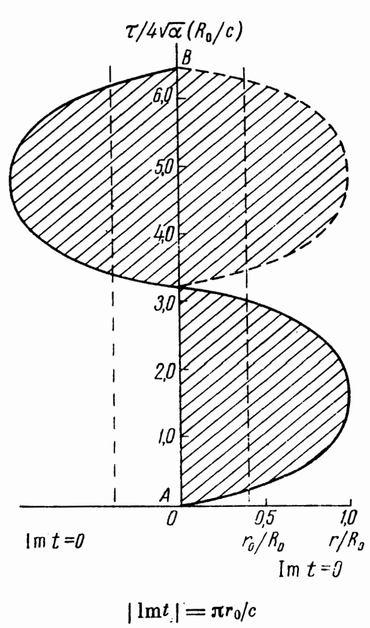

FIG. 1

Let us now consider the picture from the point of view of an external observer. The Schwarzschild system is valid only in vacuum, and owing to continuity it holds also on the surface of the pulsating sphere. Because of the invariance of the interval, we have on the surface of the sphere

$$ds^2 = \left(1 - \frac{r_0}{r}\right)c^2dt^2 + \frac{dr^2}{1 - r_0/r} = c^2d\tau^2. \tag{9}$$

We recall that the coordinate r is the same quantity in the comoving system as in the Schwarzschild system—it is defined by the condition that the area of the sphere is equal to $4\pi r^2$.[4,5] From (9) it is easy to derive the connection between t and τ:

$$t = \sqrt{1-\alpha_0}\int\frac{d\tau}{1 - r_0/r(\tau)} \tag{10}$$

This integral diverges logarithmically at $r = r_0$. This is due to the fact that the integrand has a pole at $\tau = \tau_0$, which lies on the path of integration. We need to have an analytic continuation of the integral satisfying the following conditions:

a) t is real in the R region:

b) t may be complex only in the T region:

c) time differences both within the R region and also within the T region must be real;

d) in a passage R → T → R the argument of the complex number t must return to its original value. This condition means that a change of phase by 2π in a passage R → T → R is forbidden.

We shall show that if we postulate that the path goes either above all the poles on the real axis of the complex τ plane or below all of them, these conditions are satisfied. For definiteness we shall take the path below the poles (Fig. 2, a). With this rule for going around the poles the integral (10) reduces to the integral in the sense of the principal value plus an imaginary part which arises in pass-

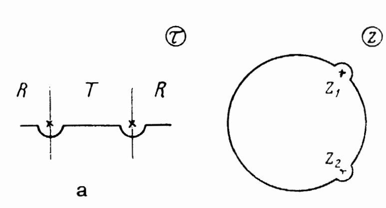

FIG. 2

ing through the R – T boundary. This imaginary part is given by

$$\operatorname{Im} t = \frac{\sqrt{1-\alpha_0}\,\pi r_0}{|v|}\operatorname{sign} v, \qquad v = \frac{dr}{d\tau}\Big|_{r=r_0} = v(r_0). \tag{11}$$

The sign of the imaginary part depends on the sign of v; it is easy to verify that the requirements a)–d) are satisfied. A shifting of poles away from the path of integration (in energy or frequency variables) is often used in physics. Since time is the conjugate variable to frequency (energy), for monochromatic radiation an imaginary term in the time is equivalent to an imaginary term in the frequency. We confine ourselves to the analogy; the question of an exact proof remains open. The important thing for us in this paper is the very fact that an analytic continuation exists which satisfies the requirements a)–d). We note also that in the T region $dt/d\tau < 0$, but this does not contradict the principle of causality.[3] The imaginary part of t is shown in Fig. 1. Since everywhere in the observable region t is real, and the imaginary part of t becomes zero again when we have passed through and beyond the T region (requirement d), there is no reason for a many-sheeted structure of the R region.[6]

Let us now find the period in the Schwarzschild system. Using the parametric representation for the half-period, we get from (10)

$$\frac{T_t}{2} = 2\frac{R_0}{c}\sqrt{\frac{1-\alpha_0}{\alpha_0}}\int_0^\pi \frac{\sin^4\beta\, d\beta}{\sin^2\beta-\alpha_0}, \qquad \alpha_0 = \frac{r_0}{R_0}. \tag{12}$$

We note that

$$\int \frac{\sin^4\beta\, d\beta}{\sin^2\beta-\alpha_0} = \frac{1+2\alpha_0}{2}\beta - \frac{\sin 2\beta}{4} + \alpha_0^2\int\frac{d\beta}{\sin^2\beta-\alpha_0}. \tag{13}$$

When taken between the limits $(0, \pi)$ the integral in the right member can be easily reduced to a contour integral around the unit circle. The poles z_1 and z_2 of the integrand then lie on the unit circle:

$$z = e^{2i\beta}, \qquad z_1 = 1 - 2\alpha_0 + 2i\sqrt{\alpha_0(1-\alpha_0)},$$

$$z_2 = 1 - 2\alpha_0 - 2i\sqrt{\alpha_0(1-\alpha_0)}. \tag{14}$$

The way of avoiding the poles which follows from Fig. 2, a is shown in Fig. 2, b. The integral is equal to zero, which for its imaginary part follows directly from the requirement d). We get finally for the period in the Schwarzschild system

$$T_t = T_\tau\sqrt{1-\alpha_0}\,(1+2\alpha_0). \tag{15}$$

We give the limiting forms of the expression (15) for small and for large α_0:

$$T_t = T_\tau(1+{}^3/_2\alpha_0), \qquad \alpha_0 \ll 1,$$

$$T_t = 3T_\tau\sqrt{1-\alpha_0}, \qquad \alpha_0 \to 1. \tag{15'}$$

The period for an external observer differs comparatively little from the period in the comoving coordinate system.

