CONCERNING ONE SINGULARITY IN PERTURBATION THEORY

V. E. TROITSKIĬ and Yu. M. SHIROKOV

Nuclear Physics Institute, Moscow State University

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It is shown within the framework of perturbation theory that the Heisenberg two-particle matrix element $\langle p \cdot k|A(x)|p' \cdot k'\rangle$ of a neutral scalar field has a singularity when $K \to 0$, where $K = p + k - p' - k'$.

1. In a previous paper[1] (referred to below as I) we studied the singularity of the Heisenberg two-particle matrix element $\langle \mathbf{p} \cdot \mathbf{k}|A(x)|\mathbf{p}' \cdot \mathbf{k}'\rangle$ of a neutral scalar field A(x) (where **p**, **p′** stand for the momenta of the particle of mass m, and **k**, **k′** stand for the momenta of the particle with mass M; we are using the in-basis). The discussion was carried out within the framework of axiomatic theory with the help of new physical quantities introduced in[2,3]—dynamical moments, a particular case of which are the integrals over currents considered recently by a number of authors.

We studied the limiting behavior at infinity in time of the operator for the dynamical moment $D(\mathbf{V}, x_0)$:

$$D(\mathbf{V}, x_0) = (1 - \mathbf{V}^2)^{1/2} \int d^3x A(\mathbf{x}, x_0 + \mathbf{V}\mathbf{x}),$$

where V is a vector parameter such that $V^2 < 1$, and it was shown that the matrix element $\langle 2|A(x)|2\rangle$ possesses a singularity as the 4-vector $K = p + k - p' - k'$ tends to zero. This singularity in the variable $w = (K_0/|K_0|)\sqrt{-K_\mu^2}$ is pole-like. The residue at the pole is related to the S-matrix off the mass shell.

The aim of the present paper is to show the presence of this singularity in perturbation theory.

2. Let us consider the two-particle matrix element of a scalar current and show that in perturbation theory it contains a diagram, which contains poles of the variable w. Let $\psi(x)$, $\chi(x)$, and $\Phi(x)$ be Heisenberg operators of three scalar fields with masses κ, M and m, respectively, and let the interaction Hamiltonian be given in the form

$$H_I = g\psi(x)(\Phi^2(x) + \chi^2(x)). \tag{1}$$

This system of interacting fields is described by the following integral equations of motion:

$$\psi(x) = \psi_{in}(x) + \int d^4y \Delta_\kappa{}^R(x-y) A(y),$$

$$\chi(x) = \chi_{in}(x) + \int d^4y \Delta_M{}^R(x-y) B(y),$$

$$\Phi(x) = \Phi_{in}(x) + \int d^4y \Delta_m{}^R(x-y) C(y), \tag{2}$$

where $\Delta_\kappa^R(x)$, $\Delta_M^R(x)$, $\Delta_m^R(x)$ are the retarded Green's functions with masses κ, M, m, and A(y), B(y), C(y) are the currents of the fields ψ, χ and Φ. From Eq. (1) we have

$$A(x) = g(\Phi^2(x) + \chi^2(x)), \quad B(x) = 2g\psi(x)\Phi(x),$$
$$C(x) = 2g\psi(x)\chi(x). \tag{3}$$

Using the iteration method one may obtain the value of the operator A(x) to any order in the interaction constant g.

We need now to extract those terms in the series which contain the pole obtained in I. To this end let us return to the considerations carried out in I.