We shall now find the parametric representation of the time $t(\beta)$ in the R region. From the very fact that the integral (10) diverges logarithmically it can be seen that $t(\beta)$ contains an infinite number of branches, which we shall number with an index m. The indefinite integral can be calculated and is given by

$$t_m(\beta) = 2\frac{R_0}{c}\sqrt{\frac{1-\alpha_0}{\alpha_0}}\left\{\frac{1+2\alpha_0}{2}\left(\beta_m - \frac{\pi}{2}\right) - \frac{\sin 2\beta_m}{4} - \frac{\alpha_0^2}{\sqrt{\alpha_0(1-\alpha_0)}}\operatorname{Arcth}\left(\sqrt{\frac{1-\alpha_0}{\alpha_0}}\operatorname{tg}\beta_m\right)\right\},$$

$$\beta_m = \beta - m\pi, \quad 0 < \beta_m < \pi, \quad \sin^2\beta > \alpha_0. \tag{16}$$

The world lines $t_m(r)$ for the case $\alpha_0 = 1/2$ are constructed in Fig. 3. It can be seen from the figure that the different branches of the world lines t(r) intersect each other. For the external observer the surface of the pulsating sphere is always in the R region. If the matter is nontransparent, the external observer sees only the outer parts of the world lines, shown in Fig. 3 as a solid line. Accordingly, the picture for an external observer is unique, in spite of the large number of branches of the function $t(\beta)$. We note that there is an analogous situation for the Lorentz contraction of a moving sphere. The Lorentz contraction is not seen, but in a coordinate system connected with the observer it exists.[7]

As can be seen from Fig. 3, the bundle of world lines $t_m(r)$ always passes closer to the boundary between the R and T regions than the externally observable world line. Unobservable world lines pass inside the matter, where the Schwarzschild metric, derived for vacuum, is not valid. Evidently the many-valuedness and the presence of an imaginary part of t are caused by the fact that the Schwarzschild solution, legitimate only for vacuum, has been continued into a region filled with matter. In like manner the integral (10) is derived on the assumption that the world line $t(\tau)$, $r(\tau)$ passes through vacuum; inside the matter this assumption is incorrect.

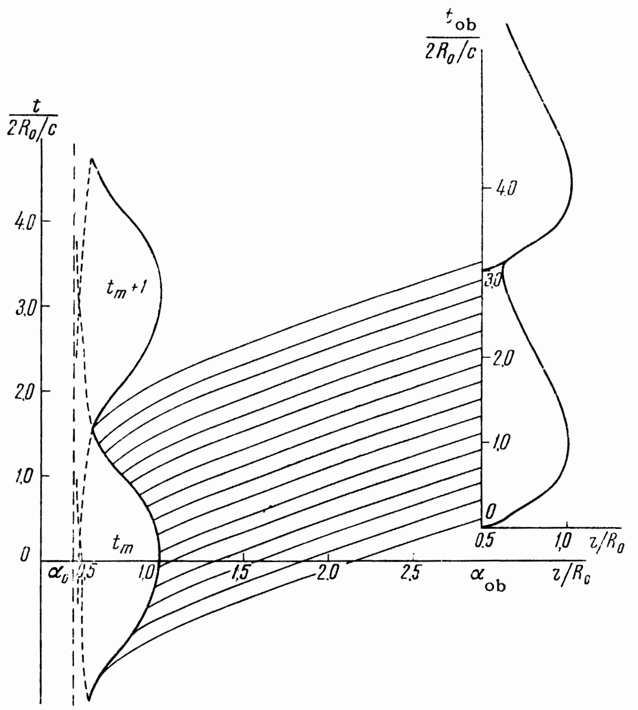

FIG. 3

The heavy solid line in Fig. 3 has kinks at the points of minimum radius ρ_0. Let us find the dependence of ρ_0 on α_0. The parameter value β_0 for which $r(\beta_0) = \rho_0$ is determined from the following transcendental equation:

$$(1+2\alpha_0)\beta_0 = \frac{\sin 2\beta_0}{2} + \frac{\alpha_0^{3/2}}{\sqrt{1-\alpha_0}} \ln \frac{\tan\beta_0 + (\alpha_0/(1-\alpha_0))^{1/2}}{\tan\beta_0 - (\alpha_0/(1-\alpha_0))^{1/2}},$$

$$\rho_0 = R_0 \sin^2\beta_0. \tag{17}$$

Let us consider various special cases. If $\alpha_0 \ll 1$, the motion along the main part of the trajectory is nonrelativistic. When we then make a change of variables in (17), we have to and including terms of order β_0^3:

$$\sqrt{\alpha_0} = k\beta_0, \quad k \ln\frac{1+k}{1-k} = 2 + \frac{2}{3k^2}, \quad k \approx 0.91,$$
$$\rho_0 = R_0\beta_0^2 = r_0 / k^2 \approx 1.2\, r_0. \tag{18}$$