The parametrization (i.e., the representation in the form of invariant functions of invariant variables) of the two-particle matrix element of a scalar current may be written in the form[3]

$$\langle \mathbf{pk}|A(x)|\mathbf{p'k'}\rangle = \delta(\mathbf{k}-\mathbf{k}')\frac{e^{-ix(p-p')}}{(2\pi)^3(4EE')^{1/2}} f_1(t)$$
$$+ \delta(\mathbf{p}-\mathbf{p}')\frac{e^{-ix(k-k')}}{(2\pi)^3(4\omega\omega')^{1/2}} f_2(t')$$
$$+ \frac{e^{-ixK}F(s, s', t, t', u, u')}{(2\pi)^3(16EE'\omega\omega')^{1/2}},$$

where

$$E = (\mathbf{p}^2 + M^2)^{1/2}, \quad E' = (\mathbf{p}'^2 + M^2)^{1/2},$$
$$\omega = (\mathbf{k}^2 + \varkappa^2)^{1/2}, \quad \omega' = (\mathbf{k}'^2 + \varkappa^2)^{1/2},$$
$$t = -(p-p')^2, \quad s = -(p+k)^2, \quad u = -(p'-k)^2,$$
$$t' = -(k-k')^2, \quad s' = -(p'+k')^2, \quad u' = -(p-k')^2,$$
$$xp = \mathbf{xp} - x_0E.$$

With the help of the operator relation for dynamic moments

$$D(\mathbf{V}, +\infty) = S^{-1}D(\mathbf{V}, -\infty)S$$

it was shown in I that the interaction form factor F may be represented in the form

$$F = \frac{\varphi(s, t, w_1, w_2, w_3)}{w - i\varepsilon} + \tilde{F}(s, t, w, w_1, w_2, w_3),$$

$$w_1 = (s - s')/w, \quad w_2 = (t - t')/w, \quad w_3 = (u - u')/w, \tag{4}$$

where $w\tilde{F} \to 0$ for $w \to 0$. According to Eq. (17) of I, the quantity φ is expressed in terms of the invariant amplitude g(s, t):

$$\frac{\varphi(s, t, w_1, w_2, w_3)}{w - i\varepsilon} = g(s, t)\left\{\frac{f_1(0)}{2pK - i\varepsilon} + \frac{f_2(0)}{2kK - i\varepsilon}\right\}$$

$$- g^*(s, t)\left\{\frac{f_1(0)}{2p'K - i\varepsilon} + \frac{f_2(0)}{2k'K - i\varepsilon}\right\}$$

$$+ 2\pi i \int \frac{d^3p''}{2E''}\frac{d^3k''}{2\omega''}\delta^4(p + k - p'' - k'')$$

$$\times g(s, t'')g(s, t''')\left\{\frac{f_1(0)}{2p''K - i\varepsilon} + \frac{f_2(0)}{2k''K - i\varepsilon}\right\}. \tag{5}$$

Here the invariant amplitude g(s, t) is related to the S-matrix in the well known way

$$\langle \mathbf{pk}|S|\mathbf{p'k'}\rangle = \delta(\mathbf{p} - \mathbf{p'})\delta(\mathbf{k} - \mathbf{k'}) - 2\pi i\delta^4(K)\frac{g(s, t)}{(16EE'\omega\omega')^{1/2}}$$

and $t'' = -(p - p'')^2$, $t''' = -(p' - p''')^2$.

Our aim is to obtain the pole expression, Eq. (5), in perturbation theory. At that we shall pay special attention to the last term on the right hand side of Eq. (5) since the presence of the pole in diagrams corresponding to the first two terms is well known.

The term of interest to us should contain, as seen from Eq. (5), the product of two invariant amplitudes and the vertex part. This will correspond to a five point function with four lines on the mass shell (two incident and two outgoing particles) and one line off the mass shell, referring to the momentum of the Fourier transform of the current. In lowest order this diagram contains a closed loop (pentagon). Thus we should be calculating the two-particle matrix element of the current A(x) in Eq. (3) accurate to fifth order in g and extract terms of the type indicated above. Direct calculation shows that there will be present five terms referring to diagrams with a pentagon. At that the four internal lines carry the retarded Green's function and the fifth carries $\delta(q_i^2 + m_i^2)$ where q_i is the momentum of the i-th internal line. The five terms will differ only in the location of this δ function on the internal loop. Let us write out one of them, J(K), omitting unimportant numerical factors. Let p and k be the 4-momenta of the incident particles, and p′ and k′ those of the outgoing particles with respectively masses m and M. Then

$$J(K) = \int \frac{d^4q\,\delta(q^2 + m^2)}{G_1G_2G_3G_4},$$

$$G_1 = (q - K)^2 + m^2 - i\varepsilon(q_0 - K_0),$$

$$G_2 = (q - K - p')^2 + \varkappa^2 - i\varepsilon(q_0 - K_0 - p_0'),$$

$$G_3 = (q - K - p' - k')^2 + M^2 - i\varepsilon(q_0 - K_0 - p_0' - k_0'),$$

$$G_4 = (q - K - p' - k' + k)^2 + \varkappa^2 - i\varepsilon(q_0 - K_0 - p_0' - k_0' + k_0). \tag{6}$$

Here the low of conservation of momentum $p + k - p' - k' - K = 0$ is explicitly taken into account.

For simplicity let us consider a coordinate system in which $\mathbf{K} = 0$. In that case taking into account the presence of the δ function under the integral sign the expression for G_1 takes on the form

$$G_1 = (K_0 - i\varepsilon)(2q_0 - K_0) \equiv (K_0 - i\varepsilon)\tilde{G}_1. \tag{7}$$

The first factor in Eq. (7) does not contain the integration variables and may be taken outside the integration sign. It is precisely the factor which gives the desired pole-like behavior provided that the remaining integral is not proportional to some power of K_0. In order to show that let us carry out first integration over q_0 with the help of the δ-function. Combining the two terms obtained in this way we reduce the integral to the form

$$J(K_0) = \frac{1}{K_0 - i\varepsilon} \times \int \frac{d^3q P(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \lambda_1, \lambda_2, q_0)}{2q_0(\alpha_1^2 - q_0^2\alpha_2^2)(\beta_1^2 - q_0^2\beta_2^2)(\gamma_1^2 - q_0^2\gamma_2^2)(\lambda_1^2 - q_0^2\lambda_2^2)} \tag{8}$$

where $q_0 = \sqrt{\mathbf{q}^2 + m^2}$; p is a polynomial in $\alpha, \beta, \gamma, \lambda, q_0$; $\alpha, \beta, \gamma, \lambda$ are determined from the relations

$$\tilde{G}_1 = \alpha_1 + \alpha_2 q_0, \quad G_2 = \beta_1 + \beta_2 q_0, \quad G_3 = \gamma_1 + \gamma_2 q_0,$$

$$G_4 = \lambda_1 + \lambda_2 q_0.$$

We are interested in the dependence of the integrand on K_0 for small K_0. To simplify further considerations we impose certain restrictions. Let $\mathbf{p'} \parallel \mathbf{k'}$, and let in addition

$$\varkappa^2 > 2m^2. \tag{9}$$

In that case the integrand may be expanded in powers of K_0. A term containing K_0 to the zeroth power will be present. This term will give in the given case the desired pole-like behavior. It should be noted that conditions (9) are not necessary for the existence of the pole, and are introduced only to simplify the discussion.

3. Let us briefly discuss the result. First of all we note that the presence of the singularity in w was shown by a method different from that conven-

tionally used in the study of Feynman diagrams. In the given case the discussion given here turns out to be simpler. As the length of the vector of one of the momenta vanishes the Gram determinant vanishes, det $(p_i p_j) = 0$ (where p_i refers to external momenta in the diagram). That type of relation usually characterizes singularities of the second type[4]. However, a special situation arises here. The point is that singularities of the second type arise in the case when the vectors of external momenta lie in the space of smaller dimensions and, consequently, the corresponding Gram determinant vanishes. As a result, in particular, the appearance of singularities of the second type depends strongly on the number of dimensions of the space. If, however, the length of the vector, corresponding to one of the external momenta, tends to zero then the determinant will tend to zero independently of the number of dimensions.

The singularity described here was obtained in I by the study of the asymptotic behavior of dynamic moments of zero rank. It is of interest to carry out analogous considerations for dynamic moments of higher ranks. This will give the possibility of obtaining singularities which are absent in conventional nonrenormalizable versions of perturbation theory.

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