The value of k is obtained by numerical computation. The relation between ρ_0 and r_0 given in (18) holds also in the relativistic region. For example, for $\alpha_0 = 0.5$ we have $R_0 = 2r_0$ and $\rho_0 = 1.16r_0$. In the extreme relativistic case, when $\alpha_0 \to 1$, setting $1 - \alpha_0 = \epsilon \ll 1$, we get the following approximate expression, correct to and including terms in ϵ^2:

$$\rho_0 = r_0\left[1 + \varepsilon - \varepsilon^2\left(\frac{9\pi^2}{32} - 1\right)\right], \quad R_0 - \rho_0 = \frac{9\pi^2}{32} r_0 \varepsilon^2. \tag{19}$$

For $\alpha_0 \to 1$ the amplitude of the pulsation is sharply diminished.

Let us now consider the question of what an external observer sees. To do so we must construct in Fig. 3 the world lines of the rays of light, which in the Schwarzschild system are determined by the equations

$$\frac{t_{ob} - t_1}{R_0/c} = \alpha_{ob} - \alpha_1 + \alpha_0 \ln\frac{\alpha_1 - \alpha_0}{\alpha_{ob} - \alpha_0},$$
$$\alpha_{ob} = \frac{r_{ob}}{R_0}, \quad \alpha_1 = \frac{r_1}{R_0}, \quad \alpha_0 = \frac{r_0}{R_0}, \tag{20}$$

where t_{ob} and r_{ob} are the coordinates of the point of observation, and t_1 and r_1 are the coordinates of the point which is the source of the light. In Fig. 3 the world lines of the source of light are shown as thin lines. These lines, tagged with numbers to correspond to the observer's time, intersect the world line of the surface of the pulsating sphere; the radius of the point of intersection is the observed distance. In the right-hand side of Fig. 3 the curve of $r(t_{ob})$ is constructed for the case $\alpha_0 = 0.5$. It follows from the construction that the visible dependence $r(t_{ob})$ is asymmetrical in time.

The investigation of the visible picture, made graphically in Fig. 3, can also be carried out analytically. Owing to mathematical difficulties[8] we shall confine ourselves to the examination of only the radial lines, which corresponds to the case in which the diffraction limit of the telescope is smaller than the angular size of the Schwarzschild sphere. The time for propagation of a light ray is

$$t_{AB} = \frac{1}{c}\int_{r_B}^{r_A} \frac{dr}{1 - r_0/r}. \tag{21}$$

The time of registration of a photon at the point A is $t_A = t_{AB} + t_B$, where t_B is the time of emission of the photon, which for the surface of the pulsating sphere is determined by Eq. (10):

$$t_H = t_B + t_{AB} = \int_{\tau_0}^{\tau} \frac{\sqrt{1-\alpha_0} - c^{-1}dr/d\tau}{1 - r_0/r}\, d\tau + \frac{1}{c}\int_{r(\tau_0)}^{r_A} \frac{dr}{1 - r_0/r},$$
$$\frac{dr}{d\tau} = \pm\sqrt{\frac{r_0}{r}}\sqrt{1 - \frac{r}{R}}. \tag{22}$$

The second term in (22) is additive and does not depend on τ; the sign of $dr/d\tau$ is determined by the direction of motion. It can be seen from (22) that:

a) For motion toward the observer there are no divergences for $r = r_0$, and the integrand is always real and positive. The order of cause and effect is the same for the external observer and a comoving observer.

b) For motion away from the observer the integrand is positive in the R region and negative in the T region, and at $r = r_0$ there is a logarithmic divergence. For the section of the trajectory in the T region t_{ob} is complex, which means that this section is unobservable.[3] Equation (22) is valid only in the case in which the passage of the light ray from the point of emission to the point of observation occurs in vacuum.

The external observer sees an expansion from the moment of passage through zero, as was pointed out in[8]. The contraction stage is visible only in the R region. After the contracting sphere has receded beyond the gravitational radius there remains outside the Schwarzschild sphere only a cloud of photons, which gradually disperses. Simultaneously the observer sees also the phase of expansion of the sphere. As soon as the expanding matter reaches the Schwarzschild sphere the gravitational field is no longer able to hold the photons back, and light is radiated away, so that the line of contraction has to be broken.

We have confined ourselves to the investigation of radial rays. Inclusion of nonradial propagation of rays should smooth out the visible behavior near the minimum value of the radius.[8,9] It can be expected that the nonradial rays will greatly smooth out the variation of the brightness. The problem of the observable variation of the brightness also requires that one take account of the physical conditions in the shell and its temperature, and is beyond the scope of the present paper.

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